Calculation of the third and higher travel–time derivatives in isotropic and anisotropic media

Luděk Klimeš

Department of Geophysics, Charles University, Ke Karlovu 3, 121 16 Praha 2, Czech Republic, Fax: +420-2-2191 1214, E-mail: klimes@seis.karlov.mff.cuni.cz

Summary

The explicit equations for the third and higher partial travel–time derivatives in both isotropic and anisotropic media are derived. The third and higher partial travel–time derivatives may be evaluated by simple numerical quadratures along rays.

Keywords

Ray theory, travel times, paraxial approximation, anisotropy, Green function.

1 Introduction

In a general heterogeneous model, travel time and its first partial derivatives may be calculated by means of numerical solution of the set of nonlinear ordinary differential equations called the ray tracing equations. The second partial travel–time derivatives may be calculated by means of numerical solution of the set of linear ordinary differential equations called the dynamic ray tracing equations. The second partial travel–time derivatives, evaluated by means of dynamic ray tracing, has found many geophysical applications. The third and higher partial travel–time derivatives may be evaluated by simple numerical quadratures along rays and, as the author believes, may also find some geophysical applications soon.

The general form of equations for the third and higher partial travel–time derivatives was suggested by Babich, Buldyrev & Molotkov (1985). In this paper, explicit equations for the third and higher partial travel–time derivatives in Cartesian coordinates are presented. The travel–time derivatives are expressed in the form of integrals along the ray, with the proper initial values. The integrands are composed of partial travel–time derivatives of lower orders than the evaluated one, and of the derivatives of the Hamiltonian, which are often composed of partial slowness (or velocity) derivatives.

Although the partial travel–time derivatives in the ray–centred coordinate system could also be calculated, most applications are assumed to require the partial derivatives rather in global Cartesian coordinates or, possibly, in some other model or configuration coordinates. That is why the principal attention is devoted to the evaluation in Cartesian coordinates. However, all equations expressed in terms of a general Hamiltonian are applicable to any coordinate system.

In: Seismic Waves in Complex 3-D Structures, Report 6 (Department of Geophysics, Charles University, Prague, September 1997)

2 Equations for the third and higher partial travel–time derivatives

We shall use the Einstein summation convention, and denote the partial derivatives by indices following a comma. For instance,

$$\tau_{,ij\dots n} = \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\cdots\frac{\partial}{\partial x_n}\tau \quad , \tag{1}$$

denotes the partial derivatives of travel time τ in coordinates x_i which may or may not be Cartesian. We also introduce the "multiindices" in order to make the equations more concise and easy to survey. For example,

$$\begin{matrix}\tau_{,rij} & \tau_{,rk} \\ {}_{,rik} & {}_{,rj} \\ {}_{,rjk} & {}_{,ri}\end{matrix} \tag{2}$$

stands for

$$\tau_{,rij}\tau_{,rk} + \tau_{,rik}\tau_{,rj} + \tau_{,rjk}\tau_{,ri} \quad . \tag{3}$$

In both isotropic and anisotropic media, we assume the travel time $\tau(x_i)$ determined by the eikonal equation of general form

$$H\left(x_i, \tau_{,j}(x_m)\right) = 0 \quad , \tag{4}$$

where function $H(x_i, p_j)$ of six phase–space coordinates is the Hamiltonian. The corresponding ray tracing equations than have the form of

$$\frac{\mathrm{d}}{\mathrm{d}\gamma_3}x_i = \frac{\partial}{\partial p_i}H(x_q, p_r) \quad , \qquad \frac{\mathrm{d}}{\mathrm{d}\gamma_3}p_i = -\frac{\partial}{\partial x_i}H(x_q, p_r) \quad , \tag{5}$$

where the independent parameter γ_3 along rays is determined by the form of the Hamiltonian.

If $\tau(x_i)$ is the solution of the eikonal equation (4), function

$$h\left(x_k\right) = H\left(x_i, \tau_{,j}(x_m)\right) \tag{6}$$

is obviously constant (equal to zero) and all its partial derivatives thus vanish,

$$\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\dots\frac{\partial}{\partial x_n}h\left(x_q\right) = 0 \quad . \tag{7}$$

In terms of the Hamiltonian, this equation reads

$$\left(\frac{\partial}{\partial x_i} + \tau_{,ia}\frac{\partial}{\partial p_a}\right)\left(\frac{\partial}{\partial x_j} + \tau_{,jb}\frac{\partial}{\partial p_b}\right)\dots\left(\frac{\partial}{\partial x_n} + \tau_{,nf}\frac{\partial}{\partial p_f}\right)H = 0 \quad . \tag{8}$$

Here the phase–space partial derivatives are evaluated at point $(x_q, p_r) = (x_q, \tau_{,r}(x_s))$.

We define the matrices of partial derivatives of coordinates x_i and of the slowness vector components $\tau_{,i}$ with respect to 3 ray parameters: ray take–off parameters γ_1, γ_2, and parameter γ_3 along rays,

$$Q_{ia} = \frac{\partial x_i}{\partial \gamma_a} \quad , \qquad P_{ia} = \frac{\partial \tau_{,i}}{\partial \gamma_a} \quad . \tag{9}$$

Let us emphasize that the definition of matrices Q_{ia} and P_{ia} depends on the parameter γ_3 along rays, which is in turn determined by the form of the Hamiltonian. Usage of matrices Q_{ia} and P_{ia}, defined for a particular γ_3, in the equations for travel–time derivatives derived from the Hamiltonian corresponding to different γ_3 must be avoided.

Matrices Q_{ia} and P_{ia} are often calculated for $\gamma_3 = \tau$. The transformation of matrix Q_{ia} to the one defined for different γ_3 is described in Section 3.2.

Equation

$$P_{ia} = \tau_{,ij}\, Q_{ja} \tag{10}$$

is a direct consequence of the above definitions. The dynamic ray tracing system

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\gamma_3} Q_{ia} &= \frac{\partial}{\partial p_i}\frac{\partial}{\partial x_j} H(x_q,p_r)\, Q_{ja} + \frac{\partial}{\partial p_i}\frac{\partial}{\partial p_j} H(x_q,p_r)\, P_{ja} \\ \frac{\mathrm{d}}{\mathrm{d}\gamma_3} P_{ia} &= -\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} H(x_q,p_r)\, Q_{ja} - \frac{\partial}{\partial x_i}\frac{\partial}{\partial p_j} H(x_q,p_r)\, P_{ja}\end{aligned} \tag{11}$$

can be obtained by differentiating the ray tracing system (5) with respect to γ_a. After insertion of (10), the first equation of the dynamic ray tracing system (11) reads

$$\frac{\mathrm{d}}{\mathrm{d}\gamma_3} Q_{ia} = Q_{ja} \left(\frac{\partial}{\partial x_j} + \tau_{,jk} \frac{\partial}{\partial p_k} \right) \frac{\partial}{\partial p_i} H \quad . \tag{12}$$

Let us now consider the covariant derivatives $\tau_{,ijk\ldots mn} Q_{ia} Q_{jb} Q_{kc} \cdots Q_{me} Q_{nf}$ of the travel–time with respect to γ_a, differentiate them along a ray with respect to γ_3, and insert equations (5) and (12) for $\frac{\mathrm{d}}{\mathrm{d}\gamma_3} x_i$ and $\frac{\mathrm{d}}{\mathrm{d}\gamma_3} Q_{ia}$, respectively,

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}\gamma_3}[\tau_{,ijk\ldots mn} Q_{ia} Q_{jb} Q_{kc} \cdots Q_{me} Q_{nf}] \\ &\qquad = \Bigg[\tau_{,rijk\ldots mn} \frac{\partial}{\partial p_r} H \\ &\qquad\quad + \tau_{,rjk\ldots mn} \left(\frac{\partial}{\partial x_i} + \tau_{,is} \frac{\partial}{\partial p_s} \right) \frac{\partial}{\partial p_r} H \\ &\qquad\quad + \tau_{,irk\ldots mn} \left(\frac{\partial}{\partial x_j} + \tau_{,js} \frac{\partial}{\partial p_s} \right) \frac{\partial}{\partial p_r} H \\ &\qquad\quad \vdots \\ &\qquad\quad + \tau_{,ijk\ldots mr} \left(\frac{\partial}{\partial x_n} + \tau_{,ns} \frac{\partial}{\partial p_s} \right) \frac{\partial}{\partial p_r} H \Bigg] Q_{ia} Q_{jb} Q_{kc} \cdots Q_{me} Q_{nf} \ .\end{aligned} \tag{13}$$

The partial travel–time derivatives may then be expressed in the form of the integral along the ray,

$$\boxed{\tau_{,ijk\ldots mn}(\gamma_3) = \tau_{,ijk\ldots mn}(\gamma_3^0) + Q^{-1}_{ai} Q^{-1}_{bj} Q^{-1}_{ck} \cdots Q^{-1}_{em} Q^{-1}_{fn} \int_{\gamma_3^0}^{\gamma_3} \mathrm{d}\gamma_3\, L_{abc\ldots ef} \quad ,} \tag{14}$$

where

$$\boxed{L_{abc\ldots ef} = K_{ijk\ldots mn} Q_{ia} Q_{jb} Q_{kc} \cdots Q_{me} Q_{nf} \quad ,} \tag{15}$$

with

$$
\begin{aligned}
K_{ijk...mn} = \tau_{,rijk...mn}\frac{\partial}{\partial p_r}H + \tau_{,rjk...mn}&\left(\frac{\partial}{\partial x_i}+\tau_{,is}\frac{\partial}{\partial p_s}\right)\frac{\partial}{\partial p_r}H \\
+\tau_{,irk...mn}&\left(\frac{\partial}{\partial x_j}+\tau_{,js}\frac{\partial}{\partial p_s}\right)\frac{\partial}{\partial p_r}H \\
&\vdots \\
+\tau_{,ijk...mr}&\left(\frac{\partial}{\partial x_n}+\tau_{,ns}\frac{\partial}{\partial p_s}\right)\frac{\partial}{\partial p_r}H \ . \qquad (16)
\end{aligned}
$$

In (14), we have denoted by Q^{-1}_{ai} the components of the matrix inverse to matrix Q_{ia}.

The travel–time derivatives on the right–hand side of (16) coincide with those terms on the left–hand side of equation (8) containing the highest–order travel–time derivatives, and are cancelled for the third and higher travel–time derivatives if subtracting, for nonzeroth derivatives, (8) from (16),

$$
\boxed{
\begin{aligned}
K_{ijk...mn} = \tau_{,rijk...mn}\frac{\partial}{\partial p_r}H + \tau_{,rjk...mn}&\left(\frac{\partial}{\partial x_i}+\tau_{,is}\frac{\partial}{\partial p_s}\right)\frac{\partial}{\partial p_r}H \\
+\tau_{,irk...mn}&\left(\frac{\partial}{\partial x_j}+\tau_{,js}\frac{\partial}{\partial p_s}\right)\frac{\partial}{\partial p_r}H \\
&\vdots \\
+\tau_{,ijk...mr}&\left(\frac{\partial}{\partial x_n}+\tau_{,ns}\frac{\partial}{\partial p_s}\right)\frac{\partial}{\partial p_r}H \\
-\left(\frac{\partial}{\partial x_i}+\tau_{,ia}\frac{\partial}{\partial p_a}\right)\left(\frac{\partial}{\partial x_j}+\tau_{,jb}\frac{\partial}{\partial p_b}\right)\dots&\left(\frac{\partial}{\partial x_n}+\tau_{,nf}\frac{\partial}{\partial p_f}\right)H \ . \qquad (17)
\end{aligned}
}
$$

The remaining travel–time derivatives in (17) are of lower orders than the derivative (14) to be calculated, and should thus have been previously evaluated.

The last term in equation (17), present for nonzeroth travel–time derivatives, contains all derivatives of Hamiltonian H from the first to the order equal to the number of indices of $K_{ij...n}$. It is composed of all possible terms of the form of

$$
H^{,ab...f}_{,...}\,\tau_{,a...}\,\tau_{,b...}\cdots\tau_{,f...} \quad , \tag{18}
$$

where we have denoted

$$
H^{;ab...f}_{,ij...n} = \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\cdots\frac{\partial}{\partial x_n}\,\frac{\partial}{\partial p_a}\frac{\partial}{\partial p_b}\cdots\frac{\partial}{\partial p_f}H \quad . \tag{19}
$$

Subscripts in these "terms" are represented by all possible combinations of the subscripts of $K_{ij...n}$, such that each $\tau_{,a...}$ has at least one of these subscripts in place of ..., whereas $H^{,ab...f}_{,...}$ need not have any subscript. $H^{;ab...f}_{,...}$ also need not have any superscript. Each possible summand is contained within the last term in equation (17) just once. The terms of forms $H^{,a}\tau_{,a...}$, $H^{;a}_{,i}\tau_{,a...}$ and $H^{,ab}\tau_{,ai}\tau_{,b...}$ in (17) cancel out for the third and higher travel–time derivatives, the other terms remain.

Equations (14), (15) and (17) may be used to evaluate the third and higher partial travel–time derivatives in both isotropic and anisotropic models by means of numerical quadrature along traced rays.

3 Examples of the integrands in general media

3.1 Zeroth, first and second travel–time derivatives

For the zeroth–order derivatives, equation (17) takes the form of

$$K = H^{,r}\tau_{,r} \quad , \tag{20}$$

which may be used to evaluate the travel time along rays for parameter γ_3 other than the travel time,

$$\frac{\partial\tau}{\partial\gamma_3} = K \quad . \tag{21}$$

Equation (17), specified for the first–order derivatives,

$$K_i = -H_{,i} + H_{,i}^{,r}\tau_{,r} + H^{,rs}\tau_{,r}\tau_{,si} \quad , \tag{22}$$

cannot be used for the calculation of the travel–time gradient, but will turn out valuable for the transformation of matrix Q_{ia} to one defined for different γ_3.

On the other hand, equation (17), specified for the second–order travel–time derivatives,

$$K_{ij} = -H_{,ij} + H^{,ab}\tau_{,ai}\tau_{,bj} \quad , \tag{23}$$

has probably found no practical application yet.

3.2 Transformation of matrix Q

Denote Q^{τ}_{ia} matrix Q_{ia} defined for $\gamma_3 = \tau$. The transformation of the third column,

$$Q_{i3} = K\, Q^{\tau}_{i3} \quad , \tag{24}$$

is obvious from the definition of Q_{ia} and equation (21).

The first two columns Q^{τ}_{iM} of the matrix defined for $\gamma_3 = \tau$ are obviously tangent to the wavefront and equal to the columns Q_{iM} projected along the rays onto the wavefront. The difference between Q_{iM} and Q^{τ}_{iM} is thus parallel with vector $\frac{\mathrm{d}}{\mathrm{d}\gamma_3}x_i = H^{,i}$ tangent to the ray,

$$\boxed{Q_{iM} = Q^{\tau}_{iM} + H^{,i}\, g_M \quad . \qquad (25)}$$

Multiplication of (25) with travel–time gradient $\tau_{,i}$ gives

$$g_M = \frac{\tau_{,i}Q_{iM}}{H^{,i}\tau_{,i}} = \frac{1}{K}\tau_{,i}Q_{iM} \quad , \tag{26}$$

see (20). Derivative of g_M along a ray is

$$\frac{\mathrm{d}}{\mathrm{d}\gamma_3}g_M = -\frac{1}{K^2}\frac{\partial K}{\partial\gamma_3}\tau_{,i}Q_{iM} + \frac{1}{K}\frac{\mathrm{d}}{\mathrm{d}\gamma_3}\left[\tau_{,i}Q_{iM}\right] = -\frac{1}{K}\frac{\partial K}{\partial\gamma_3}g_M + \frac{1}{K}K_iQ_{iM} \quad , \tag{27}$$

see (14) with (15). Now we insert (25) at the right–hand side of (27),

$$\frac{\mathrm{d}}{\mathrm{d}\gamma_3}g_M = -\frac{1}{K}\frac{\partial K}{\partial\gamma_3}g_M + \frac{1}{K}K_iH^{,i}g_M + \frac{1}{K}K_iQ^{\tau}_{iM} \quad . \tag{28}$$

Derivative of (20) along a ray is

$$\begin{aligned}\frac{\partial K}{\partial\gamma_3} &= H_{,r}^{,i}\frac{\partial x_r}{\partial\gamma_3}\tau_{,i} + H^{,ir}\tau_{,rs}\frac{\partial x_s}{\partial\gamma_3}\tau_{,i} + H^{,i}\frac{\partial\tau_{,i}}{\partial\gamma_3}\\ &= H_{,r}^{,i}H^{,r}\tau_{,i} + H^{,ir}\tau_{,rs}H^{,s}\tau_{,i} - H^{,i}H_{,i} = K_iH^{,i} \quad ,\end{aligned} \tag{29}$$

see (22). The insertion of (29) into (28) yields

$$\boxed{g_M(\gamma_3) = g_M(\gamma_3^0) + \int_{\gamma_3^0}^{\gamma_3} \mathrm{d}\gamma_3 \; \frac{1}{K} K_i Q^{\tau}_{iM} \quad .} \tag{30}$$

The initial values $g_M(\gamma_3^0)$ are determined through equation (26).

3.3 Restricting the form of the Hamiltonian

Assume that the Hamiltonian is chosen in the form of

$$H(x_q, p_r) = \overline{H}(x_q, p_r) - V(x_q) \quad , \tag{31}$$

where $\overline{H}(x_q, p_r)$ is a *homogeneous function of the second order* with respect to slowness vector p_k,

$$\overline{H}^{,a} p_a = 2\overline{H} \quad . \tag{32}$$

Then also

$$H^{,a}_{,\dots} p_a = 2\overline{H}_{,\dots} \quad . \tag{33}$$

Differentiating (33) with respect to p_b yields

$$H^{,ab}_{,\dots} p_a = \overline{H}^{,b}_{,\dots} \quad . \tag{34}$$

Insertion of (33), and subsequently (4) with (31), into (20) yields

$$K = 2\overline{H} = 2V \quad . \tag{35}$$

Insertion of (33) and (34), and subsequently (31), into (22) yields

$$K_i = -H_{,i} + 2\overline{H}_{,i} + \overline{H}^{,s} \tau_{,si} = 2V_{,i} + H_{,i} + H^{,s} \tau_{,si} = 2V_{,i} \quad , \tag{36}$$

see (8).

3.4 Third and higher travel–time derivatives

Equation (17) is important especially for the calculation of the third travel–time partial derivatives,

$$K_{ijk} = -H_{,ijk} - H^{,a}_{\substack{,ij\\,ik\\,jk}} \tau_{\substack{,ak\\,aj\\,ai}} - H^{,ab}_{\substack{,i\\,j\\,k}} \tau_{\substack{,aj\\,ai\\,ai}} \tau_{\substack{,ak\\,ak\\,aj}} - H^{,abc} \tau_{,ai} \tau_{,bj} \tau_{,ck} \quad , \tag{37}$$

the fourth travel–time partial derivatives,

$$\begin{aligned} K_{ijkl} = & - H_{,ijkl} - H^{,a}_{\substack{,ijk\\,ijl\\,ikl\\,jkl}} \tau_{\substack{,al\\,ak\\,aj\\,ai}} - H^{,a}_{\substack{,ij\\,ik\\,jk\\,il\\,jl\\,kl}} \tau_{\substack{,akl\\,ajl\\,ail\\,ajk\\,aik\\,aij}} - H^{,ab}_{\substack{,ij\\,ik\\,jk\\,il\\,jl\\,kl}} \tau_{\substack{,ak\\,aj\\,ai\\,aj\\,ai\\,ai}} \tau_{\substack{,bl\\,bl\\,bl\\,bk\\,bk\\,bj}} - H^{,ab}_{\substack{,i\\,j\\,i\\,k\\,j\\,k\\,i\\,l\\,j\\,l\\,k\\,l}} \tau_{\substack{,aj\\,ai\\,ak\\,ai\\,ak\\,aj\\,al\\,ai\\,al\\,aj\\,al\\,ak}} \tau_{\substack{,bkl\\,bkl\\,bjl\\,bjl\\,bil\\,bil\\,bjk\\,bjk\\,bik\\,bik\\,bij\\,bij}} \\ & - H^{,ab} \tau_{\substack{,aij\\,aik\\,ajk}} \tau_{\substack{,bkl\\,bjl\\,bil}} - H^{,abc}_{\substack{,i\\,j\\,k\\,l}} \tau_{\substack{,aj\\,ai\\,ai\\,ai}} \tau_{\substack{,bk\\,bk\\,bj\\,bj}} \tau_{\substack{,cl\\,cl\\,cl\\,ck}} - H^{,abc} \tau_{\substack{,aij\\,aik\\,ajk\\,ail\\,ajl\\,akl}} \tau_{\substack{,bk\\,bj\\,bi\\,bj\\,bi\\,bi}} \tau_{\substack{,cl\\,cl\\,cl\\,ck\\,ck\\,cj}} \\ & - H^{,abcd} \tau_{,ai} \tau_{,bj} \tau_{,ck} \tau_{,dl} \quad , \end{aligned} \tag{38}$$

and so on.

4 Examples of the integrands in isotropic media

In isotropic models, the Hamiltonian is a function of $p_k p_k$, but is independent of the direction of the slowness vector p_i. If the Hamiltonian is chosen in the form of

$$H(x_q, p_r) = \overline{H}(x_q, p_r) - V(x_q) \quad , \tag{39}$$

where $\overline{H}(x_q, p_r)$ is a *homogeneous function of the first order* with respect to $|p|^2 = p_k p_k$, all partial derivatives of the Hamiltonian containing the third or higher partial derivatives with respect to the slowness–vector components are zero,

$$H^{,abc\ldots}_{,\ldots} = 0 \quad , \tag{40}$$

which considerably simplify equations like (37) and (38),

$$K_{ijk} = -H_{,ijk} - H^{,a}_{\substack{,ij\\,ik\\,jk}} \tau_{\substack{,ak\\,aj\\,ai}} - H^{,ab}_{\substack{,i\\,j\\,k}} \tau_{\substack{,aj\\,ai\\,ai}} \tau_{\substack{,bk\\,bk\\,bj}} \quad , \tag{41}$$

$$K_{ijkl} = -H_{,ijkl} - H^{,a}_{\substack{,ijk\\,ijl\\,ikl\\,jkl}} \tau_{\substack{,al\\,ak\\,aj\\,ai}} - H^{,a}_{\substack{,ij\\,ik\\,jk\\,il\\,jl\\,kl}} \tau_{\substack{,akl\\,ajl\\,ail\\,ajk\\,aik\\,aij}} - H^{,ab}_{\substack{,ij\\,ik\\,jk\\,il\\,jl\\,kl}} \tau_{\substack{,ak\\,aj\\,ai\\,aj\\,ai\\,ai}} \tau_{\substack{,bl\\,bl\\,bl\\,bk\\,bk\\,bj}} - H^{,ab}_{\substack{,i\\,j\\,i\\,k\\,j\\,k\\,i\\,l\\,j\\,l\\,k\\,l}} \tau_{\substack{,aj\\,ai\\,ak\\,ai\\,ak\\,aj\\,al\\,ai\\,al\\,aj\\,al\\,ak}} \tau_{\substack{,bkl\\,bkl\\,bjl\\,bjl\\,bil\\,bil\\,bjk\\,bjk\\,bik\\,bik\\,bij\\,bij}} - H^{,ab} \tau_{\substack{,aij\\,aik\\,ajk}} \tau_{\substack{,bkl\\,bjl\\,bil}} \quad , \tag{42}$$

$$\begin{aligned} K_{ijklm} = &- H_{,ijklm} - H^{,a}_{\substack{,ijkl\\,ijkm\\,ijlm\\,iklm\\,jklm}} \tau_{\substack{,am\\,al\\,ak\\,aj\\,ai}} - H^{,a}_{\substack{,ijk\\,ijl\\,ikl\\,jkl\\,ijm\\,ikm\\,jkm\\,ijm\\,ikm\\,ilm}} \tau_{\substack{,alm\\,akm\\,ajm\\,aim\\,akl\\,ajl\\,ail\\,ajl\\,ajl\\,ajk}} - H^{,a}_{\substack{,ij\\,ik\\,jk\\,il\\,jl\\,kl\\,im\\,jm\\,km\\,lm}} \tau_{\substack{,aklm\\,ajlm\\,ailm\\,ajkm\\,aikm\\,aijm\\,ajkl\\,aikl\\,aijl\\,aijk}} - H^{,ab}_{\substack{,ijk\\,ijl\\,ikl\\,jkl\\,ijm\\,ikm\\,jkm\\,ilm\\,jlm\\,klm}} \tau_{\substack{,al\\,ak\\,aj\\,ai\\,ai\\,aj\\,ak\\,ai\\,aj\\,ai}} \tau_{\substack{,bm\\,bm\\,bm\\,bm\\,bl\\,bl\\,bl\\,bk\\,bk\\,bj}} \\ &- H^{,ab}_{\substack{,ij\\,ij\\,ij\\,ik\\,ik\\,ik\\,jk\\,jk\\,jk\\,il\\,il\\,il\\,jl\\,jl\\,jl\\,kl\\,kl\\,kl\\,im\\,im\\,im\\,jm\\,jm\\,jm\\,km\\,km\\,km\\,lm\\,lm\\,lm}} \tau_{\substack{,ak\\,al\\,am\\,aj\\,al\\,am\\,ai\\,al\\,am\\,aj\\,ak\\,am\\,ai\\,ak\\,am\\,ai\\,aj\\,am\\,aj\\,ak\\,al\\,ai\\,ak\\,al\\,ai\\,aj\\,al\\,ai\\,aj\\,ak}} \tau_{\substack{,blm\\,bkm\\,bkl\\,blm\\,bjm\\,bjl\\,blm\\,bim\\,bil\\,bkm\\,bjm\\,bjk\\,bkm\\,bim\\,bik\\,bjm\\,bim\\,bij\\,bkl\\,bjl\\,bjk\\,bkl\\,bil\\,bik\\,bjl\\,bil\\,bij\\,bjk\\,bik\\,bij}} - H^{,ab}_{\substack{,i\\,j\\,i\\,k\\,j\\,k\\,i\\,l\\,j\\,l\\,k\\,l\\,i\\,m\\,j\\,m\\,k\\,m\\,l\\,m}} \tau_{\substack{,aj\\,ai\\,ak\\,ai\\,ak\\,aj\\,al\\,ai\\,al\\,aj\\,al\\,ak\\,am\\,ai\\,am\\,aj\\,am\\,ak\\,am\\,al}} \tau_{\substack{,bklm\\,bklm\\,bjlm\\,bjlm\\,bilm\\,bilm\\,bjkm\\,bjkm\\,bikm\\,bikm\\,bijm\\,bijm\\,bjkl\\,bjkl\\,bikl\\,bikl\\,bijl\\,bijl\\,bijk\\,bijk}} - H^{,ab} \tau_{\substack{,aij\\,aik\\,ajk\\,ail\\,ajl\\,akl\\,aim\\,ajm\\,akm\\,alm}} \tau_{\substack{,bklm\\,bjlm\\,bilm\\,bjkm\\,bikm\\,bijm\\,bjkl\\,bikl\\,bijl\\,bijk}} \quad , \end{aligned} \tag{43}$$

and so on.

4.1 Integration with respect to travel time

For example, if we choose the Hamiltonian in Cartesian coordinates in a special form of

$$H(x_q, p_r) = \tfrac{1}{2} v^2 p_k p_k - \tfrac{1}{2} \quad , \tag{44}$$

$v = v(x_i)$ being the phase velocity, then $\gamma_3 = \tau$ in (14), and only partial derivatives

$$H_{,\dots} = \tfrac{1}{2} \left(v^2\right)_{,\dots} p_k p_k \quad , \qquad H_{,\dots}^{;a} = \left(v^2\right)_{,\dots} p_a \quad , \qquad H_{,\dots}^{;ab} = \left(v^2\right)_{,\dots} \delta_{ab} \tag{45}$$

of the Hamiltonian are nonzero. The travel–time derivatives on the right–hand side of equation (16) with (45) may then be cancelled for the third and higher travel–time derivatives by subtracting equation (7) with (44),

$$\boxed{K_{ijk\dots mn} = v^2 \tau_{,r} \tau_{,rijk\dots mn} + v^2 \tau_{\substack{,ri\\,rj\\ \vdots\\ ,rn}} \tau_{\substack{,rjk\dots mn\\,irk\dots mn\\ \vdots\\ ,ijk\dots mr}} + (v^2)_{\substack{,i\\,j\\ \vdots\\ ,n}} \tau_{\substack{,r\\,r\\ \vdots\\ ,r}} \tau_{\substack{,rjk\dots mn\\,irk\dots mn\\ \vdots\\ ,ijk\dots mr}} - \left(\tfrac{1}{2} v^2 \tau_{,r} \tau_{,r} - \tfrac{1}{2}\right)_{,ijk\dots mn} .} \tag{46}$$

Equations like (41), (42), (43) for the third and higher travel–time derivatives then take the form of

$$K_{ijk} = -\left(\tfrac{1}{2} v^2\right)_{,ijk} \tau_{,r} \tau_{,r} - \left(v^2\right)_{\substack{,ij\\,ik\\,jk}} \tau_{\substack{,r\\,r\\,r}} \tau_{\substack{,rk\\,rj\\,ri}} - \left(v^2\right)_{\substack{,i\\,j\\,k}} \tau_{\substack{,rj\\,ri\\,ri}} \tau_{\substack{,rk\\,rk\\,rj}} \quad , \tag{47}$$

$$\begin{aligned} K_{ijkl} = &- \left(\tfrac{1}{2} v^2\right)_{,ijkl} \tau_{,r} \tau_{,r} - \left(v^2\right)_{\substack{,ijk\\,ijl\\,ikl\\,jkl}} \tau_{\substack{,r\\,r\\,r\\,r}} \tau_{\substack{,rl\\,rk\\,rj\\,ri}} - \left(v^2\right)_{\substack{,ij\\,ik\\,jk\\,il\\,jl\\,kl}} \tau_{\substack{,r\\,r\\,r\\,r\\,r\\,r}} \tau_{\substack{,rkl\\,rjl\\,ril\\,rjk\\,rik\\,rij}} \\ &- \left(v^2\right)_{\substack{,ij\\,ik\\,jk\\,il\\,jl\\,kl}} \tau_{\substack{,rk\\,rj\\,rk\\,rj\\,ri\\,ri}} \tau_{\substack{,rl\\,rl\\,rl\\,rk\\,rk\\,rj}} - \left(v^2\right)_{\substack{,i\\,j\\,i\\,k\\,j\\,k\\,i\\,l\\,j\\,l\\,k\\,l}} \tau_{\substack{,rj\\,ri\\,rk\\,ri\\,rk\\,rj\\,rl\\,ri\\,rl\\,rj\\,rl\\,rk}} \tau_{\substack{,rkl\\,rkl\\,rjl\\,rjl\\,ril\\,ril\\,rjk\\,rjk\\,rik\\,rik\\,rij\\,rij}} - v^2 \tau_{\substack{,rij\\,rik\\,rjk}} \tau_{\substack{,rkl\\,rjl\\,ril}} \quad , \end{aligned} \tag{48}$$

and so on.

4.2 Integration with respect to $\gamma_3 = \int v^2 \mathrm{d}\tau$

Moreover, if we choose the Hamiltonian in Cartesian coordinates in a special form of

$$H(x_q, p_r) = \tfrac{1}{2} p_k p_k - \tfrac{1}{2} v^{-2} \quad , \tag{49}$$

then $\mathrm{d}\gamma_3 = v^2 \mathrm{d}\tau$ in (14), and only partial derivatives

$$H_{,\ldots} = -\left(\tfrac{1}{2} v^{-2}\right)_{,\ldots} \quad , \qquad H^{,a} = p_a \quad , \qquad H^{,ab} = \delta_{ab} \tag{50}$$

of the Hamiltonian are nonzero. Equation (16) then simplifies to

$$K_{ijk\ldots mn} = \tau_{,r}\tau_{,rijk\ldots mn} + \tau_{\substack{,ri \\ ,rj \\ \vdots \\ ,rn}} \tau_{\substack{,rjk\ldots mn \\ ,irk\ldots mn \\ \vdots \\ ,ijk\ldots mr}} \quad . \tag{51}$$

The travel–time derivatives on the right–hand side of (51) may then be cancelled for the third and higher travel–time derivatives by subtracting (7) with (49) from (51),

$$\boxed{K_{ijk\ldots mn} = \left(\tfrac{1}{2} v^{-2}\right)_{,ijk\ldots mn} + \tau_{,r}\tau_{,rijk\ldots mn} + \tau_{\substack{,ri \\ ,rj \\ \vdots \\ ,rn}} \tau_{\substack{,rjk\ldots mn \\ ,irk\ldots mn \\ \vdots \\ ,ijk\ldots mr}} - \left(\tfrac{1}{2}\tau_{,r}\tau_{,r}\right)_{,ijk\ldots mn} \quad .} \tag{52}$$

Equations like (41), (42), (43) for the third and higher travel–time derivatives then take the form of

$$K_{ijk} = \left(\tfrac{1}{2} v^{-2}\right)_{,ijk} \quad , \tag{53}$$

$$K_{ijkl} = \left(\tfrac{1}{2} v^{-2}\right)_{,ijkl} - \tau_{\substack{,rij \\ ,rik \\ ,rjk}} \tau_{\substack{,rkl \\ ,rjl \\ ,ril}} \quad , \tag{54}$$

$$K_{ijklm} = \left(\tfrac{1}{2} v^{-2}\right)_{,ijklm} - \tau_{\substack{,rij \\ ,rik \\ ,rjk \\ ,ril \\ ,rjl \\ ,rkl \\ ,rim \\ ,rjm \\ ,rkm \\ ,rlm}} \tau_{\substack{,rklm \\ ,rjlm \\ ,rilm \\ ,rjkm \\ ,rikm \\ ,rijm \\ ,rjkl \\ ,rikl \\ ,rijl \\ ,rijk}} \quad , \tag{55}$$

$$K_{ijklmn} = \left(\tfrac{1}{2} v^{-2}\right)_{,ijklmn} - \tau_{\substack{,rij \\ ,rik \\ ,rjk \\ ,ril \\ ,rjl \\ ,rkl \\ ,rim \\ ,rjm \\ ,rkm \\ ,rlm \\ ,rin \\ ,rjn \\ ,rkn \\ ,rln \\ ,rmn}} \tau_{\substack{,rklmn \\ ,rjlmn \\ ,rilmn \\ ,rjkmn \\ ,rikmn \\ ,rijmn \\ ,rjkln \\ ,rikln \\ ,rijln \\ ,rijkn \\ ,rjklm \\ ,riklm \\ ,rijlm \\ ,rijkm \\ ,rijkl}} - \tau_{\substack{,rijk \\ ,rijl \\ ,rikl \\ ,rjkl \\ ,rijm \\ ,rikm \\ ,rjkm \\ ,rilm \\ ,rjlm \\ ,rklm}} \tau_{\substack{,rlmn \\ ,rkmn \\ ,rjmn \\ ,rimn \\ ,rkln \\ ,rjln \\ ,riln \\ ,rjkn \\ ,rikn \\ ,rijn}} \quad , \tag{56}$$

$$K_{ijklmno} = \left(\tfrac{1}{2}v^{-2}\right)_{,ijklmno} - \tau_{\substack{,rij\\,rik\\,rjk\\,ril\\,rjl\\,rkl\\,rim\\,rjm\\,rkm\\,rlm\\,rin\\,rjn\\,rkn\\,rln\\,rmn\\,rio\\,rjo\\,rko\\,rlo\\,rmo\\,rno}} \tau_{\substack{,rklmno\\,rjlmno\\,rilmno\\,rjkmno\\,rikmno\\,rijmno\\,rjklno\\,riklno\\,rijlno\\,rijkno\\,rjklmo\\,riklmo\\,rijlmo\\,rijkmo\\,rijklo\\,rjklmn\\,riklmn\\,rijlmn\\,rijkmn\\,rijkln\\,rijklm}} - \tau_{\substack{,rijk\\,rijl\\,rikl\\,rjkl\\,rijm\\,rikm\\,rjkm\\,rilm\\,rjlm\\,rklm\\,rijn\\,rikn\\,rjkn\\,riln\\,rjln\\,rkln\\,rimn\\,rjmn\\,rkmn\\,rlmn\\,rijo\\,riko\\,rjko\\,rilo\\,rjlo\\,rklo\\,rimo\\,rjmo\\,rkmo\\,rlmo\\,rino\\,rjno\\,rkno\\,rlno\\,rmno}} \tau_{\substack{,rlmno\\,rkmno\\,rjmno\\,rimno\\,rklno\\,rjlno\\,rilno\\,rjkno\\,rikno\\,rijno\\,rklmo\\,rjlmo\\,rilmo\\,rjkmo\\,rikmo\\,rijmo\\,rjklo\\,riklo\\,rijlo\\,rijko\\,rklmn\\,rjlmn\\,rilmn\\,rjkmn\\,rikmn\\,rijmn\\,rjkln\\,rikln\\,rijln\\,rijkn\\,rjklm\\,riklm\\,rijlm\\,rijkm\\,rijkl}} \quad , \tag{57}$$

and so on.

5 Conclusions

Equations (14), (15) and (17) may be used to evaluate the third and higher partial travel–time derivatives in both isotropic and anisotropic models by means of a numerical quadrature along rays. For a special choice of the Hamiltonian in isotropic models, equation (17) considerably simplifies to equations like (46) or (52).

References

Babich, V.M., Buldyrev, N.J. & Molotkov, I.A. (1985): Space–time ray method. Linear and non–linear waves (in Russian). Leningrad, Leningrad Univ. Press.