Correlation functions of random media

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1 Correlation functions of material parameters

The *geological structure* is assumed to be a realization of a random medium with known statistical characteristics. This approach enables to satisfactorily account for the differences between the geological structure and the *seismic model* of the structure, based on our quite incomplete information on the structure. The statistical characteristics represent our knowledge about the deterministic features of the structure and may change with more information on the structure. The randomness corresponds to the unknown details in the geological structure and may decrease with more information on the structure.

1.1 Material parameters

The geological structure is assumed to be expressed in terms of spatial distribution $u = u(\mathbf{x})$ of several *material parameters* u. For example, the material parameters may represent the density, P and S wave velocities or slownesses, or the elements of the stiffness matrix.

The *microscale heterogeneities* of these parameters (heterogeneities on scales shorter than the wavelength) may influence the propagation of seismic waves of a particular frequency in such a way that the propagation corresponds to some frequency–dependent *effective material parameters* $\overline{u}(\mathbf{x},\omega)$ "locally homogeneous" at microscales. For the study of *macroscale heterogeneities* (e.g., seismic travel–time tomography) the material parameters $u(\mathbf{x})$ may appropriately be replaced with the effective material parameters $\overline{u}(\mathbf{x},\omega)$. The corresponding seismic model is then often called *macromodel*, particularly in seismic prospecting. Hereinafter, "material parameter $u(\mathbf{x})$" may represent either any of the material parameters $u(\mathbf{x})$ or any of the effective material parameters $\overline{u}(\mathbf{x},\omega)$ at a given frequency.

1.2 Mean value of the material parameter

Mean value

$$u_0(\mathbf{x}) = \langle u(\mathbf{x}) \rangle \tag{1.1}$$

of material parameter $u(\mathbf{x})$ describes the most probable value of the material parameter. The mean value $u_0(\mathbf{x})$ of the material parameter will be a very smooth function in most cases.

The unknown difference

$$U(\mathbf{x}) = u(\mathbf{x}) - u_0(\mathbf{x}) \tag{1.2}$$

of the material parameter from its mean value $u_0(\mathbf{x})$ is considered to be a realization of the random quantity.

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1.3 Correlation function of the material parameter

The unknown difference between the material parameter and its mean value may be characterized in terms of the *correlation function*

$$c(\mathbf{y},\mathbf{x}) = \left\langle U^*(\mathbf{y}-\tfrac{1}{2}\mathbf{x})\; U(\mathbf{y}+\tfrac{1}{2}\mathbf{x})\right\rangle \quad , \tag{1.3}$$

where * denotes the complex conjugacy. The first argument of $c(\mathbf{y},\mathbf{x})$ is the midpoint of the pair of points $\mathbf{x}_1 = \mathbf{y}-\frac{1}{2}\mathbf{x}$ and $\mathbf{x}_2 = \mathbf{y}+\frac{1}{2}\mathbf{x}$ in which the correlation of the difference U is studied, whereas the second argument is the distance of the points. The correlation function should vary very moderately with respect to the first argument, whereas it may vary very rapidly with respect to the second argument (distance of points), especially in the vicinity of $\mathbf{x}=\mathbf{0}$. The correlation function is independent of the first argument in a *statistically homogeneous random medium.*

The Fourier transform of the correlation function with respect to the second argument may be introduced, for instance, in the form of

$$\hat{c}(\mathbf{y},\mathbf{k}) = (2\pi)^{-d} \int_{-\infty}^{+\infty} \mathrm{d}\mathbf{x}\;\; \exp(-\mathrm{i}\mathbf{k}^{\mathrm{T}}\mathbf{x})\; c(\mathbf{y},\mathbf{x}) \quad , \tag{1.4}$$

where d is the Euclidian dimension of the space, e.g., $d=3$ in 3-D. Here we have denoted

$$\int_{-\infty}^{+\infty} \mathrm{d}\mathbf{k} = \int_{-\infty}^{+\infty} \mathrm{d}k_1 \int_{-\infty}^{+\infty} \mathrm{d}k_2 \;...\; \int_{-\infty}^{+\infty} \mathrm{d}k_d \quad . \tag{1.5}$$

The inverse Fourier transform of (1.4) is

$$c(\mathbf{y},\mathbf{x}) = \int_{-\infty}^{+\infty} \mathrm{d}\mathbf{k}\;\; \exp(\mathrm{i}\mathbf{k}^{\mathrm{T}}\mathbf{x})\; \hat{c}(\mathbf{y},\mathbf{k}) \quad . \tag{1.6}$$

Wigner distribution

Wigner distribution of function $\varphi(\mathbf{y})$ is

$$\varphi^{\mathrm{W}}(\mathbf{y},\mathbf{k}) = (2\pi)^{-d} \int_{-\infty}^{+\infty} \mathrm{d}\mathbf{x}\;\; \exp(-\mathrm{i}\mathbf{k}^{\mathrm{T}}\mathbf{x})\; \varphi^*(\mathbf{y}-\tfrac{1}{2}\mathbf{x})\; \varphi(\mathbf{y}+\tfrac{1}{2}\mathbf{x}) \quad . \tag{1.7}$$

It may also be expressed in terms of the Fourier transform $\widehat{\varphi}(\mathbf{p})$ of $\varphi(\mathbf{y})$,

$$\varphi^{\mathrm{W}}(\mathbf{y},\mathbf{k}) = \int_{-\infty}^{+\infty} \mathrm{d}\mathbf{p}\;\; \exp(-\mathrm{i}\mathbf{p}^{\mathrm{T}}\mathbf{y})\; \widehat{\varphi}^*(-\mathbf{k}-\tfrac{1}{2}\mathbf{p})\; \widehat{\varphi}(-\mathbf{k}+\tfrac{1}{2}\mathbf{p}) \quad , \tag{1.8}$$

see Ryzhik, Papanicolaou & Keller (1996).

The Fourier transform (1.4) of the correlation function is thus the mean value of the Wigner distribution of the unknown difference between the material parameter and its mean value,

$$\hat{c}(\mathbf{y},\mathbf{k}) = \left\langle U^{\mathrm{W}}(\mathbf{y},\mathbf{k})\right\rangle \quad , \tag{1.9}$$

and the correlation function is its inverse Fourier transform,

$$c(\mathbf{y},\mathbf{x}) = \int_{-\infty}^{+\infty} \mathrm{d}\mathbf{k}\;\; \exp(\mathrm{i}\mathbf{k}^{\mathrm{T}}\mathbf{x})\; \left\langle U^{\mathrm{W}}(\mathbf{y},\mathbf{k})\right\rangle \quad . \tag{1.10}$$

1.4 Synthetic realizations of a statistically homogeneous random medium

(a) On a regular rectangular grid, we generate a pseudorandom realization $W(\mathbf{x})$ of the *white noise* of the desired statistical distribution of the functional values, with the *unit standard deviation.*

(b) Calculate d-dimensional Fourier transform $\widehat{W}(\mathbf{k})$ of the white noise.

(c) Multiply the Fourier transform with the *spectral filter* $\widehat{F}(\mathbf{k})$,

$$\widehat{U}(\mathbf{k}) = \widehat{F}(\mathbf{k})\ \widehat{W}(\mathbf{k}) \quad . \tag{1.11}$$

(d) Calculate d-dimensional inverse Fourier transform $U(\mathbf{x})$ of the product $\widehat{U}(\mathbf{k})$. The multiplication factors of the forward and inverse Fourier transform may, of course, be different from equations (1.4) and (1.6).

(e) Add the appropriate mean value $u_0(\mathbf{x})$ to arrive at the desired pseudorandom realization,

$$u(\mathbf{x}) = u_0(\mathbf{x}) + U(\mathbf{x}) \quad . \tag{1.12}$$

The Fourier transform of the correlation function is then

$$\hat{c}(\mathbf{y},\mathbf{k}) = (2\pi)^{-d}\ \widehat{F}^*(\mathbf{k})\ \widehat{F}(\mathbf{k}) \quad , \tag{1.13}$$

where $\widehat{F}^*(\mathbf{k})\widehat{F}(\mathbf{k})$ is the power spectrum of the filter. The corresponding "homogeneous" correlation function is

$$\boxed{c(\mathbf{y},\mathbf{x}) = C(\mathbf{x}) = (2\pi)^{-d} \int_{-\infty}^{+\infty} \mathrm{d}\mathbf{k}\ \exp(\mathrm{i}\mathbf{k}^{\mathrm{T}}\mathbf{x})\ \widehat{F}^*(\mathbf{k})\ \widehat{F}(\mathbf{k}) \quad . \qquad (1.14)}$$

see (1.6).

2 Isotropic homogeneous correlation functions

Realizations $U(\mathbf{x})$ of a random medium with *isotropic correlation function* may be obtained by multiplying the Fourier transform of the realizations of a white noise of unit standard deviation by the *isotropic spectral filter*

$$\widehat{F}(\mathbf{k}) = \widehat{f}(k) \tag{2.1}$$

with

$$k = (\mathbf{k}^{\mathrm{T}}\mathbf{k})^{\frac{1}{2}} \quad , \tag{2.2}$$

and inversely Fourier transforming the products back into the space domain. Let us recall that $U(\mathbf{x})$ denotes just the random component taken with respect to the mean value, i.e., that $U(\mathbf{x})$ has zero mean value.

The power spectrum of the filter is then the function of k only,

$$P(k) = \widehat{f}^*(k)\widehat{f}(k) \quad . \tag{2.3}$$

Assuming that a white noise has the unit standard deviation, the corresponding *isotropic homogeneous correlation function* is then, see (1.14),

$$C(\mathbf{x}) = (2\pi)^{-d} \int_{-\infty}^{+\infty} \mathrm{d}k_1 \cos(k_1 x_1) ... \int_{-\infty}^{+\infty} \mathrm{d}k_d \cos(k_d x_d)\ P(k) \quad , \tag{2.4}$$

where d is the Euclidian dimension of the space.

We may rotate, before integrating, the k_1 axis into the direction of vector $\mathbf{x}$ to arrive at

$$C(\mathbf{x}) = (2\pi)^{-d} \int\limits_{-\infty}^{+\infty} \mathrm{d}k_1 \cos(k_1 x) \int\limits_{-\infty}^{+\infty} \mathrm{d}k_2 ... \int\limits_{-\infty}^{+\infty} \mathrm{d}k_d \;\; P(k) \quad , \tag{2.5}$$

where

$$x = (\mathbf{x}^{\mathrm{T}}\mathbf{x})^{\frac{1}{2}} \quad . \tag{2.6}$$

For $d > 1$, we introduce the distance r from the k_1 axis,

$$r = [(k_2)^2 + ... + (k_d)^2]^{\frac{1}{2}} \quad , \tag{2.7}$$

and recall the equation

$$V_d(r) = \frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2}+1)} \;\; r^d \tag{2.8}$$

for the volume of d–dimensional sphere of radius r. Differentiating the volume with respect to the radius, we get the surface of the sphere,

$$S_d(r) = \frac{d\ \pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2}+1)} \;\; r^{d-1} = \frac{2\ \pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} \;\; r^{d-1} \quad . \tag{2.9}$$

Integrating (2.5) over the surface of the $(d-1)$–dimensional sphere of radius r in the subspace $k_1 = constant$, and taking into account that the integrands are constant along the surface, we arrive at

$$C(\mathbf{x}) = (2\pi)^{-d} \; \frac{4\ \pi^{\frac{d-1}{2}}}{\Gamma(\frac{d-1}{2})} \int\limits_{0}^{+\infty} \mathrm{d}k_1 \cos(k_1 x) \int\limits_{0}^{+\infty} \mathrm{d}r \; r^{d-2} \; P\Big(\sqrt{(r)^2 + (k_1)^2}\Big) \quad . \tag{2.10}$$

We now switch from the integration over k_1 and r to the integration over k_1 and $k = \sqrt{(r)^2 + (k_1)^2}$,

$$C(\mathbf{x}) = (2\pi)^{-d} \; \frac{4\ \pi^{\frac{d-1}{2}}}{\Gamma(\frac{d-1}{2})} \int\limits_{0}^{+\infty} \mathrm{d}k \; k \; P(k) \int\limits_{0}^{k} \mathrm{d}k_1 \cos(k_1 x) \left[k^2 - (k_1)^2\right]^{\frac{d-3}{2}} \quad . \tag{2.11}$$

For $d > 1$, the integral with respect to k_1 may be evaluated,

$$\int\limits_{0}^{k} \mathrm{d}k_1 \cos(k_1 x) \left[k^2 - (k_1)^2\right]^{\frac{d-3}{2}} = 2^{\frac{d-4}{2}} \; \pi^{\frac{1}{2}} \; \Gamma(\tfrac{d-1}{2}) \; k^{\frac{d-2}{2}} \; x^{\frac{2-d}{2}} \; \mathrm{J}_{\frac{d-2}{2}}(kx) \quad . \tag{2.12}$$

Inserting (2.12) into (2.11), the correlation function may be expressed in terms of the Hankel transform of $k^{\frac{d-1}{2}} P(k) = k^{\frac{d-1}{2}} \widehat{f}^{*}(k)\widehat{f}(k)$,

$$\boxed{C(\mathbf{x}) = (2\pi)^{-\frac{d}{2}} \; x^{\frac{1-d}{2}} \int\limits_{0}^{+\infty} \mathrm{d}k \; \sqrt{kx} \;\; \mathrm{J}_{\frac{d-2}{2}}(kx) \; k^{\frac{d-1}{2}} \; \widehat{f}^{*}(k)\widehat{f}(k) \quad .} \tag{2.13}$$

Taking into account that

$$\mathrm{J}_{-\frac{1}{2}}(\xi) = \sqrt{\frac{2}{\pi}}\,\frac{\cos(\xi)}{\sqrt{\xi}} \quad , \tag{2.14}$$

equation (2.13) may be used for $d \geq 1$, see (2.5) for $d = 1$.

3 Examples of isotropic homogeneous correlation functions

Most (or all?) of the isotropic spectral filters (2.1) recently considered in the geophysical literature are special cases of filter

$$\boxed{\widehat{f}(k) = \kappa\ \left[a^{-2} + k^2\right]^{-\frac{d}{4}-\frac{N}{2}}\ \exp\left(-\frac{a_{\mathrm{G}}^2 k^2}{8}\right) \quad . \tag{3.1}}$$

Individual special filters may be fitted by the relevant choice of exponent N and two correlation lengths a_{G} and a.

Function (2.1) is the product of spectral filter

$$\widehat{f}_{\mathrm{S}_N}(k) = \kappa\ k^{-\frac{d}{2}-N} \quad , \tag{3.2}$$

corresponding to the self–affine random medium, with high–pass wavenumber filter

$$\widehat{f}_a(k) = \left[1 + (ak)^{-2}\right]^{-\frac{d}{4}-\frac{N}{2}} \tag{3.3}$$

and low–pass wavenumber filter

$$\widehat{f}_{a_{\mathrm{G}}}(k) = \exp\left(-\frac{a_{\mathrm{G}}^2 k^2}{8}\right) \quad . \tag{3.4}$$

Note that, with regards to the discussion in Section 3.3, the self–affine random medium with exponents

$$-\tfrac{1}{2} < N < 0 \tag{3.5}$$

seems to be suitable for the description of the material properties of the geological structures.

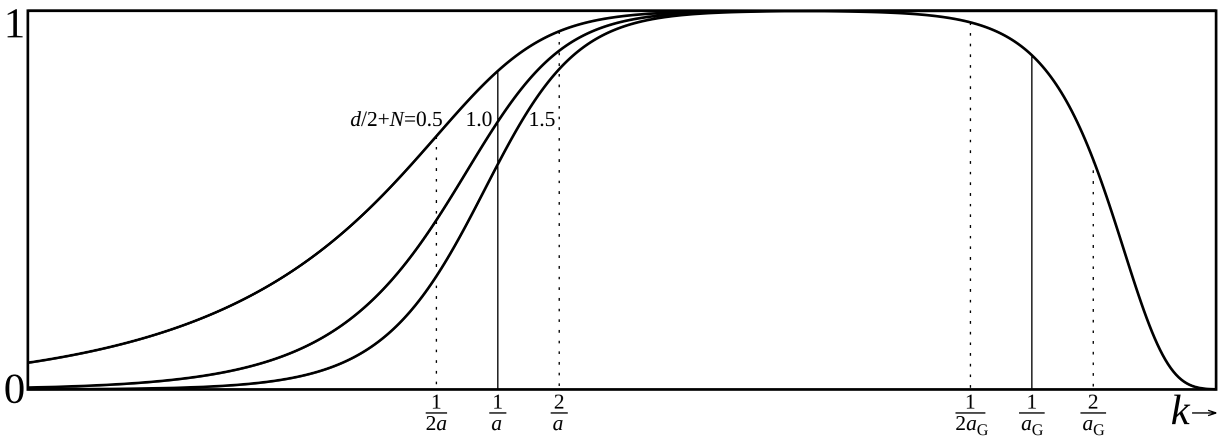

Figure 1. High-pass filter $\widehat{f}_a(k)$ controlled by correlation length a and low-pass Gaussian filter $\widehat{f}_{a_{\mathrm{G}}}(k)$ controlled by correlation length a_{G}. The horizontal wavenumber axis has a logarithmic scale.

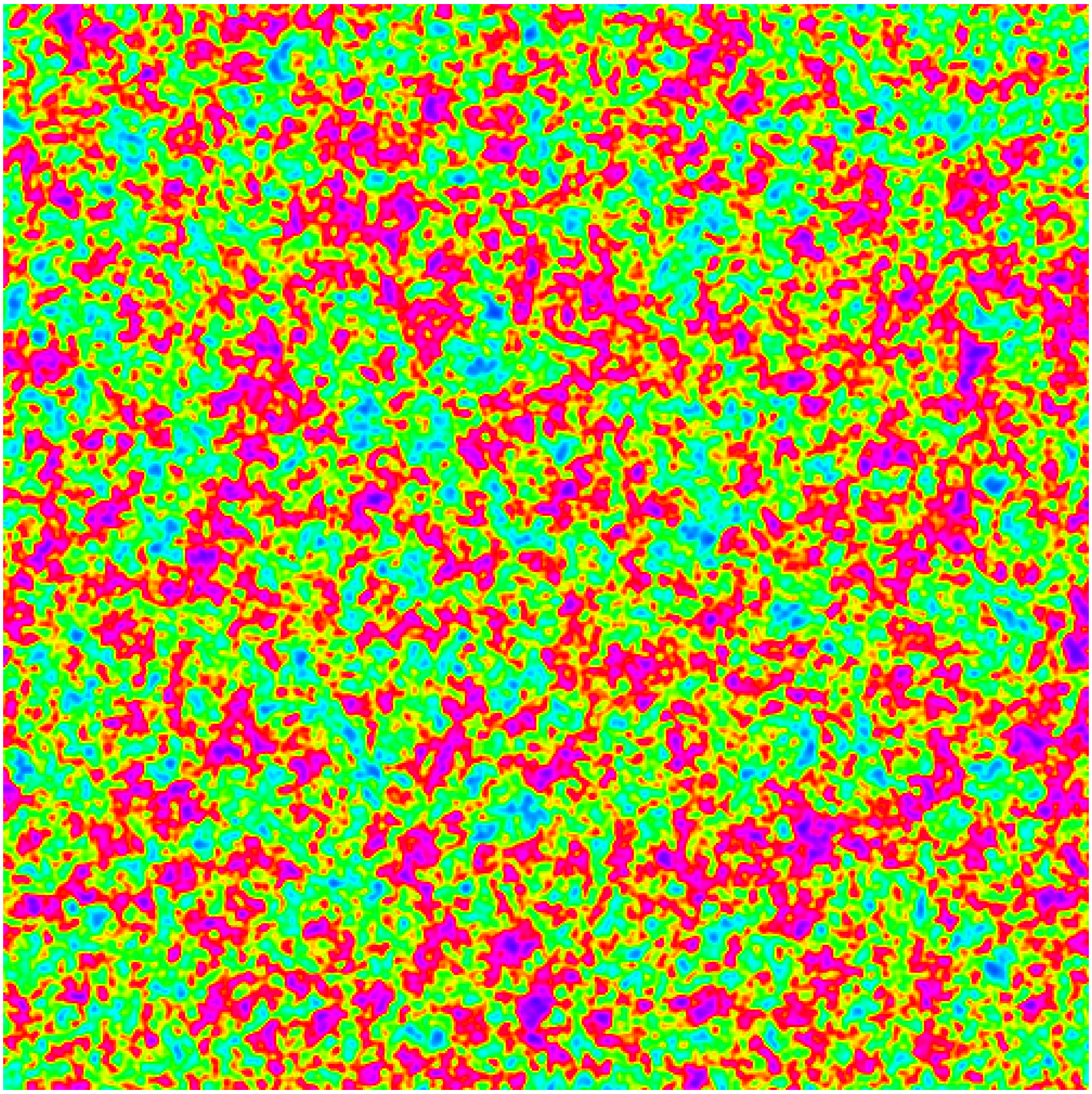

Figure 2. A 2-D representation of the random medium described by exponent $N = -0.2$ and correlation lengths $a_G = 0.005\ H$ and $a = 0.020\ H$, where H is the dimension of the square sample. Yellow colour corresponds to the mean value.

The application of the low–pass filter of correlation length a_G removes small–scale heterogeneities. The resulting random medium remains self–affine at distances considerably longer than the correlation length a_G and qualitatively describes rather the features of the geological structure seen by seismic waves of limited bandwidth than the entire geological structure.

The application of the high–pass filter of correlation length a suppresses large–scale heterogeneities. The resulting random medium remains self–affine at distances considerably shorter than the correlation length a and qualitatively corresponds rather to the difference between the material parameter and its mean value, which fits the heterogeneities considerably larger than the correlation length a, than to the geological structure itself.

If both the low–pass and high–pass filters are applied, the random medium is self–affine at distances x

$$a_{\mathrm{G}} \ll x \ll a \quad . \tag{3.6}$$

3.1 Gaussian correlation function ($N = -\frac{d}{2}$, $a = +\infty$)

$$\widehat{f}_{\mathrm{G}}(k) = \kappa \ \exp\left(-\frac{a_{\mathrm{G}}^2 k^2}{8}\right) \quad . \tag{G.1}$$

For $0 < d$, Hankel transform of $k^{\frac{d-1}{2}} \exp\left(-\frac{a_{\mathrm{G}}^2 k^2}{4}\right)$ is

$$\int\limits_0^{+\infty} \mathrm{d}k \ \sqrt{kx} \ \mathrm{J}_{\frac{d-2}{2}}(kx) \ k^{\frac{d-1}{2}} \exp\left(-\frac{a_{\mathrm{G}}^2 k^2}{4}\right) = 2^{\frac{d}{2}} \ a_{\mathrm{G}}^{-d} \ x^{\frac{d-1}{2}} \exp\left(-a_{\mathrm{G}}^{-2} x^2\right) \quad . \tag{G.2}$$

Inserting (G.1) and (G.2) into (2.13), we arrive at the Gaussian correlation function

$$C_{\mathrm{G}}(\mathbf{x}) = \kappa^2 \ \pi^{-\frac{d}{2}} \ a_{\mathrm{G}}^{-d} \ \exp\left(-a_{\mathrm{G}}^{-2} x^2\right) \quad , \tag{G.3}$$

or simply

$$C_{\mathrm{G}}(\mathbf{x}) = \sigma^2 \ \exp\left(-a_{\mathrm{G}}^{-2} x^2\right) \tag{G.4}$$

if we put

$$\kappa = \sigma \ \pi^{\frac{d}{4}} \ a_{\mathrm{G}}^{\frac{d}{2}} \quad . \tag{G.5}$$

The Gaussian correlation functions describe, in principle, low–pass filtered white noise, which is approached for $a_{\mathrm{G}} \to 0+$. The author thus does not see any relation of the Gaussian correlation functions to the properties of geological structures.

3.2 Von Kármán correlation functions ($a_{\mathrm{G}} = 0$)

For $a_{\mathrm{G}} = 0$, filter (3.1) simplifies to

$$\widehat{f}_{\mathrm{K}_N}(k) = \kappa \ \left[a^{-2} + k^2\right]^{-\frac{d}{4}-\frac{N}{2}} \quad . \tag{K.1}$$

For $0 \le d$ and $-\frac{d+1}{2} < 2N$, Hankel transform of $k^{\frac{d-1}{2}} \left[a^{-2} + k^2\right]^{-\frac{d}{2}-N}$ is

$$\int\limits_0^{+\infty} \mathrm{d}k \ \sqrt{kx} \ \mathrm{J}_{\frac{d-2}{2}}(kx) \ k^{\frac{d-1}{2}} \left[a^{-2} + k^2\right]^{-\frac{d}{2}-N}$$
$$= 2^{\frac{2-d}{2}-N} \ \left[\Gamma(\tfrac{d}{2}+N)\right]^{-1} \ a^N \ x^{\frac{d-1}{2}+N} \ \mathrm{K}_{-N}\left(\tfrac{x}{a}\right) \quad . \tag{K.2}$$

Inserting (K.1) and (K.2) into (2.13), and using relation

$$\mathrm{K}_{-N}(\xi) = \mathrm{K}_N(\xi) \tag{K.3}$$

for the MacDonald functions K_N, we arrive at the Von Kármán correlation function

$$C_{\mathrm{K}_N}(\mathbf{x}) = \kappa^2 \ 2^{1-d-N} \ \pi^{-\frac{d}{2}} \ \left[\Gamma(\tfrac{d}{2}+N)\right]^{-1} \ [ax]^N \ \mathrm{K}_N\left(a^{-1}x\right) \quad . \tag{K.4}$$

Choosing, for $N \neq 0$,

$$\kappa^2 = \sigma^2 \ 2^d \ \pi^{\frac{d}{2}} \ \left[\Gamma(|N|)\right]^{-1} \ \Gamma(\tfrac{d}{2}+N) \ a^{-2N} \quad , \tag{K.5}$$

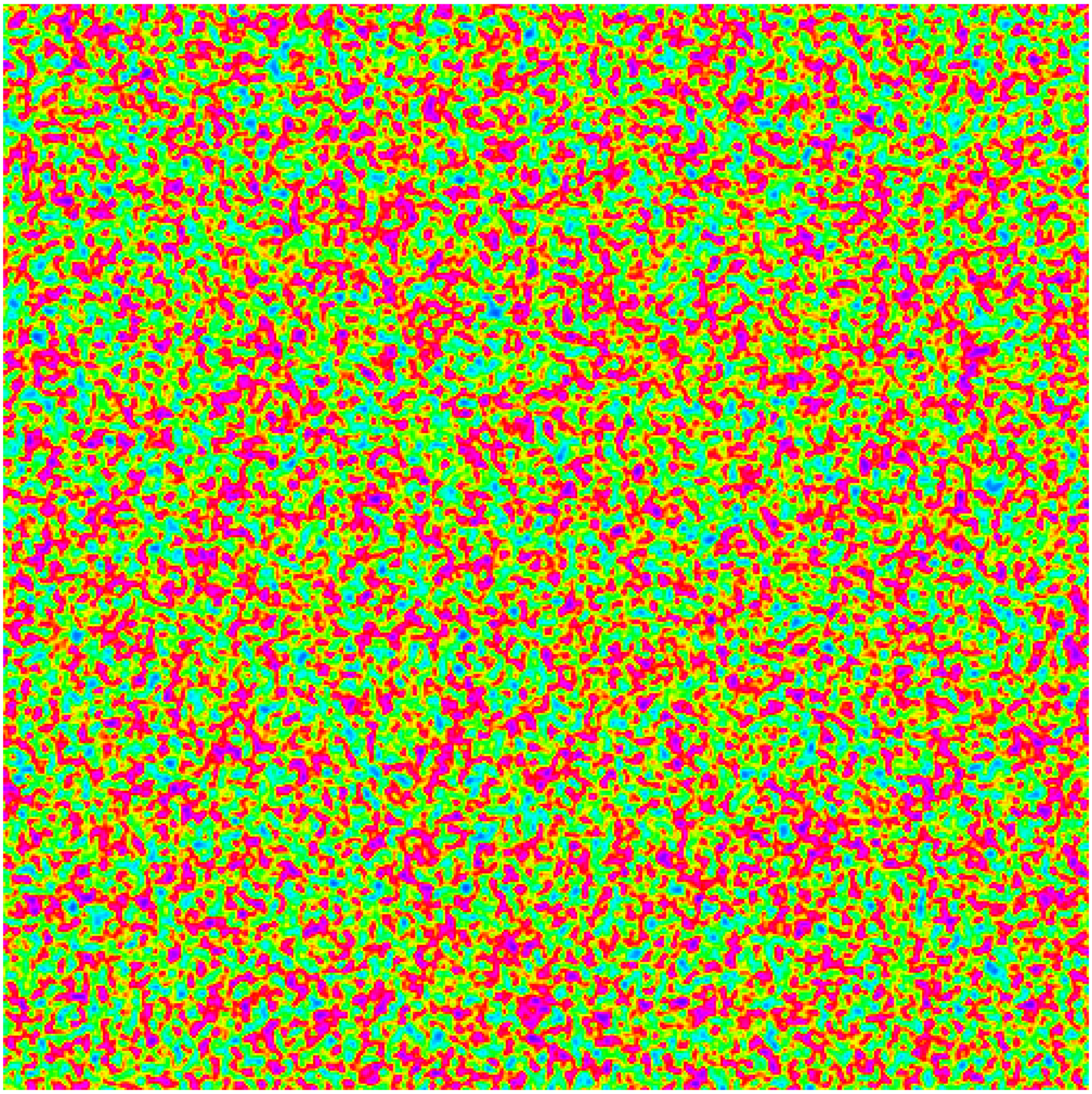

Figure 3. A 2-D representation of the Gaussian random medium ($N = -d/2 = -1$, $a = +\infty$) of correlation length $a_{\mathrm{G}} = 0.005\ H$, where H is the dimension of the square sample.

we obtain the Von Kármán correlation function in the form of

$$C_{\mathrm{K}_N}(\mathbf{x}) = \sigma^2\ \frac{2}{\Gamma(|N|)}\ \left[\frac{x}{2a}\right]^N\ \mathrm{K}_N\left(\frac{x}{a}\right) \quad . \tag{K.6}$$

Right–hand side of equation $(K.6)$ is σ^2 times the Von Kármán function of argument $a^{-1}x$, see Frankel & Clayton (1986).

The asymptotic behaviour of the MacDonald functions K_N for very large arguments is

$$\mathrm{K}_N(\xi) \underset{\xi\to+\infty}{\approx} \sqrt{\frac{\pi}{2}}\ \frac{\exp(-\xi)}{\sqrt{\xi}} \tag{K.7}$$

and, consequently, the asymptotic behaviour of the Von Kármán correlation functions

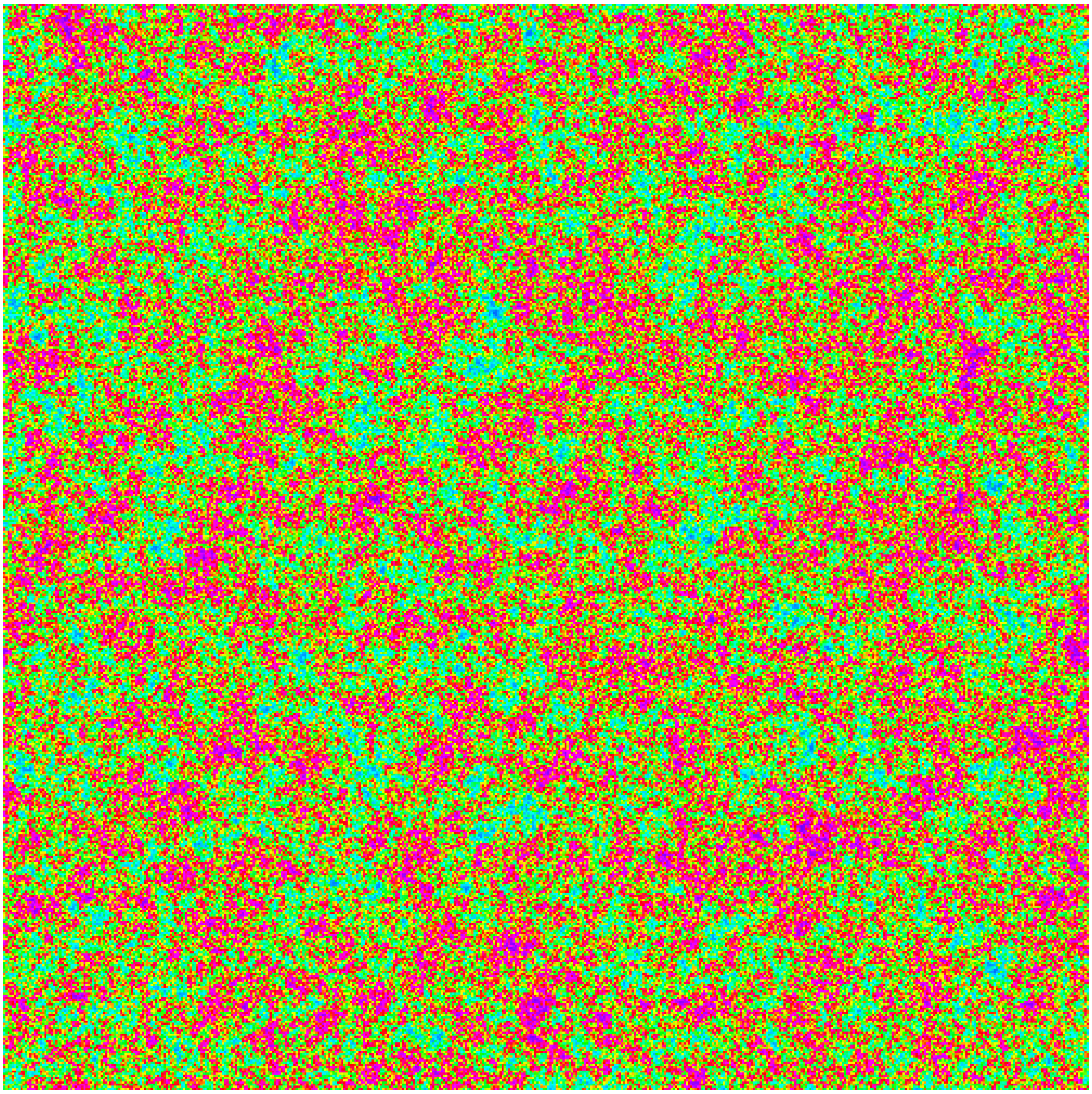

Figure 4. A 2-D representation of the von Kármán random medium ($a_{\mathrm{G}} = 0$) described by exponent $N = -0.2$ and correlation length $a = 0.020\ H$, where H is the dimension of the square sample.

$(K.6)$ for very large arguments is

$$\underset{x \to +\infty}{C_{\mathrm{K}_N}(\mathbf{x})} \approx \sigma^2\ \sqrt{\pi}\ \left[\Gamma(|N|)\right]^{-1}\ \left[\frac{x}{2a}\right]^{N-\frac{1}{2}}\ \exp\left(-\frac{x}{a}\right) \quad . \qquad (K.8)$$

The asymptotic behaviour of the MacDonald functions K_N for very small arguments is

$$\underset{\xi \to 0+}{\mathrm{K}_N(\xi)} \approx \tfrac{1}{2}\Gamma(|N|)\ \left[\frac{\xi}{2}\right]^{-|N|} \qquad (K.9)$$

and, consequently, the asymptotic behaviour of the Von Kármán correlation functions $(K.6)$ for very small arguments is

$$N > 0: \quad \underset{x \to 0+}{C_{\mathrm{K}_N}(\mathbf{x})} \approx \sigma^2 \quad ,$$

$$N < 0: \quad \underset{x \to 0+}{C_{\mathrm{K}_N}(\mathbf{x})} \approx \sigma^2 \left[\frac{x}{2a} \right]^{2N} \quad . \tag{K.10}$$

The Von Kármán medium is thus self–affine at distances considerably shorter than the correlation length a.

In seismic practice, the Von Kármán correlation functions may be appropriate for the self–affine random medium in which the large heterogeneities (larger than the correlation length) are fitted well enough by the seismic model $u_0(\mathbf{x})$. In such a case, the Von Kármán correlation functions describe small, unresolved heterogeneities contained within the difference between the geological structure and its seismic model, standing for the mean value.

3.2.1 Exponential correlation function ($N = \frac{1}{2}$, $a_{\mathrm{G}} = 0$)

For the special case of $N = \frac{1}{2}$, equation ($K.1$) takes the form of

$$\widehat{f}_{\mathrm{E}}(k) = \widehat{f}_{\mathrm{K}_{\frac{1}{2}}}(k) = \kappa \left[a^{-2} + k^2 \right]^{-\frac{d+1}{4}} \quad . \tag{E.1}$$

For $0 < d$, Hankel transform of $k^{\frac{d-1}{2}} \left[a^{-2} + k^2 \right]^{-\frac{d+1}{2}}$ is

$$\int_0^{+\infty} \mathrm{d}k \, \sqrt{kx} \, \mathrm{J}_{\frac{d-2}{2}}(kx) \, k^{\frac{d-1}{2}} \left[a^{-2} + k^2 \right]^{-\frac{d+1}{2}} = 2^{-\frac{d}{2}} \, \pi^{\frac{1}{2}} \, \left[\Gamma(\tfrac{d+1}{2}) \right]^{-1} \, a \, x^{\frac{d-1}{2}} \exp(-a^{-1}x) \quad . \tag{E.2}$$

Inserting ($E.1$) and ($E.2$) into (2.13), we arrive at the

$$C_{\mathrm{E}}(\mathbf{x}) = C_{\mathrm{K}_{\frac{1}{2}}}(\mathbf{x}) = \kappa^2 \, 2^{-d} \, \pi^{\frac{1-d}{2}} \, \left[\Gamma(\tfrac{d+1}{2}) \right]^{-1} \, a \, \exp(-a^{-1}x) \quad . \tag{E.3}$$

Choosing

$$\kappa^2 = \sigma^2 \, 2^d \, \pi^{\frac{d-1}{2}} \, \Gamma(\tfrac{d+1}{2}) \, a^{-1} \quad , \tag{E.4}$$

we obtain the exponential correlation function in the form of

$$C_{\mathrm{E}}(\mathbf{x}) = C_{\mathrm{K}_{\frac{1}{2}}}(\mathbf{x}) = \sigma^2 \, \exp(-a^{-1}x) \quad . \tag{E.5}$$

Equations ($E.4$) and ($E.5$) are the special cases of ($K.4$) and ($K.6$) respectively, with

$$\mathrm{K}_{\frac{1}{2}}(\xi) = \sqrt{\tfrac{\pi}{2}} \, \xi^{-\frac{1}{2}} \, \exp(-\xi) \quad . \tag{E.6}$$

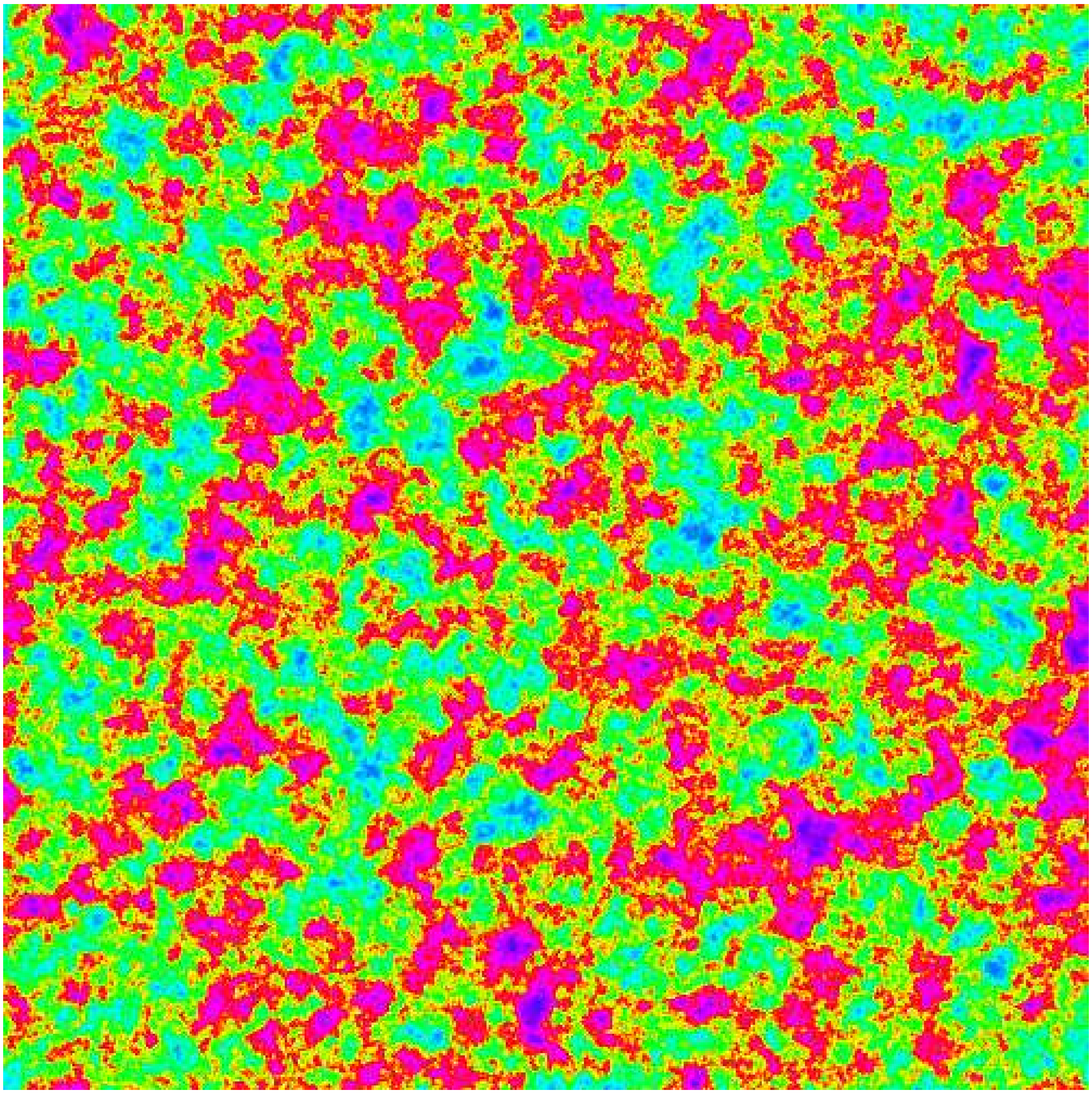

Figure 5. A 2-D representation of the exponential medium ($N = 1/2$, $a_{\mathrm{G}} = 0$) described by correlation length $a = 0.020\ H$, where H is the dimension of the square sample.

3.2.2 Zeroth Von Kármán correlation function ($N = 0$, $a_{\mathrm{G}} = 0$)

The right–hand side of (K.5) limits to infinity for $N \to 0$. This may be removed by rescaling (K.5) by the factor of $\frac{1}{2}\Gamma(|N|)$,

$$\kappa^2 = \tilde{\sigma}^2\ 2^{d-1}\ \pi^{\frac{d}{2}}\ \Gamma(\tfrac{d}{2} + N)\ a^{-2N} \quad . \qquad (Z.1)$$

The rescaled correlation function (K.6) then takes the form of

$$C_{\mathrm{K}_N}(\mathbf{x}) = \tilde{\sigma}^2\ \left[\frac{x}{2a}\right]^N\ \mathrm{K}_N\left(\frac{x}{a}\right) \quad . \qquad (Z.2)$$

For the special case of $N = 0$,

$$C_{\mathrm{K}_0}(\mathbf{x}) = \tilde{\sigma}^2\ \mathrm{K}_0\left(a^{-1}x\right) \quad . \qquad (Z.3)$$

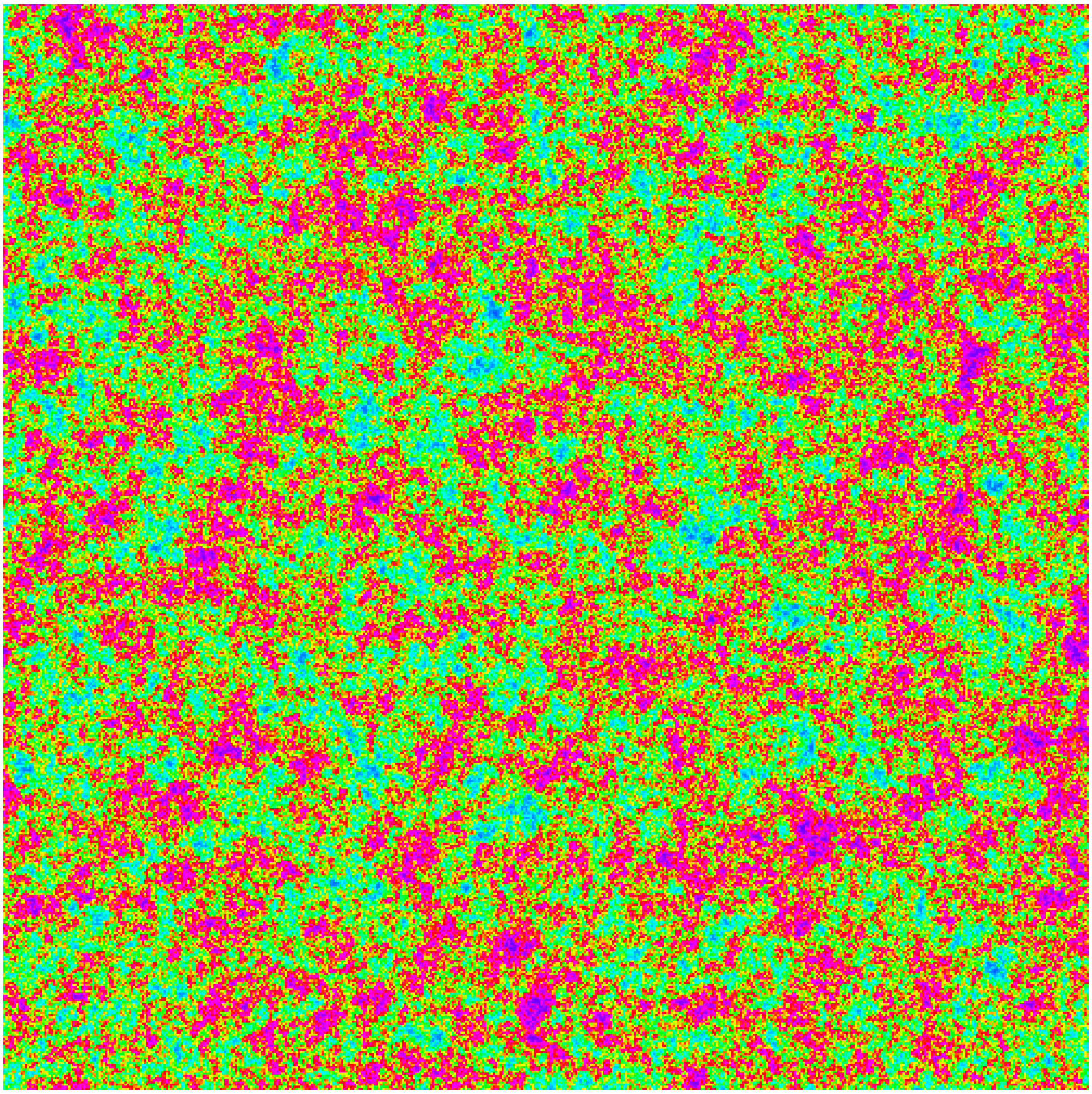

Figure 6. A 2-D representation of the von Kármán random medium ($a_{\mathrm{G}} = 0$) described by exponent $N = 0$ and correlation length $a = 0.020\ H$, where H is the dimension of the square sample.

3.3 Self–affine random medium ($a_{\mathrm{G}} = 0$, $a = +\infty$)

Von Kármán medium is not self–affine over all scales, it is asymptotically self–affine at small distances $x \ll a$. Self–affine random medium may thus be obtained as the limiting case of the Von Kármán medium for $a \to +\infty$,

$$\widehat{f}_{\mathrm{S}_N}(k) = \kappa\ k^{-\frac{d}{2}-N} \quad , \tag{$S.1$}$$

see ($K.1$). For $-\frac{d}{2} < 2N < 0$, Hankel transform of $k^{\frac{d-1}{2}} k^{-d-2N}$ is

$$\int\limits_0^{+\infty} \mathrm{d}k\ \sqrt{kx}\ \mathrm{J}_{\frac{d-2}{2}}(kx)\ k^{-\frac{d+1}{2}-2N} = 2^{-\frac{d}{2}-2N}\ \frac{\Gamma(|N|)}{\Gamma(\frac{d}{2}+N)}\ x^{\frac{d-1}{2}+2N} \quad . \tag{$S.2$}$$

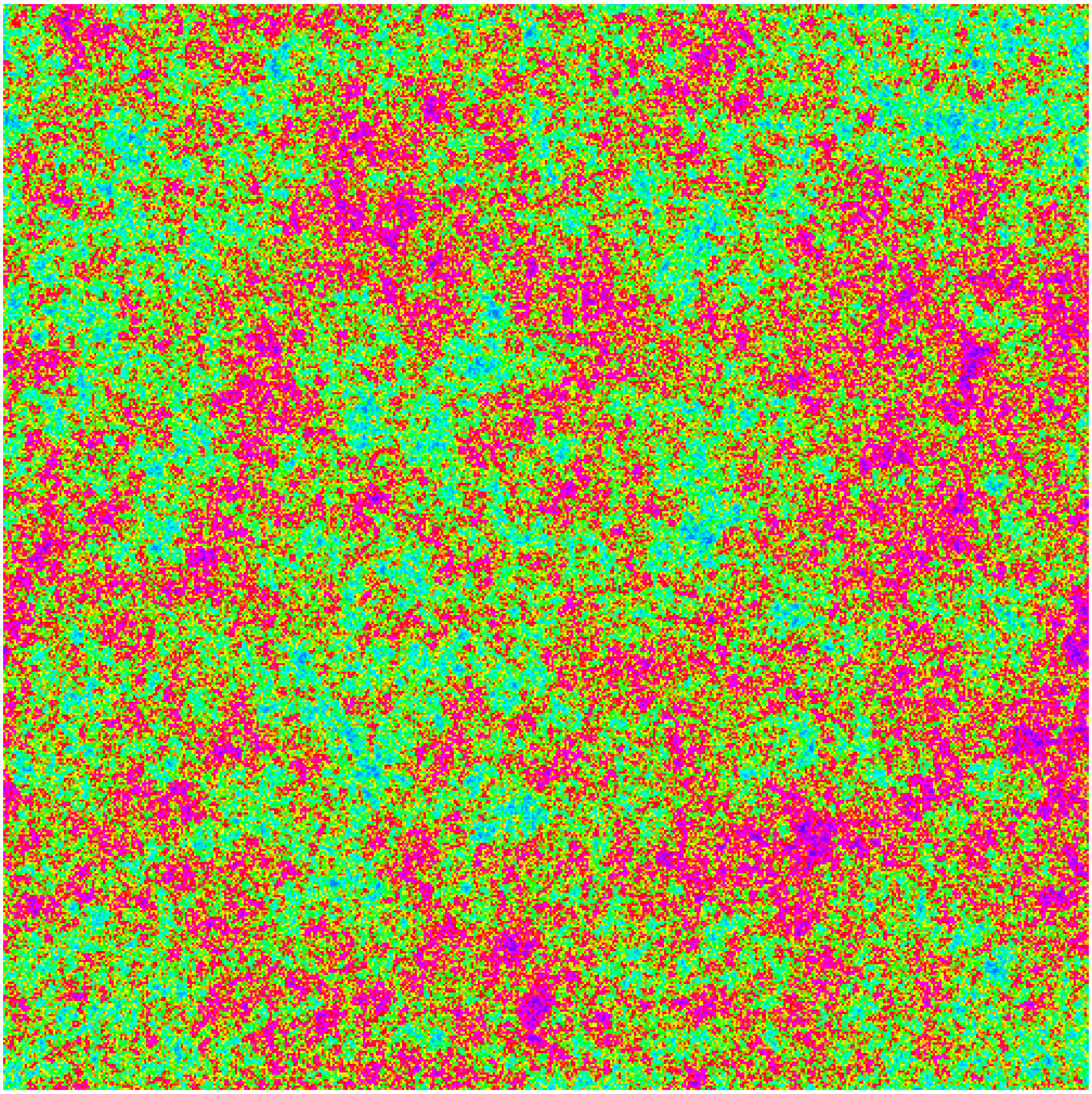

Figure 7. A 2-D representation of the self–affine random medium ($a_{\rm G} = 0$, $a = +\infty$) described by exponent $N = -0.2$.

Inserting (S.1) and (S.2) into (2.13), we arrive at the power–law correlation function

$$C_{{\rm S}_N}(\mathbf{x}) = \kappa^2 \; 2^{-d-2N} \; \pi^{-\frac{d}{2}} \; \frac{\Gamma(|N|)}{\Gamma(\frac{d}{2}+N)} \; x^{2N} \tag{S.3}$$

of the self–affine random medium. Choosing

$$\kappa^2 = \sigma^2 \; 2^d \; \pi^{\frac{d}{2}} \; \left[\Gamma(|N|)\right]^{-1} \; \Gamma(\tfrac{d}{2}+N) \; \left[\tfrac{2}{L}\right]^{2N} \quad , \tag{S.4}$$

we obtain the power–law correlation function in the form of

$$C_{{\rm S}_N}(\mathbf{x}) = \sigma^2 \; \left[\frac{x}{L}\right]^{2N} \quad . \tag{S.5}$$

Here L is some reference distance supplemented to preserve the correct physical units in the expressions. Multiplication factor $\left(\frac{2}{L}\right)^{2N}$ at the right–hand side of (S.4) substitutes here multiplication factor a^{-2N} approaching zero at the right–hand side of $(K.5)$ for $a \to +\infty$, see $(K.10)$.

The self–affine random medium seems to be the most appropriate description of the material properties of the geological structures which contain heterogeneities of all scales. Since correlation function $(S.5)$ should be descending with distance x,

$$N < 0 \quad . \tag{$S.6$}$$

The line integrals of correlation function $(S.5)$ across $\mathbf{x} = \mathbf{0}$, representing travel–time variances along infinitely narrow rays, are finite for

$$-\tfrac{1}{2} < N \tag{$S.7$}$$

(Klimeš 1996). Although fat rays corresponding to finite frequencies admit for $-\frac{d}{2} < N$, the travel–time variances would be considerably increasing with frequency for $N \leq -\frac{1}{2}$. Such a strong velocity dispersion does not seem realistic to the author.

3.4 Laguerr correlation functions ($a = +\infty$)

On the other hand, the effective material parameters may be appropriately described in terms of the self–affine random medium convolved with the Gaussian smoothing function,

$$\widehat{f}_{\mathrm{L_N}}(k) = \kappa\ k^{-\frac{d}{2}-N}\ \exp\left(-\frac{a_{\mathrm{G}}^2 k^2}{8}\right) \quad , \tag{$L.1$}$$

see $(G.1)$ and $(S.1)$.

The corresponding correlation function may be analytically expressed in terms of the generalized Laguerr polynomials, unfortunately with both noninteger subscript and superscript. That is why the explicit equation for the correlation function is not presented here, assuming most of the calculations be carried out in the wavenumber domain.

3.5 Low–pass filtered self–affine random medium

Box–window low–pass filtered power spectrum $(S.1)$ is

$$\begin{aligned} k < K: &\quad \widehat{f}_{\mathrm{S}_N\rceil}(k) = \kappa\ k^{-\frac{d}{2}-N} \quad , \\ k > K: &\quad \widehat{f}_{\mathrm{S}_N\rceil}(k) = 0 \quad . \end{aligned} \tag{$F.1$}$$

Correlation function (2.13) then reads

$$C_{\mathrm{S}_N\rceil}(\mathbf{x}) = \kappa^2\ (2\pi)^{-\frac{d}{2}}\ x^{\frac{1-d}{2}} \int_0^K \mathrm{d}k\ \sqrt{kx}\ \ \mathrm{J}_{\frac{d-2}{2}}(kx)\ k^{\frac{-d-1}{2}-2N} \quad . \tag{$F.2$}$$

Changing the integration variable, this may be turned into

$$C_{\mathrm{S}_N\rceil}(\mathbf{x}) = \kappa^2\ (2\pi)^{-\frac{d}{2}}\ x^{\frac{1-d}{2}}\ K^{\frac{-d+1}{2}-2N} \int_0^1 \mathrm{d}l\ \sqrt{l\ Kx}\ \ \mathrm{J}_{\frac{d-2}{2}}(l\ Kx)\ l^{\frac{-d-1}{2}-2N} \quad . \tag{$F.3$}$$

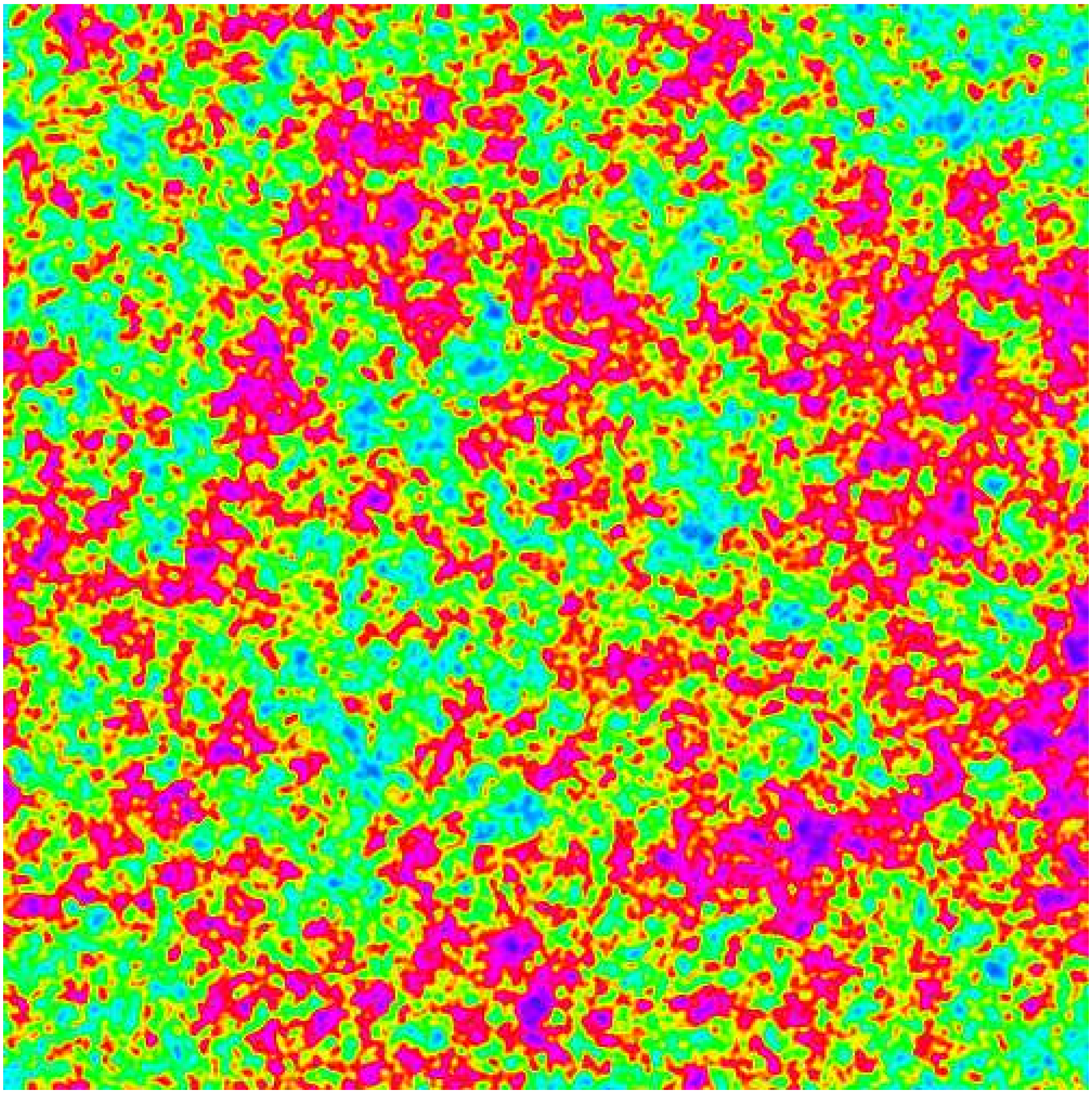

Figure 8. A 2-D representation of the Laguerr random medium ($a = +\infty$) described by exponent $N = -0.2$ and correlation length $a_{\mathrm{G}} = 0.005\ H$, where H is the dimension of the square sample.

3.5.1 Low–pass filtered self–affine medium: $N = -1$

For the special case of $N = -1$,

$$\begin{aligned} k < K: &\quad \widehat{f}_{\mathrm{S}_{-1}]}(k) = \kappa\ k^{1-\frac{d}{2}} \quad , \\ k > K: &\quad \widehat{f}_{\mathrm{S}_{-1}]}(k) = 0 \quad , \end{aligned} \tag{F.4}$$

the Hankel transform of $l^{\frac{-d+2}{2}+\frac{1}{2}}$ in equation $(F.3)$ may be evaluated analytically to yield

$$C_{\mathrm{S}_{-1}]}(\mathbf{x}) = \kappa^2\ (2\pi)^{-\frac{d}{2}}\ x^{\frac{1-d}{2}}\ K^{\frac{-d+1}{2}+2} \left[\frac{2^{\frac{-d+2}{2}+1} (Kx)^{\frac{d-2}{2}-\frac{3}{2}}}{\Gamma\left(\frac{d-2}{2}\right)} - \frac{1}{\sqrt{Kx}}\ \mathrm{J}_{\frac{d}{2}}(Kx) \right] \quad , \tag{F.5}$$

which finally reads

$$C_{\mathrm{S}_{-1}]}(\mathbf{x}) = \kappa^2 \ (2\pi)^{-\frac{d}{2}} \ \left[\frac{2^{2-\frac{d}{2}} \ x^{-2}}{\Gamma\left(\frac{d-2}{2}\right)} - K^{2-\frac{d}{2}} \ x^{-\frac{d}{2}} \ J_{\frac{d}{2}}(Kx) \right] \quad . \tag{F.6}$$

In three–dimensional space, $d = 3$, this equation reads

$$C_{\mathrm{S}_{-1}]}(\mathbf{x}) = \kappa^2 \ (2\pi)^{-\frac{3}{2}} \ \left[\frac{\sqrt{2}}{\sqrt{\pi} \ x^2} - \frac{\sqrt{2}}{\sqrt{\pi}} \ \frac{\cos(Kx)}{x^2} \right] = \frac{\kappa^2}{2\,\pi^2} \ \frac{1}{x^2} \ [1 - \cos(Kx)] \quad . \tag{F.7}$$

For

$$\kappa^2 = \sigma^2 \ 2 \ \pi^2 \ L^2 \quad , \tag{F.8}$$

see $(S.4)$, it takes the form of

$$C_{\mathrm{S}_{-1}]}(\mathbf{x}) = \sigma^2 \ \frac{L^2}{x^2} \ [1 - \cos(Kx)] \quad . \tag{F.9}$$

In two–dimensional space, $d = 2$, the special case discussed here limits to the low–pass filtered white noise of the next section.

3.5.2 Low–pass filtered white noise

For the special case of $N = -\frac{d}{2}$ equation $(F.1)$ reads

$$k < K: \quad \widehat{f}_{\mathrm{W}]}(k) = \widehat{f}_{\mathrm{S}_{-\frac{d}{2}}]}(k) = \kappa \quad ,$$
$$k > K: \quad \widehat{f}_{\mathrm{W}]}(k) = \widehat{f}_{\mathrm{S}_{-\frac{d}{2}}]}(k) = 0 \tag{W.1}$$

and the Hankel transform of $l^{\frac{d-2}{2}+\frac{1}{2}}$ in equation $(F.3)$ may be evaluated analytically to arrive at

$$C_{\mathrm{W}]}(\mathbf{x}) = \kappa^2 \ (2\pi)^{-\frac{d}{2}} \ x^{\frac{1-d}{2}} \ K^{\frac{d+1}{2}} \ \frac{1}{\sqrt{Kx}} \ \mathrm{J}_{\frac{d}{2}}(Kx) \quad , \tag{W.2}$$

which finally reads

$$C_{\mathrm{W}]}(\mathbf{x}) = \kappa^2 \ (2\pi)^{-\frac{d}{2}} \ \left(\frac{K}{x}\right)^{\frac{d}{2}} \ \mathrm{J}_{\frac{d}{2}}(Kx) \quad . \tag{W.3}$$

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References

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