Notes on waves in layered media to accompany program Rmatrix

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1. Introduction

These notes describe the theory behind program Rmatrix, which computes the reflection and transmission properties of a stack of plane, homogeneous, generally–anisotropic layers. Rmatrix returns the reflection/transmission "matrices" for the stack, a concept due to B. L. N. Kennett and explored thoroughly for the isotropic case in the book by Kennett (1983). These matrices can be used to form synthetic seismograms for body or surface waves, perhaps those excited by sources buried in the layering. Although these notes do reach the point–source solution for the anisotropic layers, many properties of the solution are not discussed. Some aspects will be quite analogous to the isotropic case (for which Kennett 1983 is recommended) and those new effects due to the anisotropy largely remain to be investigated, which is why Rmatrix was written.

2. Basic equations

In a standard notation the stress–strain relation takes the form

$$\sigma_{ij} = c_{ijkl}\partial_l u_k$$

and the equation of motion for a impulsive point force at $\mathbf{x}_s$ may be written

$$\partial_j \sigma_{ij} = \rho \partial_t^2 u_i + m_i \delta(\mathbf{x} - \mathbf{x}_s)\delta(t) . \tag{2.1}$$

The material properties vary only with depth x_3, which for convenience will also be referred to as z, and the source will taken to be at $\mathbf{x}_s = (0, 0, z_s)$. Greek subscripts α, β, ..., will be used below to denote the indices 1, 2 of the lateral coordinates. The Fourier transforms wrt to time and the lateral coordinates are defined by

$$u(p_1, p_2, \omega) = \iiint_{-\infty}^{\infty} u(x_1, x_2, t) e^{-\mathrm{i}\omega(p_1 x_1 + p_2 x_2 - t)}\, dx_1 dx_2 dt . \tag{2.2}$$

After Fourier transformation the equations of motion may be written in the matrix form

$$\frac{d\mathbf{y}}{dz} = \mathrm{i}\omega \mathbf{A}\mathbf{y} + \mathbf{s}\delta(z - z_s) e^{-\mathrm{i}\omega p_\alpha x_{\alpha s}} \tag{2.3a}$$

where the 6–vector of displacement and normal stress (traction) is

$$\mathbf{y} = \begin{pmatrix} u_i \\ t_i \end{pmatrix}, \qquad t_i = \frac{1}{\mathrm{i}\omega}(\sigma_{13}, \sigma_{23}, \sigma_{33})^T , \tag{2.3b}$$

the force 6-vector is

$$\mathbf{s} = (0, 0, 0, m_1, m_2, m_3)^T , \tag{2.3c}$$

the "system matrix" is

$$\mathbf{A} = \begin{pmatrix} \mathbf{T}^T & \mathbf{C}_{33}^{-1} \\ \mathbf{S} & \mathbf{T} \end{pmatrix} , \tag{2.3d}$$

with

$$\mathbf{T} = -p_\gamma\, \mathbf{C}_{\gamma 3}\, \mathbf{C}_{33}^{-1} , \qquad \mathbf{S} = \rho \mathbf{I} - p_\gamma p_\alpha \left(\mathbf{C}_{\alpha\gamma} - \mathbf{C}_{\alpha 3}\, \mathbf{C}_{33}^{-1}\, \mathbf{C}_{3\gamma} \right) , \tag{2.3e}$$

and the matrix $\mathbf{C}_{ij}$ is defined by

$$(C_{ij})_{kl} = c_{kilj} . \tag{2.3f}$$

Equation (2.3) is equivalent to equation (2.10) of Woodhouse (1974) except that i and frequency ω have been factored out of the $\mathbf{A}$ by using a slightly different displacement–stress vector.

The six eigenvalues of $\mathbf{A}$ represent the 6 possible normal slownesses p_3 (three upgoing and three downgoing waves) for the given tangential slownesses. The six eigenvectors are the displacements and interface tractions associated with each possible wavetype.

The solution of (2.3) in a homogeneous layer takes the form

$$\mathbf{y}(z) = \mathbf{N}e^{i\omega\mathbf{q}z}\,\mathbf{c}\,, \tag{2.4}$$

where $\mathbf{N}$ is the 6×6 matrix whose columns are the eigenvectors of $\mathbf{A}$, $\mathbf{q}$ is the diagonal matrix of corresponding eigenvalues and $\mathbf{c}$ is a column vector of constants call the wavevector for the layer. The elements of $\mathbf{c}$ are the amplitudes of the upgoing and downgoing waves and the assumed ordering of the columns of $\mathbf{N}$ is indicated by the example

$$\mathbf{c} = (P_D\,, S_{1D}\,, S_{2D}\,, P_U\,, S_{1U}\,, S_{2U})^T\,, \tag{2.5}$$

where the subscripts indicate down– and up–going waves and the prefix *quasi* is assumed in the anisotropic case. Each eigenvector or column of $\mathbf{N}$ may be normalized so that the displacement elements form a unit vector. Then, for a single upgoing quasi-P wave with unit amplitude the corresponding wavevector $\mathbf{c}$ would have the form $(0, 0, 0, 1, 0, 0)^T$.

3. Single interface

The boundary conditions require that $\mathbf{y}$ be continuous across the interface so the wavevector c_- just above the interface and that c_+ just below the interface are related by

$$\mathbf{N}(z+)\,\mathbf{c}_+ = \mathbf{N}(z-)\,\mathbf{c}_- \,. \tag{3.1}$$

The set of all possible unit–amplitude incident wavetypes and directions is incorporated in the 6×6 matrix equation,

$$\mathbf{N}(z+)\begin{pmatrix} \mathbf{R}_U & \mathbf{T}_D \\ \mathbf{I} & 0 \end{pmatrix} = \mathbf{N}(z-)\begin{pmatrix} 0 & \mathbf{I} \\ \mathbf{T}_U & \mathbf{R}_D \end{pmatrix} . \tag{3.2}$$

This defines the reflection and transmission coefficents $\mathbf{R}_U$, $\mathbf{T}_U$, etc. Rearranging (3.2) yields

$$\mathbf{N}^{-1}(z+)\mathbf{N}(z-) \;=\; \mathbf{Q}(z+,z-) \;=\; \begin{pmatrix} \mathbf{T}_D - \mathbf{R}_U\mathbf{T}_U^{-1}\mathbf{R}_D & \mathbf{R}_U\mathbf{T}_U^{-1} \\ -\mathbf{T}_U^{-1}\mathbf{R}_D & \mathbf{T}_U^{-1} \end{pmatrix} . \tag{3.3}$$

The matrix $\mathbf{Q}$ is the *interface* or *scattering* matrix for the boundary and has the effect of translating between the wavevectors in the upper and lower medium. The single–interface reflection and transmission coefficients can now be determined by inverting (3.3) using the 3×3 partitions of $\mathbf{Q}$ to obtain

$$\begin{pmatrix} \mathbf{T}_U & \mathbf{R}_U \\ \mathbf{R}_D & \mathbf{T}_D \end{pmatrix} = \begin{pmatrix} \mathbf{Q}_{22}^{-1} & \mathbf{Q}_{12}\,\mathbf{Q}_{22}^{-1} \\ -\mathbf{Q}_{22}^{-1}\mathbf{Q}_{21} & \mathbf{Q}_{11} - \mathbf{Q}_{12}\,\mathbf{Q}_{22}^{-1}\,\mathbf{Q}_{21} \end{pmatrix} . \tag{3.4}$$

Sometimes it is useful to consider the alternative or inverse form of the scatterer

$$\mathbf{N}^{-1}(z-)\mathbf{N}(z+) \;=\; \mathbf{Q}(z-,z+) \;=\; \begin{pmatrix} \mathbf{T}_D^{-1} & -\mathbf{T}_D^{-1}\mathbf{R}_U \\ \mathbf{R}_D\mathbf{T}_D^{-1} & \mathbf{T}_U - \mathbf{R}_D\mathbf{T}_D^{-1}\mathbf{R}_U \end{pmatrix} . \tag{3.5}$$

4. Stack of homogeneous layers

Equation (3.1) can be generalised to describe a finite depth interval (z_1, z_2)

$$\mathbf{N}(z_2)\,\mathbf{c}_2 = \mathbf{P}(z_2, z_1)\mathbf{N}(z_1)\,\mathbf{c}_1\,, \tag{4.1}$$

where $\mathbf{P}$ is the displacement–stress propagator. Hence, the wavevectors $\mathbf{c}_{1,2}$ are connected by the "wave propagator"

$$\mathbf{Q}(z_2, z_1) \;=\; \mathbf{N}^{-1}(z_2)\mathbf{P}(z_2, z_1)\mathbf{N}(z_1) \;=\; \begin{pmatrix} \mathbf{T}_D - \mathbf{R}_U\mathbf{T}_U^{-1}\mathbf{R}_D & \mathbf{R}_U\mathbf{T}_U^{-1} \\ -\mathbf{T}_U^{-1}\mathbf{R}_D & \mathbf{T}_U^{-1} \end{pmatrix} \tag{4.2a}$$

and the reflection/transmission matrices here are now those for the entire region between z_1 and z_2 (i.e. including reverberations). The form (4.2a) is suitable for the case where $z_2 > z_1$ (i.e. z_2 is deeper) and the corresponding inverse scatterer is

$$\mathbf{Q}(z_1, z_2) \;=\; \mathbf{N}^{-1}(z_1)\mathbf{P}(z_1, z_2)\mathbf{N}(z_2) \;=\; \begin{pmatrix} \mathbf{T}_D^{-1} & -\mathbf{T}_D^{-1}\mathbf{R}_U \\ \mathbf{R}_D\mathbf{T}_D^{-1} & \mathbf{T}_U - \mathbf{R}_D\mathbf{T}_D^{-1}\mathbf{R}_U \end{pmatrix}. \tag{4.2b}$$

It follows from (4.2) and the chain rule for the propagator $\mathbf{P}$ that the wave propagator also satisfies a chain rule

$$\mathbf{Q}(z_3, z_1) \;=\; \mathbf{Q}(z_3, z_2)\,\mathbf{Q}(z_2, z_1)\,. \tag{4.3}$$

The wave propagator across a homogeneous layer of thickness $h = z_2 - z_1$ is

$$\mathbf{E}(z_2, z_1) = \begin{pmatrix} \mathbf{E}_D & 0 \\ 0 & \mathbf{E}_U \end{pmatrix}, \tag{4.4a}$$

where

$$\mathbf{E}_D = \;\mathrm{diag}\left[e^{\mathrm{i}\omega q_{D1} h}, e^{\mathrm{i}\omega q_{D2} h}, e^{\mathrm{i}\omega q_{D3} h}\right] \qquad\qquad \mathbf{E}_U = \;\mathrm{diag}\left[e^{\mathrm{i}\omega q_{U1} h}, e^{\mathrm{i}\omega q_{U2} h}, e^{\mathrm{i}\omega q_{U3} h}\right]. \tag{4.4b}$$

Now consider the reflection matrices at the top z_{1+} of a homogeneous layer overlying a stack of layers from z_{2-} down to z_{3+}. Let the reflection matrices for the overall region (z_{1+}, z_{3+}) be denoted by an overbar $\bar{\mathbf{R}}_U$, etc. and those for the region (z_{2-}, z_{3+}) be without an overbar. From the chain rule (4.3) we have $\mathbf{Q}(z_3+, z_1+) = \mathbf{Q}(z_3+, z_2-)\mathbf{Q}(z_2-, z_1+)$ and then from relation (4.2a) we find

$$\begin{pmatrix} \bar{\mathbf{T}}_D - \bar{\mathbf{R}}_U\bar{\mathbf{T}}_U^{-1}\bar{\mathbf{R}}_D & \bar{\mathbf{R}}_U\bar{\mathbf{T}}_U^{-1} \\ -\bar{\mathbf{T}}_U^{-1}\bar{\mathbf{R}}_D & \bar{\mathbf{T}}_U^{-1} \end{pmatrix} = \begin{pmatrix} \mathbf{T}_D - \mathbf{R}_U\mathbf{T}_U^{-1}\mathbf{R}_D & \mathbf{R}_U\mathbf{T}_U^{-1} \\ -\mathbf{T}_U^{-1}\mathbf{R}_D & \mathbf{T}_U^{-1} \end{pmatrix} \begin{pmatrix} \mathbf{E}_D & 0 \\ 0 & \mathbf{E}_U \end{pmatrix}. \tag{4.5}$$

Hence we have the addition rules for adding a new layer to the stack (with no interface at the top of that layer as yet)

$$\bar{\mathbf{T}}_U = \mathbf{E}_U^{-1}\mathbf{T}_U \qquad \bar{\mathbf{R}}_D = \mathbf{E}_U^{-1}\mathbf{R}_D\mathbf{E}_D \qquad \bar{\mathbf{R}}_U = \mathbf{R}_U \qquad \bar{\mathbf{T}}_D = \mathbf{T}_D\mathbf{E}_D\,. \tag{4.6}$$

Now consider the next interface (z_{1-}, z_{1+}). The chain rule (4.3) for wave propagators implies

$$
\begin{pmatrix} \bar{\bar{\mathbf{T}}}_D - \bar{\bar{\mathbf{R}}}_U \bar{\bar{\mathbf{T}}}_U^{-1} \bar{\bar{\mathbf{R}}}_D & \bar{\bar{\mathbf{R}}}_U \bar{\bar{\mathbf{T}}}_U^{-1} \\ -\bar{\bar{\mathbf{T}}}_U^{-1} \bar{\bar{\mathbf{R}}}_D & \bar{\bar{\mathbf{T}}}_U^{-1} \end{pmatrix}
$$

$$
= \begin{pmatrix} \bar{\mathbf{T}}_D - \bar{\mathbf{R}}_U \bar{\mathbf{T}}_U^{-1} \bar{\mathbf{R}}_D & \bar{\mathbf{R}}_U \bar{\mathbf{T}}_U^{-1} \\ -\bar{\mathbf{T}}_U^{-1} \bar{\mathbf{R}}_D & \bar{\mathbf{T}}_U^{-1} \end{pmatrix} \begin{pmatrix} \mathbf{T}_D - \mathbf{R}_U \mathbf{T}_U^{-1} \mathbf{R}_D & \mathbf{R}_U \mathbf{T}_U^{-1} \\ -\mathbf{T}_U^{-1} \mathbf{R}_D & \mathbf{T}_U^{-1} \end{pmatrix}, \tag{4.7}
$$

where a double overbar represents the net result for the region (z_{1-}, z_{3+}) and the coefficients without an overbar are those for the top interface in isolation. The addition rules for including the new interface in the stack are therefore

$$
\begin{aligned}
\bar{\bar{\mathbf{T}}}_U &= \mathbf{T}_U(\mathbf{I} - \bar{\mathbf{R}}_D \mathbf{R}_U)^{-1} \bar{\mathbf{T}}_U && (4.8a) \\
\bar{\bar{\mathbf{R}}}_D &= \mathbf{R}_D + \mathbf{T}_U(\mathbf{I} - \bar{\mathbf{R}}_D \mathbf{R}_U)^{-1} \bar{\mathbf{R}}_D \mathbf{T}_D && (4.8b) \\
&= \mathbf{R}_D + \mathbf{T}_U \bar{\mathbf{R}}_D(\mathbf{I} - \mathbf{R}_U \bar{\mathbf{R}}_D)^{-1} \mathbf{T}_D && (4.8c) \\
\bar{\bar{\mathbf{R}}}_U &= \bar{\mathbf{R}}_U + \bar{\mathbf{T}}_D \mathbf{R}_U(\mathbf{I} - \bar{\mathbf{R}}_D \mathbf{R}_U)^{-1} \bar{\mathbf{T}}_U && (4.8d) \\
&= \bar{\mathbf{R}}_U + \bar{\mathbf{T}}_D(\mathbf{I} - \mathbf{R}_U \bar{\mathbf{R}}_D)^{-1} \mathbf{R}_U \bar{\mathbf{T}}_U && (4.8e) \\
\bar{\bar{\mathbf{T}}}_D &= \bar{\mathbf{T}}_D[\mathbf{I} + \mathbf{R}_U(\mathbf{I} - \bar{\mathbf{R}}_D \mathbf{R}_U)^{-1} \bar{\mathbf{R}}_D] \mathbf{T}_D \\
&= \bar{\mathbf{T}}_D(\mathbf{I} - \mathbf{R}_U \bar{\mathbf{R}}_D)^{-1} \mathbf{T}_D\,. && (4.8f)
\end{aligned}
$$

5. Free surface

The stress–displacement equation

$$\begin{pmatrix} \mathbf{u}_0 \\ 0 \end{pmatrix} = \mathbf{P}(0, z_L)\mathbf{N}(z_L) \begin{pmatrix} \mathbf{c}_D \\ 0 \end{pmatrix} \tag{5.1}$$

corresponds to downgoing waves at the depth z_L and a stress–free condition at $z = 0$. It is convenient to introduce 3×3 partitions of the eigenvector matrix defined according to

$$\mathbf{N} = \begin{pmatrix} \mathbf{M}_D & \mathbf{M}_U \\ \mathbf{N}_D & \mathbf{N}_U \end{pmatrix} . \tag{5.2}$$

Then, using the definition (4.2b) of the wave propagator, the stress–free condition implies that

$$0 = (\mathbf{N}_{D0} + \mathbf{N}_{U0}\mathbf{R}_D)\,\mathbf{T}_D^{-1}\mathbf{c}_D \tag{5.3}$$

and the condition for a nonzero value of $\mathbf{c}_D$ is

$$0 = \det \left|(\mathbf{N}_{D0} + \mathbf{N}_{U0}\mathbf{R}_D)\,\mathbf{T}_D^{-1}\right| . \tag{5.4}$$

This is one form of the dispersion relation for surface waves.

6. Symmetries

The system matrix $\mathbf{A}$ (2.3d) contains a number of symmetries that can be used to reduce the amount of work done in calculations and to check results. These symmetries have been noted by many authors (e.g. Woodhouse 1974; Kennett et al. 1978; Fryer & Frazer 1984; Chapman & Orcutt 1985; Garmany 1983; Thomson, Clarke & Garmany 1986; Chapman 1994).

The eigenvalue problem,

$$\mathbf{A}\mathbf{N} = \mathbf{N}\mathbf{q} \tag{6.1}$$

has the related equations

$$\mathbf{N}^T\mathbf{A}^T = \mathbf{q}\mathbf{N}^T \qquad \mathbf{N}^{-1}\mathbf{A} = \mathbf{q}\mathbf{N}^{-1}\,. \tag{6.2}$$

By using the fact that $\mathbf{A}$ exhibits the symmetry

$$(\mathbf{J}_1\mathbf{A}) - (\mathbf{J}_1\mathbf{A})^T = 0 \qquad \mathbf{J}_1 = \begin{pmatrix} 0 & \mathbf{I} \\ \mathbf{I} & 0 \end{pmatrix}\,, \tag{6.3}$$

it is found from the first of (6.2) that

$$\left(\mathbf{N}^T\mathbf{J}_1\right)\mathbf{A} = \mathbf{q}\left(\mathbf{N}^T\mathbf{J}_1\right)\,. \tag{6.4}$$

On comparing this with the second of (6.2) the inverse of $\mathbf{N}$ is identified as

$$\mathbf{N}^{-1} = \mathbf{d}\mathbf{N}^T\mathbf{J}_1\,, \tag{6.5}$$

where $\mathbf{d}$ is a diagonal matrix that depends on how $\mathbf{N}$ is normalized. It is possible to normalize so that $\mathbf{d}$ is the identity matrix, but more generally it may be found from the trivial diagonal inverse

$$\mathbf{d} = \left(\mathbf{N}^T\,\mathbf{J}_1\,\mathbf{N}\right)^{-1}\,. \tag{6.6}$$

For a stack of generally anisotropic lossy layers, a propagation invariant can be found by considering the system (2.1) and the related system obtained by replacing (p_1, p_2) with $(-p_1, -p_2)$ (Chapman 1994):

$$\frac{d\mathbf{y}}{dz} = \mathrm{i}\omega\mathbf{A}\mathbf{y} \qquad \text{and} \qquad \frac{d\mathbf{y}'}{dz} = \mathrm{i}\omega\mathbf{A}'\mathbf{y}'\,, \tag{6.7}$$

where

$$\mathbf{A}'(p_1, p_2) = \mathbf{A}(-p_1, -p_2) = \begin{pmatrix} -\mathbf{T}^T & \mathbf{C}_{33}^{-1} \\ \mathbf{S} & -\mathbf{T} \end{pmatrix}\,. \tag{6.8}$$

The symmetry

$$\mathbf{J}'\mathbf{A}' - (\mathbf{J}'\mathbf{A})^T = 0 \tag{6.9}$$

applies, where

$$\mathbf{J}' = \begin{pmatrix} 0 & \mathbf{I} \\ -\mathbf{I} & 0 \end{pmatrix} . \tag{6.10}$$

Hence for fundamental matrices $\mathbf{F}$ and $\mathbf{F}'$

$$\frac{d}{dz}\left[\mathbf{F}'^T\mathbf{J}'\mathbf{F}\right] = \mathrm{i}\omega\left[\mathbf{F}'^T\mathbf{A}'^T\mathbf{J}'\mathbf{F} + \mathbf{F}'^T\mathbf{J}'\mathbf{A}\mathbf{F}\right] = 0$$

and this leads to the invariant

$$\mathbf{F}'^T(z)\mathbf{J}'\mathbf{F}(z) = constant \qquad\qquad \mathbf{P}'^T(z_2, z_1)\mathbf{J}'\mathbf{P}(z_2, z_1) = \mathbf{J}' , \tag{6.11}$$

the latter version being for a propagator.

Now consider the eigenvalue problem for the reversed system

$$\mathbf{A}'\mathbf{N}' = \mathbf{N}'\mathbf{q}' . \tag{6.12}$$

The point symmetry of anisotropic materials means that the eigenvalues have the property $\mathbf{q}(-p_1, -p_2) = -\mathbf{q}(p_1, p_2)$ and the group velocity also has the same property. The latter means that the first three columns of $\mathbf{N}'$ correspond to upgoing waves and the last three to downgoing waves. By steps analogous to those which lead to (6.5) and (6.6), it now follows from the reversal symmetry (6.9) that the eigenvector matrices are related by

$$\mathbf{N}'^T\mathbf{J}'\mathbf{N} = \mathbf{d}' , \tag{6.13a}$$

a diagonal matrix. On multiplying this from the right by the earlier result (6.5) we find the additional relation

$$\mathbf{N}' = \mathbf{J}'\mathbf{J}_1\mathbf{N}\mathbf{d}\mathbf{d}' \tag{6.13b}$$

and we note that $\mathbf{J}'\mathbf{J}_1$ is diagonal too. For later use it is convenient to introduce the principle 3×3 partitions of $\mathbf{d}'$ defined by $\mathbf{d}' = (\mathbf{d}_A \oplus \mathbf{d}_B)$.

Next consider the reflection/transmission problem for a stack of layers

$$\mathbf{N}(z_2)\begin{pmatrix} \mathbf{R}_U & \mathbf{T}_D \\ \mathbf{I} & 0 \end{pmatrix} = \mathbf{P}(z_2, z_1)\mathbf{N}(z_1)\begin{pmatrix} 0 & \mathbf{I} \\ \mathbf{T}_U & \mathbf{R}_D \end{pmatrix} \tag{6.14a}$$

and that for the reversed system

$$\mathbf{N}'(z_2)\begin{pmatrix} \mathbf{I} & 0 \\ \mathbf{R}'_U & \mathbf{T}'_D \end{pmatrix} = \mathbf{P}'(z_2, z_1)\mathbf{N}'(z_1)\begin{pmatrix} \mathbf{T}'_U & \mathbf{R}'_D \\ 0 & \mathbf{I} \end{pmatrix} , \tag{6.14b}$$

where the ordering of columns of $\mathbf{N}'$ has been noted.

Multiplying (6.14a) from the left by the tranpose of (6.14b), using the invariant (6.11) and (6.13a) then leads to

$$\begin{pmatrix} \mathbf{I} & 0 \\ \mathbf{R}'_U & \mathbf{T}'_D \end{pmatrix}^T \mathbf{d}'_2 \begin{pmatrix} \mathbf{R}_U & \mathbf{T}_D \\ \mathbf{I} & 0 \end{pmatrix} = \begin{pmatrix} \mathbf{T}'_U & \mathbf{R}'_D \\ 0 & \mathbf{I} \end{pmatrix}^T \mathbf{d}'_1 \begin{pmatrix} 0 & \mathbf{I} \\ \mathbf{T}_U & \mathbf{R}_D \end{pmatrix} , \tag{6.15}$$

where $\mathbf{d}'_{1,2} = \mathbf{d}'(z_{1,2})$. This can be recast in the form

$$\boldsymbol{\mathcal{R}}'^T = \begin{pmatrix} -\mathbf{d}'_{2A} & 0 \\ 0 & \mathbf{d}'_{1B} \end{pmatrix} \boldsymbol{\mathcal{R}} \begin{pmatrix} \mathbf{d}'^{-1}_{2B} & 0 \\ 0 & -\mathbf{d}'^{-1}_{1A} \end{pmatrix}, \tag{6.16}$$

where

$$\boldsymbol{\mathcal{R}} = \begin{pmatrix} \mathbf{R}_U & \mathbf{T}_D \\ \mathbf{T}_U & \mathbf{R}_D \end{pmatrix}. \tag{6.17}$$

Relation (6.16) is known as reciprocity symmetry (Chapman 1994). It is the most general relation as it applies for a stack of possibly lossy ($\mathbf{A}$ complex) arbitrarily anisotropic layers.

Other symmetries exist for special situations. If the medium is perfectly elastic $\mathbf{A}^\dagger = \mathbf{A}^T$ and when combined with the $\mathbf{J}_1$ symmetry this leads to a propagation invariant. For any fundamental matrix $\mathbf{F}$

$$\frac{d\mathbf{F}}{dz} = \mathrm{i}\omega\mathbf{A}\mathbf{F} \qquad \text{and} \qquad \frac{d\mathbf{F}^\dagger}{dz} = -\mathrm{i}\omega\mathbf{F}^\dagger\mathbf{A}^\dagger = -\mathrm{i}\omega\mathbf{F}^\dagger\mathbf{A}^T,$$

so that J_1 symmetry implies

$$\frac{d}{dz}\left[\mathbf{F}^\dagger\mathbf{J}_1\mathbf{F}\right] = -\mathrm{i}\omega\left[\mathbf{F}^\dagger\mathbf{A}^T\mathbf{J}_1\mathbf{F} - \mathbf{F}^\dagger\mathbf{J}_1\mathbf{A}^T\mathbf{F}\right] = 0$$

or

$$\mathbf{F}^\dagger(z)\mathbf{J}_1\mathbf{F}(z) = \mathit{constant} \qquad \text{and} \qquad \mathbf{P}^\dagger(z_2,z_1)\mathbf{J}_1\mathbf{P}(z_2,z_1) = \mathbf{J}_1\,. \tag{6.18}$$

For propagating waves in lossless layers the eigenvalues $\mathbf{q}$ and eigenvectors $\mathbf{N}$ in (6.1) are real. The relation (6.6) can then be replaced by

$$\mathbf{N}^T\mathbf{J}_1\mathbf{N} = \mathbf{N}^\dagger\mathbf{J}_1\mathbf{N} = \mathbf{d}^{-1}\,. \tag{6.19}$$

Recalling the wave propagator $\mathbf{Q}$ definition in terms of $\mathbf{P}$ (4.2) we see that the second of (6.18) takes the form

$$[\mathbf{N}(z_2)\mathbf{Q}(z_2,z_1)\mathbf{N}^{-1}(z_1)]^\dagger\mathbf{J}_1[\mathbf{N}(z_2)\mathbf{Q}(z_2,z_1)\mathbf{N}^{-1}(z_1)] = \mathbf{J}_1$$

and rearranging this gives

$$\mathbf{Q}^\dagger(z_2,z_1)\mathbf{d}^{-1}(z_2)\mathbf{Q}(z_2,z_1) = \mathbf{d}^{-1}(z_1)\,, \tag{6.20}$$

where (6.19) has been used. The wave propagator relates to the reflection/transmission properties of the stack via (4.2). On using this fact in (6.20) and rearranging the information one finds, after some work, that

$$\boldsymbol{\mathcal{R}}^\dagger \begin{pmatrix} -\mathbf{d}^{-1}_{2A} & 0 \\ 0 & \mathbf{d}^{-1}_{1B} \end{pmatrix} \boldsymbol{\mathcal{R}} = \begin{pmatrix} \mathbf{d}^{-1}_{2B} & 0 \\ 0 & -\mathbf{d}^{-1}_{1A} \end{pmatrix}, \tag{6.21}$$

where $\mathbf{d}_{A,B}$ are the principle blocks of $\mathbf{d}$ in (6.19). Relation (6.21) can be called the unitarity of the reflection/transmission matrices for propagating waves in anisotropic lossless layers.

In the case of a horizontal mirror plane there is the additional symmetry

$$\mathbf{J}_2\mathbf{A} - (\mathbf{J}_2\mathbf{A})^T = 0\,, \tag{6.22}$$

where

$$\mathbf{J}_2 = \begin{pmatrix} 0 & \mathbf{X} \\ -\mathbf{X} & 0 \end{pmatrix} \qquad\qquad \mathbf{X} = \mathrm{diag}[1,1,-1]\,. \tag{6.23}$$

This symmetry leads to the invariant

$$\mathbf{F}^T(z)\mathbf{J}_2\mathbf{F}(z) = constant \qquad\qquad \mathbf{P}^T(z_2,z_1)\mathbf{J}_2\mathbf{P}(z_2,z_1) = \mathbf{J}_2\,, \tag{6.24}$$

which holds in the presence of anelasticity ($\mathbf{A}$ complex). Steps analogous to those leading to the reciprocity relation (6.16), but now using $\mathbf{J}_2$ symmetry, lead to what is essentially the special form of (6.16) which applies in the transversely isotropic possibly lossy stack of layers.

Lastly, for a single interface separating possibly lossy anisotropic media, the simple eigenvector inverse result (6.5,6) leads to a relation analogous to the unitarity result (6.21), but with the Hermitian conjugate replaced by the ordinary transpose. It is reasonable that this single–interface unitarity should hold for lossy media (whereas (6.21) does not), since no attenuation losses occur across an infinitely thin interface (and a propagation invariant such as (6.18) is not needed).

An important special case of a single interface is the free surface at $z = 0$, for which the local reflection coefficient is denoted $\mathbf{R}_U^{f0}$. It satisfies the condition that

$$\mathbf{N}(0)\begin{pmatrix} \mathbf{R}_U^{f0} \\ \mathbf{I} \end{pmatrix}$$

has vanishing stress components. Hence pre– and post–multiplication of the inverse of (6.6) by $(\mathbf{R}_U^{f0}, \mathbf{I})$ gives a rhs equal to zero. Similarly pre–multiplication of (6.13a) by $(\mathbf{R'}_U^{f0}, \mathbf{I})$ and post–mulitplication by $(\mathbf{R}_U^{f0}, \mathbf{I})$ gives a lhs equal to zero. These equations therefore give the symmetry properties of the free–surface coefficient and its reversed system counterpart

$$(\mathbf{R}_U^{f0})^T\mathbf{d}_A^{-1}\mathbf{R}_U^{f0} + \mathbf{d}_B^{-1} = 0 \qquad\qquad (\mathbf{R'}_U^{f0})^T\mathbf{d}'_B + \mathbf{d}'_A\mathbf{R}_U^{f0} = 0\,, \tag{6.25}$$

which, bearing in mind the last paragraph, apply for general anisotropy and complex $\mathbf{A}$.

Further investigation of symmetry properties is possible. For example, the restriction to propagating waves in the derivation of the unitarity relation (6.21) could be removed by studying the complex eigenvalues, particularly for tranversely isotropic media where the eigenvalues for up– and downgoing waves are simply related. The description given here provides a fairly good overview, however.

7. Source excitation

Kennett (1983) describes the effect of a source using the reflection matrix formulation and this approach reveals the reverberations in the stack very clearly.

Let $\mathbf{B}_S(z)$ be a fundamental matrix such that

$$\mathbf{B}_S(z_S) = \mathbf{N}(z_S). \tag{7.1}$$

so that it corresponds to unit down– and upgoing waves at the source depth z_S. Its partitions are denoted

$$\mathbf{B}_S = \begin{pmatrix} \boldsymbol{W}_{DS} & \boldsymbol{W}_{US} \\ \boldsymbol{T}_{DS} & \boldsymbol{T}_{US} \end{pmatrix}. \tag{7.2}$$

The free–surface condition is satisfied by the combination

$$\boldsymbol{W}_{1S} = \boldsymbol{W}_{US} + \boldsymbol{W}_{DS}\mathbf{R}_U^{fS}, \tag{7.3}$$

where $\mathbf{R}_U^{fS}$ is the overall reflection coefficient from the source depth to the surface which satisfies the stress–free condition

$$\boldsymbol{T}_{1S} = \boldsymbol{T}_{US} + \boldsymbol{T}_{DS}\mathbf{R}_U^{fS} = 0. \tag{7.4}$$

At the bottom of the stack z_L only downgoing waves exist in the homogeneous half space. The displacement and stress combinations

$$\boldsymbol{W}_{2S} = \boldsymbol{W}_{DS} + \boldsymbol{W}_{US}\mathbf{R}_D^{SL} \qquad \boldsymbol{T}_{2S} = \boldsymbol{T}_{DS} + \boldsymbol{T}_{US}\mathbf{R}_D^{SL}, \tag{7.5}$$

correspond to downgoing waves only at z_L if the reflection matrix $\mathbf{R}_D^{SL}$ and transmission matrix $\mathbf{T}_D^{SL}$ satisfy

$$\boldsymbol{W}_{DS} + \boldsymbol{W}_{US}\mathbf{R}_D^{SL} = \mathrm{M}_L\mathbf{T}_D^{SL} \qquad \boldsymbol{T}_{US} + \boldsymbol{T}_{DS}\mathbf{R}_D^{SL} = \mathrm{N}_L\mathbf{T}_D^{SL}, \tag{7.6}$$

where M_L and N_L are the downgoing displacement and stress partitions of the eigenvector matrix $\mathbf{N}$ in the lower half space.

Linear combinations of the displacement matrices $\boldsymbol{W}_{1S}(z)$ and $\boldsymbol{W}_{2S}(z)$ and the corresponding stress matrices $\boldsymbol{T}_{1S}(z)$ and $\boldsymbol{T}_{2S}(z)$ allow us to represent the total solution in the presence of a source. The combinations must satisfy the jump conditions

$$\begin{aligned} \boldsymbol{W}_{1S}\boldsymbol{\nu}_1 - \boldsymbol{W}_{2S}\boldsymbol{\nu}_2 &= \mathbf{0} \\ \boldsymbol{T}_{1S}\boldsymbol{\nu}_1 - \boldsymbol{T}_{2S}\boldsymbol{\nu}_2 &= -\mathbf{m}, \end{aligned} \tag{7.7}$$

and the solutions to these simultaneous equations can be obtained using the propagation invariants of §6.

Noting the initial value (7.1) of fundamental matrix $\mathbf{B}_S$, and its reversed–system analogue, one finds from invariant (6.11) and result (6.13a) that

$$\begin{pmatrix} \boldsymbol{W'}_{US} & \boldsymbol{W'}_{DS} \\ \boldsymbol{T'}_{US} & \boldsymbol{T'}_{DS} \end{pmatrix}^T \mathbf{J'} \begin{pmatrix} \boldsymbol{W}_{DS} & \boldsymbol{W}_{US} \\ \boldsymbol{T}_{DS} & \boldsymbol{T}_{US} \end{pmatrix} = \mathbf{d}'_S \tag{7.8}$$

where $\mathbf{d}'_S = \mathbf{d}'(z_S)$. In a notation of Kennett (1983) this will be written

$$\begin{pmatrix} < \boldsymbol{W'}_{US}, \boldsymbol{W}_{DS} > & < \boldsymbol{W'}_{US}, \boldsymbol{W}_{US} > \\ < \boldsymbol{W'}_{DS}, \boldsymbol{W}_{DS} > & < \boldsymbol{W'}_{DS}, \boldsymbol{W}_{US} > \end{pmatrix} = \mathbf{d}'_S\,, \tag{7.9a}$$

where

$$< \boldsymbol{W'}_{US}, \boldsymbol{W}_{DS} >= \boldsymbol{W'}^T_{US}\boldsymbol{T}_{DS} - \boldsymbol{T'}^T_{US}\boldsymbol{W}_{DS}\,, \qquad \text{etc.} \tag{7.9b}$$

The extension of this to the combinations (7.3,4,5) is found by noting that, for example,

$$\begin{aligned} < \boldsymbol{W'}_{1S}, \boldsymbol{W}_{2S} > &=< \boldsymbol{W'}_{US}, \boldsymbol{W}_{DS} > +(\mathbf{R'}^{fS}_U)^T < \boldsymbol{W'}_{DS}, \boldsymbol{W}_{DS} > + < \boldsymbol{W'}_{US}, \boldsymbol{W}_{US} > \mathbf{R}^{SL}_D \\ &\qquad + (\mathbf{R'}^{fS}_U)^T < \boldsymbol{W'}_{DS}, \boldsymbol{W}_{US} > \mathbf{R}^{SL}_D \\ &= \mathbf{d}'_{SA} + (\mathbf{R'}^{fS}_U)^T \mathbf{d}'_{SB}\mathbf{R}^{SL}_D \\ &= \mathbf{d}'_{SA}(\mathbf{I} - \mathbf{R}^{fS}_U\mathbf{R}^{SL}_D)\,, \end{aligned} \tag{7.10a}$$

where the reflection coefficient reciprocity (6.16) has been used in the last step to remove the dependence on the reversed system. The other results are that

$$< \boldsymbol{W'}_{1S}, \boldsymbol{W}_{1S} >=< \boldsymbol{W'}_{2S}, \boldsymbol{W}_{2S} >= \mathbf{0} \qquad\qquad < \boldsymbol{W'}_{2S}, \boldsymbol{W}_{1S} >= \mathbf{d}'_{SB}(\mathbf{I} - \mathbf{R}^{SL}_D\mathbf{R}^{fS}_U)\,. \tag{7.10b}$$

The excitation equations (7.7) can be decoupled by using results (7.10) and one finds that

$$\langle \boldsymbol{W'}_{2S}, \boldsymbol{W}_{1S}\rangle\, \boldsymbol{\nu}_1 = -(\boldsymbol{W'}_{2S})^T(z_S)\mathbf{m} \qquad\qquad \langle \boldsymbol{W'}_{1S}, \boldsymbol{W}_{2S}\rangle\, \boldsymbol{\nu}_2 = (\boldsymbol{W'}_{1S})^T(z_S)\mathbf{m}\,. \tag{7.11}$$

Once these are solved for $\boldsymbol{\nu}_{1,2}$ we have the displacement–stress vector at any depth in the form

$$\mathbf{y}(z) = \begin{pmatrix} \boldsymbol{W}_{1S}(z,z_S) \\ \boldsymbol{T}_{1S}(z,z_S) \end{pmatrix} \boldsymbol{\nu}_1 + \mathbf{P}(z,z_S)\begin{pmatrix} 0 \\ \mathbf{m} \end{pmatrix} H(z - z_S) \tag{7.12a}$$

$$= \begin{pmatrix} \boldsymbol{W}_{2S}(z,z_S) \\ \boldsymbol{T}_{2S}(z,z_S) \end{pmatrix} \boldsymbol{\nu}_2 - \mathbf{P}(z,z_S)\begin{pmatrix} 0 \\ \mathbf{m} \end{pmatrix} H(z_S - z)\,. \tag{7.12b}$$

For example, the displacement above the source which satisfies the boundary and source conditions can be written

$$\begin{aligned} \mathbf{u}(p_x,p_y,z,\omega) &= -\boldsymbol{W}_{1S}(z)\,\langle \boldsymbol{W'}_{2S}, \boldsymbol{W}_{1S}\rangle^{-1}\,(\boldsymbol{W'}_{2S}(z_S))^T\mathbf{m} \\ &= -\boldsymbol{W}_{1S}(z)(\mathbf{I} - \mathbf{R}^{SL}_D\mathbf{R}^{fS}_U)^{-1}\mathbf{d'}^{-1}_{SB}(\boldsymbol{W'}_{2S}(z_S))^T\mathbf{m}\,. \end{aligned} \tag{7.13}$$

The final response is

$$\mathbf{u}(x_1, x_2, z, t) = \frac{1}{(2\pi)^3} \iiint \mathbf{u}(p_1, p_2, z, \omega) e^{i\omega(p_1 x_1 + p_2 x_2 - t)} \, dp_1 dp_2 d\omega \,. \tag{7.14}$$

The layer response (7.12) can be expanded in terms of rays as described by Kennett (1983, see in particular §7.2.1 and equations (7.25) to (7.35) there).

The surface wave contribution comes from poles defined by

$$\det | < \boldsymbol{W}'_{1S}, \boldsymbol{W}_{2S} > | = 0 \qquad \det | < \boldsymbol{W}'_{2S}, \boldsymbol{W}_{1S} > | = 0$$

or, by (7.10),

$$\det |\mathbf{I} - \mathbf{R}_U^{fS} \mathbf{R}_D^{SL}| = 0 \qquad \det |\mathbf{I} - \mathbf{R}_D^{SL} \mathbf{R}_U^{fS}| = 0 \tag{7.15}$$

and it can be shown that the first form of the condition (7.15) implies the second form. At these poles unique solutions $\boldsymbol{\nu}_{1,2}$ of the homogeneous forms ($\mathbf{m} = 0$) of (7.11) can be found, giving the mode eigenfunctions.

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