Comparison of ray methods with the exact solution in the 1-D anisotropic "twisted crystal" model

Petr Bulant & Luděk Klimeš
Department of Geophysics, Charles University, Ke Karlovu 3, 121 16 Praha 2, Czech Republic, Fax: +420-2-2191 1214,
E-mails: bulant@seis.karlov.mff.cuni.cz, klimes@seis.karlov.mff.cuni.cz
Ivan Pšenčík
Geophysical Institute, Academy of Sciences of Czech Republic, Boční II 1401, 141 31 Praha 4, Czech Republic, FAX: +420-2-72761549, E-mail: ip@ig.cas.cz

Summary

The exact analytical solution for the plane S wave, propagating along the axis of spirality in the simple 1-D anisotropic "twisted crystal" model, is known. In this contribution, the exact solution and eight ray–theory approximate solutions of different provenance are numerically compared. The eight approximate solutions are: (a1) coupling ray theory solution calculated by evaluating the analytical solution of the coupling ray theory equations, (a2) coupling ray theory solution calculated by 3-D ray tracing package CRT, (a3) coupling ray theory solution calculated by 3-D ray tracing package ANRAY, (b1) zero–order quasi–isotropic approximation to the coupling ray theory calculated by evaluating the analytical solution of the quasi–isotropic ray theory equations, (b2) zero–order quasi–isotropic approximation calculated by 3-D ray tracing package ANRAY, (c1) anisotropic ray theory solution calculated by evaluating the analytical solution of the anisotropic ray theory equations, (d1) isotropic ray theory solution calculated by evaluating the analytical solution of the isotropic ray theory equations, (d2) isotropic ray theory solution calculated by 3-D ray tracing package CRT.

Keywords

Coupling ray theory, quasi–isotropic approximation, anisotropic ray theory, isotropic ray theory, validity conditions of ray methods.

1 Introduction

The *"twisted crystal"* model is created of a homogeneous anisotropic elastic material by uniformly helicoidally twisting the x_1x_2 coordinate plane along the x_3 axis. It is one of the simplest models useful to demonstrate the limits of applicability of the zero–order isotropic and anisotropic ray theories and to test the coupling ray theory, which is the generalization of both the zero–order isotropic and anisotropic ray theories and provides the continuous transition between them.

The great advantage of this model is that the exact analytical solution for the plane S wave propagating along the axis of spirality is known (Lakhtakia 1994, Klimeš 1999).

In this contribution, the exact solution and eight ray–theory approximate solutions of different provenance are numerically compared. The approximate solutions correspond to the zero–order isotropic and anisotropic ray theories, to the coupling ray theory of Coates & Chapman (1990), and to the zero–order approximation of the quasi–isotropic approach to the coupling ray theory according to Pšenčík (1997, 1998a,

In: Seismic Waves in Complex 3-D Structures, Report 8 (Department of Geophysics, Charles University, Prague, 1999), pp. 119–126

1998b). The eight approximate solutions are:
(a1) coupling ray theory solution calculated by evaluating of the analytical solution of the coupling ray theory equations,
(a2) coupling ray theory solution calculated by 3-D ray tracing package CRT,
(a3) coupling ray theory solution calculated by 3-D ray tracing package ANRAY,
(b1) zero–order quasi–isotropic approximation to the coupling ray theory calculated by evaluating the analytical solution of the quasi–isotropic ray theory equations,
(b2) zero–order quasi–isotropic approximation calculated by 3-D ray tracing package ANRAY,
(c1) anisotropic ray theory solution calculated by evaluating the analytical solution of the anisotropic ray theory equations,
(d1) isotropic ray theory solution calculated by evaluating the analytical solution of the isotropic ray theory equations,
(d2) isotropic ray theory solution calculated by 3-D ray tracing package CRT.

The objectives of the numerical comparison are (a) to demonstrate the applicability and accuracy of the coupling ray theory, (b) to debug the computer codes for the coupling ray theory, (c) to find the source of discrepancies between the numerical results of Bulant & Klimeš (1998) and Pšenčík (1998b), (d) to demonstrate the differences between various implementations of the coupling ray theory.

2 Model for the numerical comparison

We use the "twisted crystal" model designed by Vavryčuk (1999). Parameter ε, used by Klimeš (1999) to describe the degree of anisotropy, is

$$\varepsilon = \frac{-\gamma\sin^2(\theta)}{1+\gamma\sin^2(\theta)} \tag{1}$$

in the notation of Vavryčuk (1999). The selected numeric values are

$$\gamma\sin^2(\theta) = 0.15 \times 0.75 = 0.1125 \quad . \tag{2}$$

The arithmetic average v_0^2 of density–normalized elastic parameters a_{1313} and a_{2323} is

$$v_0^2 = a_{44}[1+\gamma\sin^2(\theta)] \tag{3}$$

in the notation of Vavryčuk (1999). The selected numeric values are

$$v_0^2 = 6.0\,\mathrm{km}^2\mathrm{s}^{-2} \times [1+0.15\times 0.75] = 6.675\,\mathrm{km}^2\mathrm{s}^{-2} \quad . \tag{4}$$

The square of the reference isotropic velocity used in package ANRAY is

$$v_R^2 = 6.9\,\mathrm{km}^2\mathrm{s}^{-2} \quad . \tag{5}$$

Parameter K describing the rotation of the crystal axes along the x_3 axis has the value of

$$K = 0.032\,\mathrm{km}^{-1} \quad . \tag{6}$$

The source–receiver distance corresponds to the crystal axes rotated by π radians,

$$x_3 = \frac{\pi}{K} \simeq 98.17477 \quad . \tag{7}$$

Note that Vavryčuk (1999) used $\frac{1}{6}$, $\frac{2}{6}$ and $\frac{3}{6}$ of the above value, respectively.

The "small" parameters in this model then take the values of

$$\left|\frac{\varepsilon}{2}\right| = \frac{9}{178} \simeq 0.0506 \tag{8}$$

and

$$\left|\frac{\Delta v}{v_R}\right| = \left|\frac{v_0 - v_R}{v_R}\right| \simeq 0.0163 \quad . \tag{9}$$

The *resonant frequency* (Klimeš 1999, section 3.4) is

$$F = \left|\frac{v_0 K}{2\pi}\right| \simeq 0.0132\,\mathrm{Hz} \tag{10}$$

and the *coupling frequency* (Klimeš 1999, section 4.1) is

$$\left|\frac{2}{\varepsilon}\right| F \simeq 0.260\,\mathrm{Hz} \quad . \tag{11}$$

The anisotropic ray theory travel times are

$$\tau_1 \simeq 36.212310\,\mathrm{s} \quad , \qquad \tau_2 \simeq 40.079682\,\mathrm{s} \quad . \tag{12}$$

Their arithmetic average, which is the best isotropic travel time, is

$$\tau \simeq 38.145996\,\mathrm{s} \quad . \tag{13}$$

3 Relative differences of the one–way propagator matrices

For the numerical comparison, we define the relative difference between one–way propagator matrix $\mathbf{U}$ and reference one–way propagator matrix $\mathbf{U}_0$ as

$$\Delta = \sqrt{\tfrac{1}{2}\mathrm{Tr}\left([\mathbf{U}_0 - \mathbf{U}]^{+}[\mathbf{U}_0 - \mathbf{U}]\right)} \quad . \tag{14}$$

This definition is equivalent to equation (87) of Klimeš (1999), except for the resonant frequencies $f \in (F\sqrt{1-|\varepsilon|}, F\sqrt{1+|\varepsilon|})$, but numerically no visible difference between the definitions has been observed even at resonant frequencies.

If only the first columns $\mathbf{u}$ and $\mathbf{u}_0$ of propagator matrices $\mathbf{U}$ and $\mathbf{U}_0$ are available for comparison (package ANRAY), we define the relative difference analogously as

$$\Delta = \sqrt{[\mathbf{u}_0 - \mathbf{u}]^{+}[\mathbf{u}_0 - \mathbf{u}]} \quad . \tag{15}$$

4 Numerical comparison

The numerical comparison was divided into two steps:
(A) The comparison of the results of 3-D ray tracing packages ANRAY and CRT with the corresponding numerically evaluated analytical solutions in order to check the equations and to debug both the 3-D code and one–purpose programs for the analytical solutions.
(B) The comparison of the exact solution with the analytical solutions of the equations for the zero–order isotropic and anisotropic ray theories, the coupling ray theory of Coates & Chapman (1990), and the zero–order quasi–isotropic approximation to the coupling ray theory according to Pšenčík (1997, 1998a, 1998b).

We used frequency interval $\langle 0.0\,\mathrm{Hz}, 2.6\,\mathrm{Hz}\rangle$ with frequency step $\Delta f = 0.0125\,\mathrm{Hz}$ for the comparison with the results of 3-D ray tracing packages ANRAY and CRT, aiming to use the Gabor signal centred at 1.3 Hz for future comparison with the high–order anisotropic ray theory or the finite differences in the time domain.

We used frequency interval $\langle 0.001\,\mathrm{Hz}, 10\,\mathrm{Hz}\rangle$, with frequency step $\Delta f = 0.00025\,\mathrm{Hz}$ in subinterval $\langle 0.001\,\mathrm{Hz}, 0.1\,\mathrm{Hz}\rangle$, step $\Delta f = 0.0025\,\mathrm{Hz}$ in subinterval $\langle 0.1\,\mathrm{Hz}, 1\,\mathrm{Hz}\rangle$ and step $\Delta f = 0.025\,\mathrm{Hz}$ in subinterval $\langle 1\,\mathrm{Hz}, 10\,\mathrm{Hz}\rangle$, for the comparison of the different methods, based on the analytical solutions.

4.1 Package CRT, isotropic and coupling ray methods

The numerically evaluated analytical solution of the coupling ray–theory equations was compared with the results of package CRT version 5.20. This enabled to find several bugs in source code file `wan.for` and fix them in version 5.30. Unfortunately, the fixed bugs did not visibly influence the seismograms presented as an example by Bulant & Klimeš (1998) and the differences from the results of Pšenčík (1998b) thus remain unexplained.

Relative differences between the numerical results of package CRT version 5.30 (isotropic and coupling ray methods) and the corresponding analytical solutions are at the level of rounding errors, i.e., below 0.001. Let us note that the phase term $2\pi f\tau$ corresponding to travel time τ amounts the order of 10^3. Since the absolute error of the phase term projects to the relative error in the one–way propagator matrix, the relative differences below 0.001 correspond to the relative travel–time errors below 0.000001.

4.2 Package ANRAY, coupling and quasi–isotropic ray methods

Program `weakan.for` of package ANRAY, designed for the zero–order approximation of the quasi–isotropic approach in version 4.10, was upgraded to include also the coupling ray theory of Coates & Chapman (1990) in version 4.20. The numerically evaluated analytical solutions of the coupling and quasi–isotropic ray methods then have been compared with the results of package ANRAY.

The coupling equations are numerically solved by the Euler method. The error of the Euler method for a harmonic wave of frequency f with integration step $\Delta\tau$ is

$$\Delta_{\mathrm{Euler}} \approx \frac{1}{2}\frac{|\tau_2-\tau_1|}{2\tau}\,2\pi f\,\Delta\tau \quad , \tag{16}$$

see travel times (12) and (13). The error due to the accumulation of the travel–time rounding errors is roughly

$$\Delta_{\mathrm{round}} \approx \frac{1}{2}\frac{\tau}{\Delta\tau}\,2\pi f\;\tau\,\delta \quad , \tag{17}$$

where δ is the relative rounding error, something like $\delta=2^{-24}$ on a PC. The optimum step along the ray is then

$$\Delta\tau = \tau\sqrt{\frac{2\tau}{|\tau_2-\tau_1|}\,\delta} \quad . \tag{18}$$

The differences for step $\Delta\tau = 0.025\,\mathrm{s}$ at frequency $f = 2.6\,\mathrm{Hz}$ are about 2% for the zero–order quasi–isotropic approximation and about 2.5% for the coupling ray theory, which is in a good agreement with estimation (17) giving 2.8%.

4.3 Comparison of four analytical ray–theory solutions with the exact solution

The relative differences of the analytical solutions of the equations for the zero–order isotropic and anisotropic ray theories, the coupling ray theory of Coates & Chapman (1990), and the zero–order quasi–isotropic approximation to the coupling ray theory according to Pšenčík (1997, 1998a, 1998b) from the exact solution are plotted in the log–log scale in Figure 1.

Note that the isotropic ray theory is applied to the isotropic model with the best propagation velocity as suggested by Klimeš (1999) and that the results of the zero–order quasi–isotropic approximation depend on the reference velocity.

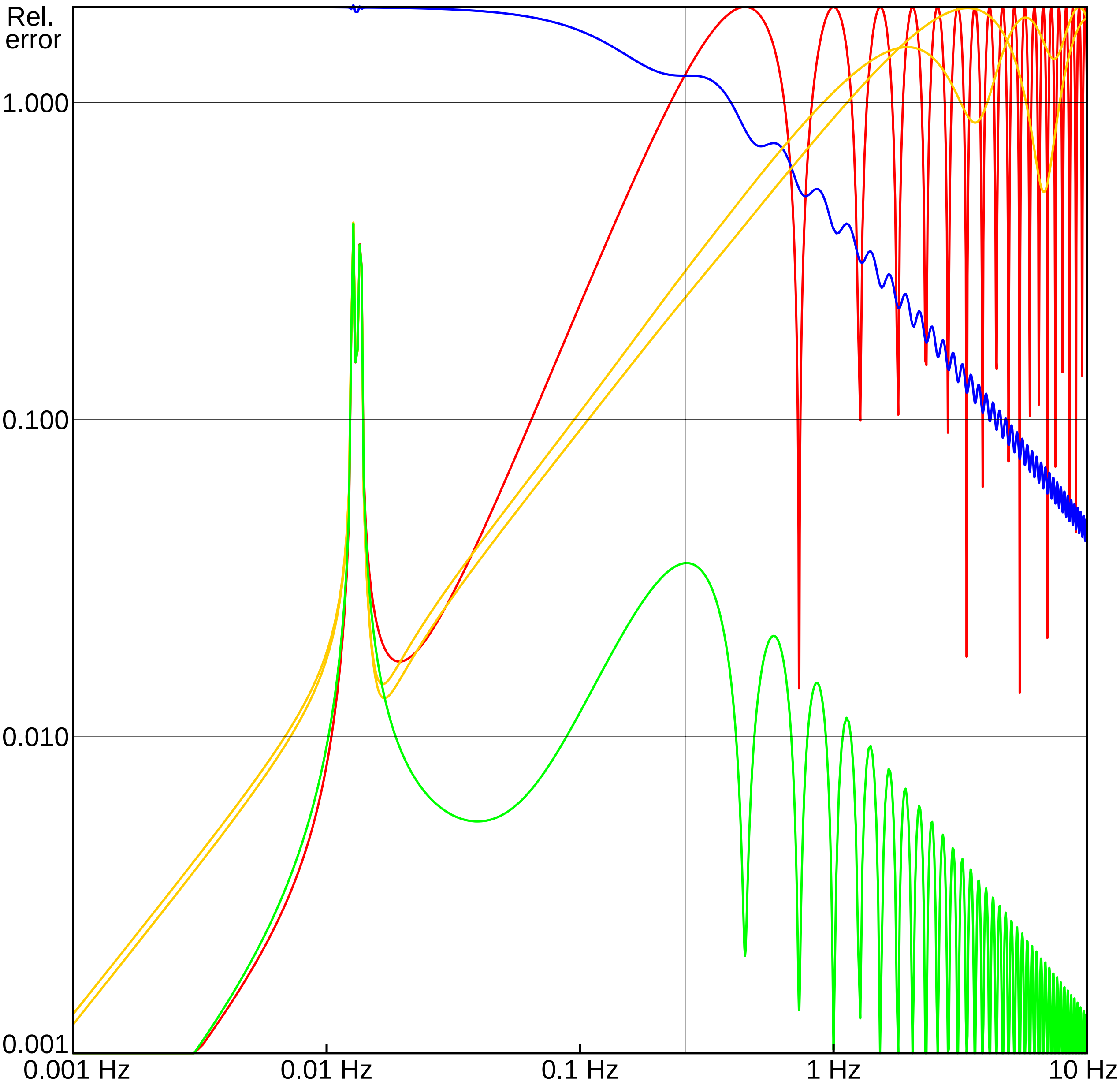

Figure 1. The relative differences of the coupling ray theory of Coates & Chapman (1990) [*green*], zero–order quasi–isotropic approximation to the coupling ray theory according to Pšenčík (1997, 1998a, 1998b) [*yellow*], zero–order anisotropic ray theory [*blue*] and zero–order isotropic ray theory [*red*] from the exact solution. Two vertical lines denote the resonant and coupling frequencies (10), (11). The relative error of 200% is reached, e.g., for an opposite polarization or an opposite phase. The *upper* (up to 1 Hz) quasi–isotropic curve corresponds to reference velocity (5) used in package ANRAY, the *lower* quasi–isotropic curve corresponds to reference velocity (4).

The differences from the exact solution correspond to the theoretical discussion of Klimeš (1999). The zero–order quasi–isotropic approximation to the coupling ray theory cannot bridge the gap between the isotropic and anisotropic ray theories, there are frequencies where both zero–order quasi–isotropic and anisotropic methods reach the relative error of 60%. On the other hand, the coupling ray theory of Coates & Chapman (1990) gives excellent results in this model, except for the resonant frequencies, which are far outside the validity regions of the ray theories.

Note that the only difference between the coupling ray theory and the zero–order

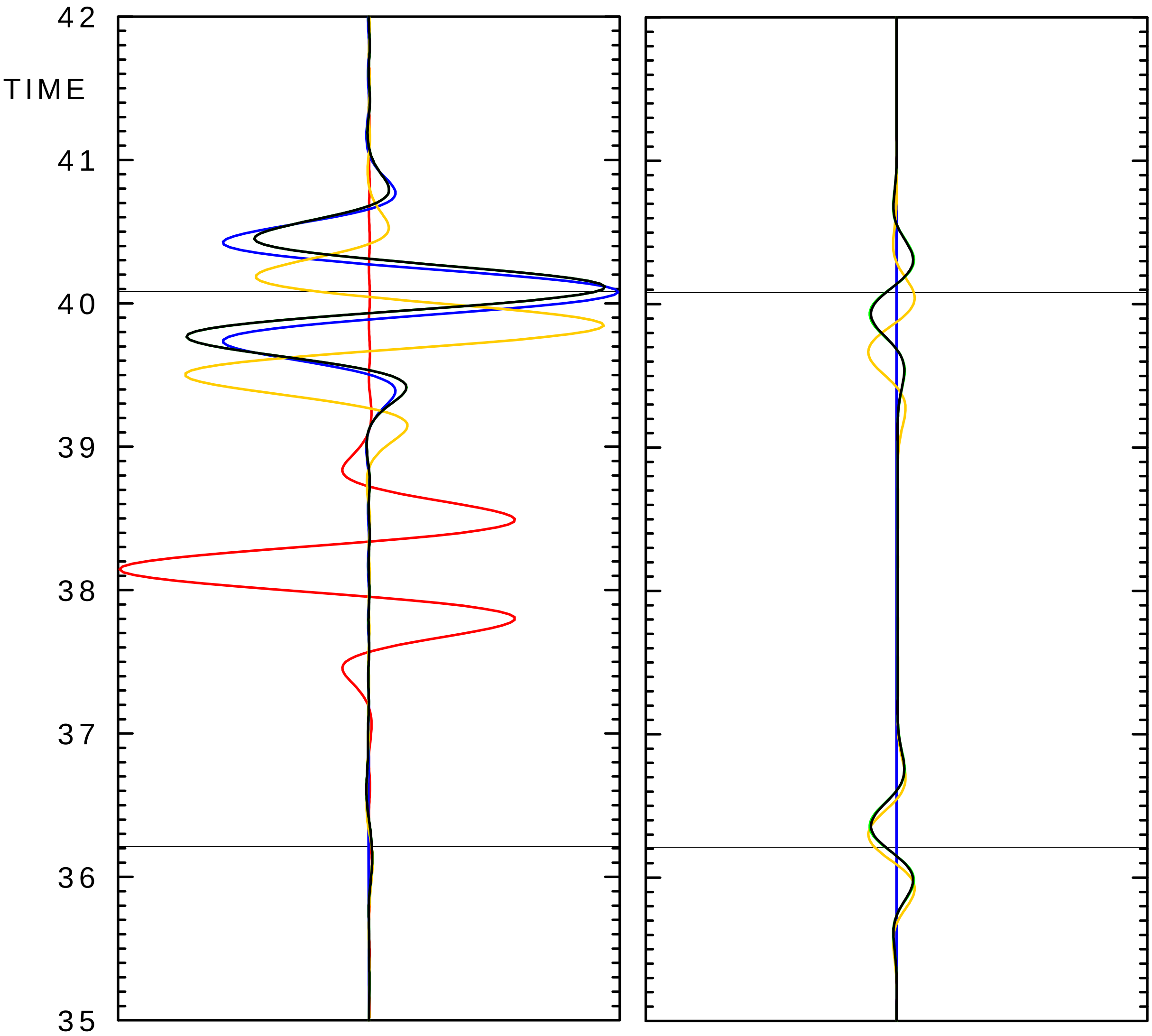

Figure 2. Synthetic seismograms for the initial displacement in the x_1 direction modulated by the Gabor signal of prevailing frequency 1.3 Hz. The x_1 displacement is on the left, x_2 displacement on the right. The exact solution is *black*, coupling ray theory of Coates & Chapman (1990) is *green* and is mostly hidden under the black exact solution, zero–order quasi–isotropic approximation to the coupling ray theory according to Pšenčík (1997, 1998a, 1998b) is *yellow*, zero–order anisotropic ray theory is *blue*, zero–order isotropic ray theory is *red* and is covered by blue on the right. Two horizontal lines denote the anisotropic ray theory travel times (12).

approximation of the quasi–isotropic approach is the calculation of the anisotropic ray theory travel times used in the coupling equations. Whereas the travel times are calculated by the numerical quadratures of the corresponding slownesses along the reference ray (Bulant & Klimeš 1998, equation 2), the quasi–isotropic approximation just corrects the reference travel time by the linear perturbation with respect to the density–normalized elastic parameters. If the reference ray corresponds to a polarization selected according to the zero–order anisotropic ray theory, the travel time of the selected polarization is exact and the coupling ray theory may be used also at high frequencies because only the travel–time difference describing the coupling due the low–frequency scattering is approximate. The high–frequency accuracy of the coupling ray theory may thus be deteriorated at most by the *common ray approximation* for both S–wave polarizations, the accuracy of which should further be studied.

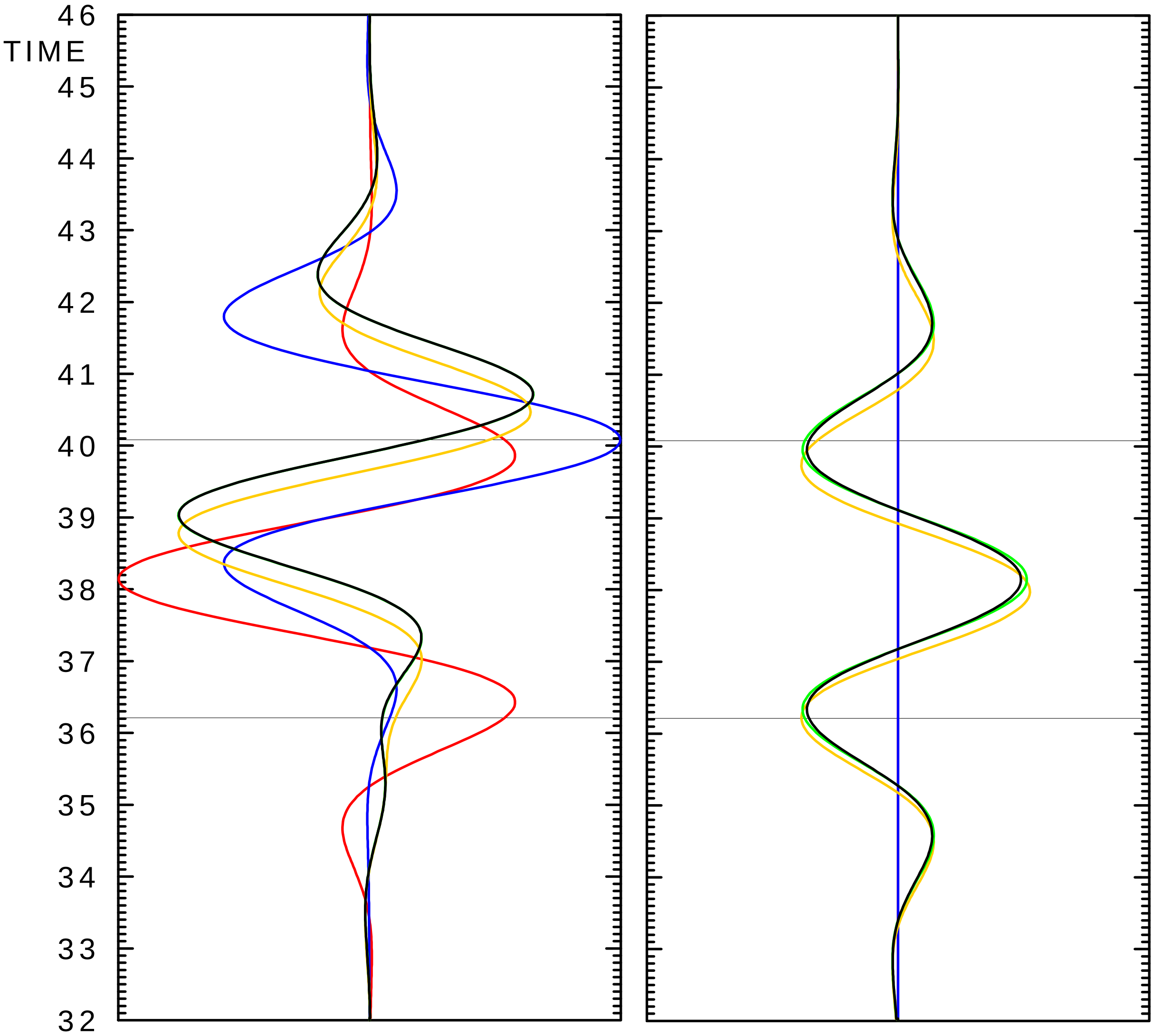

Figure 3. Synthetic seismograms for the initial displacement in the x_1 direction modulated by the Gabor signal of prevailing frequency equal to coupling frequency 0.26 Hz. The x_1 displacement is on the left, x_2 displacement on the right. The exact solution is *black*, coupling ray theory of Coates & Chapman (1990) is *green* and is partly hidden under the black exact solution, zero–order quasi–isotropic approximation to the coupling ray theory according to Pšenčík (1997, 1998a, 1998b) is *yellow*, zero–order anisotropic ray theory is *blue*, zero–order isotropic ray theory is *red* and is covered by blue on the right. Two horizontal lines denote the anisotropic ray theory travel times (12).

4.4 Synthetic seismograms for five analytical solutions

The synthetic seismograms corresponding to the exact solution and to the analytical solutions of the equations for the coupling ray theory of Coates & Chapman (1990), the zero–order quasi–isotropic approximation of the coupling ray theory according to Pšenčík (1997, 1998a, 1998b) and the zero–order isotropic and anisotropic ray theories are shown in Figures 2 and 3. The reference velocity given by (5) is used.

The initial displacement at $x_3 = 0$ has the direction of the x_1 axis and the time dependence in the form of symmetric Gabor signal

$$\exp\left(-\left[\frac{2\pi f_0 t}{4}\right]^2\right)\cos\left(2\pi f_0 t\right) \tag{16}$$

with prevailing frequency

$$f_0 = 1.3\,\mathrm{Hz} \tag{17}$$

for Figure 2, filtered by the cosine band–pass filter described by frequencies

$$0.00\,\mathrm{Hz} \quad , \qquad 0.13\,\mathrm{Hz} \quad , \qquad 2.47\,\mathrm{Hz} \quad , \qquad 2.60\,\mathrm{Hz} \quad . \tag{18}$$

As the prevailing frequency in Figure 2 is five times greater than the coupling frequency, the two S wave arrivals are clearly split and the only visible difference between the exact solution and the anisotropic ray theory is the slightly different polarization. The coupling ray theory is nearly exact.

Analogous seismograms for the prevailing frequency equal to the coupling frequency,

$$f_0 = 0.26\,\mathrm{Hz} \quad , \tag{19}$$

are shown in Figure 3. The cosine band–pass filter is changed to

$$0.00\,\mathrm{Hz} \quad , \qquad 0.026\,\mathrm{Hz} \quad , \qquad 2.47\,\mathrm{Hz} \quad , \qquad 2.60\,\mathrm{Hz} \quad . \tag{20}$$

References

Bulant, P. & Klimeš, L. (1998): Coupling ray theory in weakly anisotropic media. In: Seismic Waves in Complex 3-D Structures, Report 7, pp. 215–223, Dep. Geophys., Charles Univ., Prague.

Coates, R.T. & Chapman, C.H. (1990): Quasi–shear wave coupling in weakly anisotropic 3-D media. Geophys. J. int., **103**, 301–320.

Klimeš, L. (1999): Analytical one–way plane–wave solution in the 1-D anisotropic "twisted crystal" model. In: Seismic Waves in Complex 3-D Structures, Report 8, pp. 103–118, Dep. Geophys., Charles Univ., Prague.

Lakhtakia, A. (1994): Elastodynamic wave propagation in a continuously twisted structurally chiral medium along the axis of spirality. J. acoust. Soc. Am., **95**, 597–600.

Pšenčík, I. (1997): Green functions for inhomogeneous weakly anisotropic media. In: Seismic Waves in Complex 3-D Structures, Report 6, pp. 263–282, Dep. Geophys., Charles Univ., Prague.

Pšenčík, I. (1998a): Green's function for inhomogeneous weakly anisotropic media. Geophys. J. int., **135**, 279–288.

Pšenčík, I. (1998b): Quasi–shear waves in the zero–order approximation of the quasi–isotropic approach. Preliminary results. In: Seismic Waves in Complex 3-D Structures, Report 7, pp. 225–266, Dep. Geophys., Charles Univ., Prague.

Vavryčuk, V. (1999): Applicability of the ray theory for S–wave propagation in elastic weakly anisotropic inhomogeneous media. In: Seismic Waves in Complex 3-D Structures, Report 8, pp. 127–149, Dep. Geophys., Charles Univ., Prague.