Green functions for inhomogeneous weakly anisotropic media [1]

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Summary

Formulae for the zero-order principal term plus the first-order additional term $(0 + 1A)$ of the qP and qS wave Green functions in the so-called quasi-isotropic (QI) approximation are derived for an unbounded inhomogeneous weakly anisotropic medium. The basic idea of this approximation is the assumption that the deviation of the tensor of elastic parameters of a weakly anisotropic medium from the tensor of elastic parameters of a nearby "background" isotropic medium is of the order ω^{-1} for $\omega \to \infty$. Under this assumption, the procedure of constructing the Green functions consists of two steps: (i) calculation of rays, travel times, the geometrical spreading and polarization vectors in the background isotropic medium; (ii) calculation of corrections of travel times, amplitudes and polarization vectors due to the deviation of the weakly anisotropic medium from the isotropic background.

The zero-order QI approximation is especially important for study of qS waves. It removes the well-known problems of the standard ray method for anisotropic media in regions, in which the difference between the phase velocities of qS waves is small. This is the case of weakly anisotropic media as well as of singular regions of qS waves such as vicinities of, for example, kiss and intersection singularities. In such situations, frequency-dependent amplitudes of the qS waves in the zero-order QI approximation are obtained by a numerical solution of two coupled first-order ordinary differential equations along a ray in the background isotropic medium. When variations of phase velocities along a ray are considerably smaller than their differences (the medium is weakly inhomogeneous or homogeneous), approximate closed-form solutions of the two differential equations can be found. The results of the QI approximation describe then two decoupled qS waves, which can be studied separately by the standard ray method for anisotropic media. In the limit of infinitely weak anisotropy, the formulae for the QI approximation smoothly converge to formulae for isotropic media. The formulae for the zero-order QI approximation are regular everywhere with exception of singular regions of the ray method for isotropic media. The accuracy of the QI approximation can be increased by considering the first-order additional terms of the QI approximation.

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It is shown that the two coupled equations are equivalent to the equations of the coupling ray theory (CRT) based on a simplification of a coupling volume integral. Use of a different vectorial framework along rays in the background medium in the QI approximation than in the CRT avoids some serious problems of the CRT approach. The QI approximation with the first-order additional terms is expected to yield results of comparable or better quality than the CRT.

Key words: Anisotropy, body waves, elastodynamics, Green's function, inhomogeneous media, mode coupling, polarization, ray theory, seismic-wave propagation, shear-wave splitting.

1 Introduction

It is well known that the application of the standard ray method (Červený 1972; Červený, Molotkov & Pšenčík 1977) to the propagation of seismic waves in inhomogeneous anisotropic media is connected with certain difficulties unknown in studies of the wave propagation in isotropic media. These problems are connected with the propagation of qS waves; the computation of the qP waves can be performed without any additional difficulties to those known from isotropic media. The difficulties with qS waves arise *locally* in a vicinity of the qS wave singularities such as, for example, intersection or kiss singularities or *globally* in media with weak anisotropy. The cause of the difficulties in both cases is the close proximity or even coincidence of phase velocities of the two qS waves. This leads to problems with the application of ray tracing and dynamic ray tracing equations, see Shearer & Chapman (1989), Gajewski & Pšenčík (1990), Bakker (1996).

The mentioned problems are also well known from studies of propagation of electromagnetic waves in inhomogeneous anisotropic media. Kravtsov & Orlov (1980) explain the difficulties by the implicit assumption of the standard ray method that it describes *independent* waves. In regions, in which the two qS waves propagate with nearly the same phase velocities, the two waves cannot be considered any more as two independent waves. In other words, when searching for an asymptotic solution of the equation of motion, two small parameters must be taken into account in regions of proximity of the qS wave phase velocities. In addition to the small parameter $\epsilon_1 = \frac{c}{\omega L}$ for $\omega \to \infty$, where ω is a circular frequency, c is a phase velocity and L is a characteristic length (the shortest of the distances, on which the quantities related to the wave propagation change by amount comparable with their size), another small parameter ϵ_2, which characterizes the strength of anisotropy (or the proximity of the phase velocities of qS waves) must be considered. For $\epsilon_2 > \epsilon_1$ the qS waves can be dealt with separately and the standard ray method can be used. For $\epsilon_2 \leq \epsilon_1$, a modified approach must be used. For the latter situation, Kravtsov (1968) proposed the so-called *quasi-isotropic approximation* or briefly the QI approximation. The approach is based on the assumption that the deviation of the elastic tensor of a weakly anisotropic medium from the elastic tensor of a nearby isotropic medium is, formally, of the order ω^{-1} or less for $\omega \to \infty$, see Kravtsov (1968), Kravtsov & Orlov (1980), Kravtsov, Naida & Fuki (1996). Note that a similar procedure has been successfully applied by Moczo, Bard & and Pšenčík (1987) and Gajewski & Pšenčík (1992) to problems concerning the wave propagation in media with weak absorption.

The QI approximation was modified for the elastodynamic case by several authors, see Naida (1977), Sharafutdinov (1994), Zillmer, Kashtan & Gajewski (1997). For its application to the elastic plane waves, see Kiselev (1994). In the zero-order approximation, the QI approximation yields two coupled ordinary differential equations for the determination of the amplitudes of the qS wave along a ray in the nearby background isotropic medium. The equations must be solved numerically along each considered ray. The solutions of the coupled equations are regular everywhere including regions of qS wave singularities. In fact, the coupled equations can be used even in regions of qS wave singularities in media, which cannot be considered weakly anisotropic. In the limit of infinitely weak anisotropy, the solutions smoothly converge to the well-known solutions for isotropic media .

The problem of qS wave singularities and of qS wave propagation in weakly anisotropic media was studied thoroughly by Chapman & Shearer (1989) and Coates & Chapman (1990), respectively. In the former case, the authors developed the so-called *connection* formulae allowing to traverse singular regions. Before and behind these regions, the standard ray method for inhomogeneous anisotropic media can be used. Using a simplification of a coupling volume integral, Coates & Chapman (1990) developed a *coupling ray theory* (CRT) for inhomogeneous weakly anisotropic media. The CRT is based, similarly as the QI approximation, on the solution of two coupled ordinary differential equations, which yield quantities necessary for the determination of qS wave amplitudes and which must be solved repeatedly for all the considered frequencies. Coates & Chapman (1990) showed that CRT is applicable not only to weakly anisotropic media but also in qS wave singular regions.

In this paper, the basic equations of the QI approximation for the elastodynamic case are first reviewed in Sec.2. It is shown that the QI approximation can be also used for the study of the qS wave propagation in isolated regions in strongly anisotropic media, in which the qS waves propagate with nearly the same phase velocities. In Sec.3, the application of the QI approximation is then illustrated on the simple case of qP waves in an unbounded inhomogeneous arbitrary but weakly anisotropic medium. The sum of the zero-order term and the first-order additional terms, $(0 + 1A)$ term, of the qP wave Green function in the QI approximation is presented. In Sec.4.1, the QI approximation is applied to the qS waves in an unbounded inhomogeneous arbitrary weakly anisotropic medium. Two coupled ordinary differential equations found earlier by the above mentioned authors by using the QI approximation are rederived. It is shown that when the variations of the medium are small and anisotropy becomes stronger, the two coupled equations can be *decoupled* and their solutions – the zero-order QI approximation of the qS wave ray amplitudes – can be written in the form of simple *closed-form* expressions similar to that derived for the qP wave. In such situations, there is no need for the numerical solution of the coupled equations. The accuracy of the zero-order QI approximation is increased by supplementing it by the first-order additional terms. In this way, the $(0+1A)$ term of the qS wave Green function is obtained. In Sec.4.2, it is shown that the zero-order QI approximation is equivalent to the CRT of Coates & Chapman (1990) (see also Zillmer et al.1997). Since

Coates & Chapman (1990) made several numerical tests illustrating the applicability of the CRT to weakly anisotropic media as well as to qS wave singular regions, the result of Sec.4.2 has important consequences for the applicability of the formulae presented in this paper. In Sec.5, the results obtained in the paper are discussed from various aspects.

Component notation is used consistently throughout the paper. The lowercase indices attain values 1, 2 and 3, the uppercase indices attain only values 1 and 2. The Einstein summation convention is used for the repeated indices. The following notation for partial derivatives with respect to spatial coordinates is used: $u_{i,jk} = \partial^2 u_i/\partial x_j \partial x_k$.

2 Quasi-isotropic ray approximation

Homogeneous elastodynamic equation for an inhomogeneous anisotropic medium reads

$$(c_{ijkl}u_{k,l})_{,j} - \rho \ddot{u}_i = 0. \tag{1}$$

Here c_{ijkl} is a tensor of the fourth rank called *tensor of elastic parameters.* It has the following symmetry properties

$$c_{ijkl} = c_{jikl} = c_{ijlk} = c_{klij}. \tag{2}$$

Let us seek the time-harmonic solution of the elastodynamic equation (1) in the form of an asymptotic high frequency ray series:

$$u_k(x_j, \omega, t) = e^{-i\omega(t-\tau(x_j))} \sum_{n=0}^{\infty} U_k^{(n)}(x_j) \frac{1}{(-i\omega)^n} \quad . \tag{3}$$

Here ω is the circular frequency and i is the imaginary unit. The scalar real-valued function $\tau(x_j)$ is the *eikonal* or *phase function* and the vectorial, generally complex-valued functions $U_i^{(n)}(x_j)$ are the *amplitude coefficients* of the ray series. The phase function and the amplitude coefficients are the quantities to be found. Inserting (3) into (1), we get

$$\sum_{n=0}^{\infty} \omega^{n+2}[N_i(U_k^{(n)}) - M_i(U_k^{(n-1)}) + L_i(U_k^{(n-2)})] = 0 \tag{4}$$

for $n \geq 0$. In (4), we introduced formally

$$U_k^{(-1)} = 0, \quad U_k^{(-2)} = 0. \tag{5}$$

The differential vectorial operators N_i, M_i and L_i are given by the relations, see e.g. Červený et all.(1977):

$$N_i(U_k^{(n)}) = c_{ijkl}U_k^{(n)}\tau_{,l}\tau_{,j} - \rho U_i^{(n)}, \tag{6a}$$

$$M_i(U_k^{(n)}) = (c_{ijkl}U_k^{(n)}\tau_{,l})_{,j} + c_{ijkl}U_{k,l}^{(n)}\tau_{,j}, \tag{6b}$$

$$L_i(U_k^{(n)}) = (c_{ijkl}U_{k,l}^{(n)})_{,j}. \tag{6c}$$

Formally, the tensor of elastic parameters can be written in the following form

$$c_{ijkl} = c^0_{ijkl} + \Delta c_{ijkl}. \tag{7}$$

In (7), c^0_{ijkl} denote the elastic parameters of a background *isotropic* medium. The parameters c^0_{ijkl} are given by the formula

$$c^0_{ijkl} = \lambda \delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}). \tag{8}$$

The symbols λ, μ and ρ denote the Lamé's parameters and the density of the background isotropic medium. The quantities Δc_{ijkl} in (7) denote small deviations of elastic parameters from isotropy: $\epsilon_2 = \frac{|\Delta c_{ijkl}|}{||c^0_{ijkl}||} \ll 1$.

In the following, we consider the *quasi-isotropic (QI) approximation* assuming that the quantities Δc_{ijkl} in Eq.(7) are formally of the order ω^{-1} or less for $\omega \to \infty$. Inserting (7) into (4), taking into account the above assumption and putting zero the coefficients with ω^2 and ω, we get

$$N_i(U_k^{(0)}) = 0, \tag{9a}$$

$$N_i(U_k^{(1)}) - M_i(U_k^{(0)}) - i\omega \Delta c_{ijkl} U_k^{(0)} \tau_{,l}\tau_{,j} = 0, \tag{9b}$$

where

$$N_i(U_k^{(n)}) = (\lambda+\mu)\tau_{,j}\tau_{,i}U_j^{(n)} + \mu\tau_{,j}\tau_{,j}U_i^{(n)} - \rho U_i^{(n)}, \tag{10a}$$

$$\begin{aligned} M_i(U_k^{(0)}) = {} & (\lambda+\mu)[\tau_{,j}U_{j,i}^{(0)} + \tau_{,i}U_{j,j}^{(0)} + \tau_{,ij}U_j^{(0)}] + \mu[2\tau_{,j}U_{i,j}^{(0)} + \tau_{,jj}U_i^{(0)}] \\ & + \lambda_{,i}\tau_{,j}U_j^{(0)} + \mu_{,j}\tau_{,j}U_i^{(0)} + \mu_{,j}\tau_{,i}U_j^{(0)}. \end{aligned} \tag{10b}$$

The vectorial equation (9a) yields the eikonal equations for the determination of the phase functions $\tau(x_j)$ of the two wave modes propagating in the *isotropic* medium c^0_{ijkl}. The eikonal equations have the form

$$\tau_{,i}\tau_{,i} = \frac{1}{v^2} \tag{11}$$

with

$$v = \alpha = \sqrt{\frac{\lambda+2\mu}{\rho}} \quad \text{or} \quad v = \beta = \sqrt{\frac{\mu}{\rho}}, \tag{12}$$

where α is the P wave velocity and β is the S wave velocity in the isotropic medium described by (8). The eikonal equations can be solved with the use of the ray tracing equations for isotropic media, see e.g. Červený et al. (1977):

$$\frac{dx_i}{d\tau} = v^2\tau_{,i}, \qquad \frac{d\tau_{,i}}{d\tau} = -\frac{1}{v}\frac{\partial v}{\partial x_i}. \tag{13}$$

The quantity τ is the travel time along a ray and $\tau_{,i}$ is the slowness vector. We can see that the ray trajectories are affected only by the distribution of elastic parameters in the background isotropic medium. The weak anisotropy has *no effect* on the rays.

Note that Eq.(9a) also specifies the orientation of the polarization of waves in the zero-order QI approximation. It indicates that the polarization of the qP waves in a weakly anisotropic medium in the zero-order QI approximation is the same as the polarization of P waves in the background isotropic medium, i.e. the qP wave polarization vectors are tangent to the ray. In case of qS waves, the polarization vectors in a weakly anisotropic medium are confined to the plane perpendicular to the ray in the background isotropic medium (like the polarization vectors of S waves).

In the following, we are going to work with several orthogonal vectorial bases connected with rays in isotropic media. The most important basis is the vectorial basis of the ray-centred coordinate system q_i, see e.g., Popov & Pšenčík (1978), Červený et al.(1977) or Červený (1985). The basis consists of one unit vector $e_i^{(3)}$ tangent to a ray and two mutually perpendicular unit vectors $e_i^{(1)}$ and $e_i^{(2)}$ situated in a plane perpendicular to the ray. The vector $e_i^{(3)}$ can be determined from the slowness vector $\tau_{,i}$,

$$e_i^{(3)} = v\tau_{,i}. \tag{14}$$

The symbol v denotes either P or S wave velocity in the isotropic medium. The vectors $e_i^{(I)}$ (I=1 or 2) can be determined from the equations, see e.g. Popov & Pšenčík (1978), Červený (1985)

$$\frac{de_i^{(I)}}{d\tau} = (v_{,k}e_k^{(I)})e_i^{(3)}. \tag{15}$$

The next two vector bases to be also considered are closely related to the previous one. The vector $e_i^{(3)}$ is common for all the three bases. In one basis, the vectors $e_i^{(I)}$ are substituted by the normal and the binormal to the ray. In the other basis, the vectors $e_i^{(I)}$ are substituted by two mutually perpendicular unit vectors $g_i^{(I)}$, which specify the polarization of the qS waves in the zero-order QI approximation. In the following, we call them the qS wave polarization vectors. The polarization of the qP wave is specified by the vector $e_i^{(3)}$ in the zero-order QI approximation.

The amplitude coefficient $U_i^{(n)}$ can thus be expressed as follows:

$$U_i^{(n)} = U_1^{f(n)}f_i^{(1)} + U_2^{f(n)}f_i^{(2)} + U_3^{f(n)}e_i^{(3)}, \tag{16}$$

where $f_i^{(I)}$ are two mutually perpendicular unit vectors situated in the plane perpendicular to the ray. They can belong to any of the above discussed vectorial bases. The variation of the vectors $f_i^{(I)}$ along rays can thus be arbitrary. The superscript "f" indicates, which of the above mentioned bases is used for the expansion of $U_i^{(n)}$. In what follows, we adopt the notation used in the standard ray method for isotropic media. For qP waves, the component $U_3^{f(n)}e_i^{(3)}$ is called the *principal component* and $U_1^{f(n)}f_i^{(1)} + U_2^{f(n)}f_i^{(2)}$ the *additional component*. For qS waves, the *principal component* is $U_1^{f(n)}f_i^{(1)} + U_2^{f(n)}f_i^{(2)}$ and the *additional component* is $U_3^{f(n)}e_i^{(3)}$. In this way, Eq.(9a) indicates that in the zero-order QI approximation, qP and qS waves have non-zero only principal components. The zero-order additional components are zero. To simplify notation, we use sometimes also $f_i^{(3)} = e_i^{(3)}$ in the following.

In order to satisfy Eq.(9b), we require

$$[N_i(U_k^{(1)}) - M_i(U_k^{(0)}) - i\omega\Delta c_{ijkl}U_k^{(0)}\tau_{,l}\tau_{,j}]f_i^{(m)} = 0 \tag{17}$$

for any m. Substitution of $U_k^{(1)}$ specified by (16) into $N_i(U_k^{(1)})$ given by (10a) and multiplication of $N_i(U_k^{(1)})$ by the vector $f_i^{(m)}$ yields

$$N_i(U_k^{(1)})f_i^{(m)} = \frac{\alpha^2 - v^2}{v^2}\rho U_3^{f(1)} e_i^{(3)} f_i^{(m)} + \frac{\beta^2 - v^2}{v^2}\rho U_J^{f(1)}\delta_{Jm}. \tag{18}$$

Inserting Eq.(18) into Eq.(17), we get for qP waves

$$M_i(U_k^{(0)})e_i^{(3)} = -i\omega\rho\alpha^{-2}B_{33}U_3^{f(0)}, \tag{19a}$$

$$U_J^{f(1)} = \frac{\alpha^2}{\rho(\beta^2 - \alpha^2)}[M_i(U_k^{(0)})f_i^{(J)} + i\omega\rho\alpha^{-2}B_{3J}U_3^{f(0)}]. \tag{19b}$$

For qS waves we get

$$M_i(U_k^{(0)})f_i^{(K)} = -i\omega\rho\beta^{-2}B_{JK}U_J^{f(0)}, \quad K = 1, 2, \tag{20a}$$

$$U_3^{f(1)} = \frac{\beta^2}{\rho(\alpha^2 - \beta^2)}[M_i(U_k^{(0)})e_i^{(3)} + i\omega\rho\beta^{-2}B_{J3}U_J^{f(0)}]. \tag{20b}$$

The symbols B_{mn} denote elements of an important matrix appearing in formulae related to weakly anisotropic media, see e.g. Backus (1965), Jech & Pšenčík (1989), Pšenčík & Gajewski (1996),

$$B_{mn} = \rho^{-1}\Delta c_{ijkl} f_i^{(m)} e_j^{(3)} e_l^{(3)} f_k^{(n)}. \tag{21}$$

The equations (19a) and (20a) are the transport equations for the zero-order principal terms. The equations (19b) and (20b) yield the first-order additional terms. Comparing them with the corresponding equations for isotropic media, we can see that the differences are caused by the presence of the terms containing the elements of the matrix B_{mn}. Eqs.(19) and (20) were obtained under the assumption of Δc_{ijkl} in Eq.(7) being of the order ω^{-1} or less for $\omega \rightarrow \infty$. We can see that the same equations could be derived under the assumption of B_{mn} being of the same order. Small values of B_{mn} indicate proximity of the phase velocities to the values of velocities in the background isotropic medium and in case of qS waves, the proximity of the phase velocities of these two waves. The above assumption thus extends the region of applicability of the QI approximation also to regions of small differences between phase velocities of the qS waves.

3 qP waves

Everywhere in this section, we assume $f_i^{(I)} = e_i^{(I)}$, i.e., we consider the vectors $f_i^{(I)}$ to be the basis vectors of the ray centered coordinate system. Components of the vectorial ray amplitude in this frame are thus denoted by "e" instead of "f". Let us determine the zero-order principal component of the qP wave

$$U_i^{(0)} = \mathcal{A}e_i^{(3)}, \tag{22}$$

where $\mathcal{A} = U_3^{e(0)}$ denotes the scalar amplitude in the zero-order QI approximation. The amplitude can be generally complex-valued. Inserting (22) into (10b) and the result into (19a), yields, after some algebra, the transport equation

$$2\rho\frac{d\mathcal{A}}{d\tau} + \rho\alpha^2\mathcal{A}\tau_{,jj} + (\rho\alpha^2)_{,i}\tau_{,i}\mathcal{A} + i\omega\rho\alpha^{-2}B_{33}^e\mathcal{A} = 0. \tag{23}$$

Superscript "e" of the term B^e_{33} indicates that $f_i^{(I)}$ are substituted by $e_i^{(I)}$ in (21). Equation (23) differs from the transport equation for isotropic media by the presence of the last term on the lefthand side. It can be further rewritten into the form

$$\frac{d}{d\tau}(\mathcal{A}\sqrt{\rho\alpha\mathcal{J}}) + \frac{1}{2}i\omega\alpha^{-2}B^e_{33}\mathcal{A}\sqrt{\rho\alpha\mathcal{J}} = 0. \tag{24}$$

Here the symbol $\mathcal{J}$ denotes the Jacobian of the transformation from the ray to the Cartesian coordinates

$$\mathcal{J} = \frac{\partial(x_1, x_2, x_3)}{\partial(\gamma_1, \gamma_2, s)}. \tag{25}$$

The Jacobian is evaluated in the background *isotropic* medium and thus weak anisotropy has *no effect* on it. The ray coordinates γ_I specify a ray, the ray coordinate s specifies the length along the ray. The elements of the Jacobian (25) can be obtained as a solution of the dynamic ray tracing, see e.g. Červený (1972), Gajewski & Pšenčík (1990).

The solution of Eq.(24) can be written in the form

$$\mathcal{A}(\tau) = \frac{\psi(\gamma_1, \gamma_2, \tau_0)}{\sqrt{\rho(\tau)\alpha(\tau)\Omega_M(\tau)}}\exp[-\frac{i\omega}{2}\int_{\tau_0}^{\tau}\alpha^{-2}B^e_{33}(\xi)d\xi]. \tag{26}$$

The function $\psi(\gamma_1, \gamma_2, \tau_0)$ is constant along the ray and depends on the type of a considered source. The quantity Ω_M is related to the Jacobian $\mathcal{J}$ and to the relative geometrical spreading obtained from the dynamic ray tracing with especially chosen initial conditions, see Eq.(A.5) in Pšenčík & Teles (1996). Here and in what follows, the points on a selected ray are parameterized by the travel times calculated along the ray path in the background isotropic medium.

Let us now consider the wavefield generated by a point force f_k acting at the point x_{0m}. By comparing the formula (26) with the expression for the point force in an anisotropic medium given by Pšenčík & Teles (1996), we find that in the zero-order QI approximation the function $\psi(\gamma_1, \gamma_2, \tau_0)$ reads

$$\psi(\gamma_1, \gamma_2, \tau_0) = \frac{e_k^{(3)}(\tau_0)f_k}{4\pi\sqrt{\rho(\tau_0)\alpha(\tau_0)}}, \tag{27}$$

Inserting (27) into (26), we get

$$\mathcal{A}(\tau) = \mathcal{A}_0(\tau, \tau_0)\exp(i\omega\Delta\tau), \tag{28}$$

where

$$\mathcal{A}_0(\tau, \tau_0) = \frac{e_k^{(3)}(\tau_0)f_k}{4\pi\sqrt{\rho(\tau_0)\alpha(\tau_0)\rho(\tau)\alpha(\tau)\Omega_M(\tau)}} \tag{29a}$$

and

$$\Delta\tau = -\frac{1}{2}\int_{\tau_0}^{\tau}\alpha^{-2}(\xi)B^e_{33}(\xi)d\xi. \tag{29b}$$

We can see that in the zero-order QI approximation, the amplitude $\mathcal{A}$ depends slightly ($\Delta\tau$ is small) on the frequency ω. Note that the time shift $\Delta\tau$ was predicted independently by Červený (1982) and Hanyga (1982), see also Červený & Jech (1982).

If we specify the force f_k as a unit vector oriented successively along all the three coordinate axes x_n, i.e. $f_k = \delta_{kn}$, we can write the zero-order QI approximation $G_{in}^{(0)}$ of the qP wave Green function for an unbounded inhomogeneous weakly anisotropic medium as follows

$$G_{in}^{(0)}(\tau,\tau_0,\omega) = \mathcal{A}(\tau)e_i^{(3)}(\tau)e^{i\omega\tau} = \frac{e_n^{(3)}(\tau_0)e_i^{(3)}(\tau)}{4\pi\sqrt{\rho(\tau_0)\alpha(\tau_0)\rho(\tau)\alpha(\tau)\Omega_M(\tau)}}\exp[i\omega(\tau+\Delta\tau)]. \quad (30)$$

Here $\tau = \tau(x_m)$, $\tau_0 = \tau(x_{0m})$ and $\Delta\tau$ is given by (29b). We can see that the formula (30) differs from the corresponding formula for the isotropic background (see e.g. Eisner & Pšenčík, 1996) only by the time shift $\Delta\tau$. Thus, in the zero-order QI approximation, the deviation Δc_{ijkl} of the weakly anisotropic medium from the isotropic background c^0_{ijkl} affects only the arrival time of the qP wave. The amplitude and the polarization are not affected. The polarization of the Green function in the zero-order QI approximation is thus *linear*. Due to the reciprocity of the relative geometrical spreading function Ω_M and the element B^e_{33} of the matrix B^e_{mn}, the qP wave Green function $G_{in}^{(0)}$ is *reciprocal*.

Eisner & Pšenčík (1996) showed that the accuracy of the ray Green function in a homogeneous isotropic medium considerably increases if the leading vectorial term of the ray series is considered. By the leading vectorial term they understood the sum of the zero-order principal term and of the first-order additional term of the ray series. Since, by assumption, the medium considered in this text only slightly differs from isotropic and the variations of its parameters are also assumed to be small, we can expect similar behaviour of the Green function in the QI approximation. Of course, numerical tests are necesary to confirm this hypothesis.

The leading vectorial term $G_{in}^{(L)}$ of the qP wave Green function in the QI approximation in an inhomogeneous weakly anisotropic medium reads

$$G_{in}^{(L)}(\tau,\tau_0,\omega) = e^{i\omega\tau}(\mathcal{A}e_i^{(3)} + \frac{\mathcal{B}_I^1 e_i^{(I)}}{(-i\omega)}). \quad (31)$$

Here we denoted by $\mathcal{B}_I^1$ the first-order additional components, $\mathcal{B}_I^1 = U_I^{e(1)}$. Inserting (10b) into (19b), after some algebra we get for $\mathcal{B}_I^1$

$$\mathcal{B}_I^1 = [-\alpha(\frac{\partial\mathcal{A}_0}{\partial q_I} + i\omega\mathcal{A}_0\frac{\partial\Delta\tau}{\partial q_I}) + \frac{\mathcal{A}_0 e_i^{(I)}}{\beta^2-\alpha^2}X_i + \frac{i\omega}{\beta^2-\alpha^2}\mathcal{A}_0 B^e_{I3}]\exp(i\omega\Delta\tau), \quad (32)$$

where

$$X_i = (\alpha^2-\beta^2)\alpha_{,i} - 4\alpha\beta\beta_{,i} + \alpha(\alpha^2-2\beta^2)\rho^{-1}\rho_{,i}. \quad (33)$$

All the quantities in (32) and (33) with the exception of the derivatives with respect to q_I can be evaluated without problem. The derivatives with respect to q_I can be evaluated approximately using the procedure described by Eisner & Pšenčík (1996). Inspecting

Eq.(32), we can see that in the leading vectorial term the weak anisotropy is responsible, in addition to the travel time correction of the zero-order principal component, for the correction Δg_i of the qP wave polarization vector in the zero-order QI approximation $e_i^{(3)}$

$$\Delta g_i = \alpha \frac{\partial \Delta\tau}{\partial q_I} e_i^{(I)} + \frac{B_{I3}^e e_i^{(I)}}{\alpha^2 - \beta^2}. \tag{34}$$

The first term on the RHS represents a correction due to the perturbation of the phase front. The second term on the RHS represents a correction of the polarization vector due to the deviation of the weakly anisotropic medium from the isotropic background. It was predicted by the use of the perturbation theory, see Jech & Pšenčík (1989). Note that both terms lead to the deviation of the polarization vector from the direction $e_i^{(3)}$ but the polarization remains linear. Both terms are non-zero even in homogeneous media. Remaining terms in (32), if non-zero, lead to elliptical polarization of the qP ray Green function. From Eqs.(31)-(33) we can see that the effects of weak anisotropy and inhomogeneity on the polarization vectors are of the same order.

4 qS waves

As indicated after Eq.(16), the principal component of a qS wave has the form

$$U_i^{(0)} = \mathcal{B} f_i^{(1)} + \mathcal{C} f_i^{(2)}. \tag{35}$$

Here $\mathcal{B} = U_1^{f(0)}$ and $\mathcal{C} = U_2^{f(0)}$ denote scalar zero-order ray amplitudes with respect to the vectors $f_i^{(I)}$. The amplitudes can be generally complex-valued. Inserting (35) into (10b) and the result into (20a), we get, after some algebra, two coupled transport equations for $\mathcal{B}$ and $\mathcal{C}$:

$$\begin{aligned} 2\rho\frac{d\mathcal{B}}{d\tau} + \rho\beta^2\tau_{,jj}\mathcal{B} + (\rho\beta^2)_{,i}\tau_{,i}\mathcal{B} + 2\rho\frac{df_j^{(2)}}{d\tau}f_j^{(1)}\mathcal{C} + i\omega\rho\beta^{-2}(B_{11}\mathcal{B} + B_{12}\mathcal{C}) = 0, \\ 2\rho\frac{d\mathcal{C}}{d\tau} + \rho\beta^2\tau_{,jj}\mathcal{C} + (\rho\beta^2)_{,i}\tau_{,i}\mathcal{C} + 2\rho\frac{df_j^{(1)}}{d\tau}f_j^{(2)}\mathcal{B} + i\omega\rho\beta^{-2}(B_{12}\mathcal{B} + B_{22}\mathcal{C}) = 0. \end{aligned} \tag{36}$$

The elements of the matrix B_{mn} are defined in Eq.(21).

If the vectors $f_i^{(I)}$ in Eq.(35) are the normal and the binormal to the considered ray, then the coupled system of two equations (36) is an elastodynamic form of quasi-isotropic equations of Kravtsov (1968), Kravtsov & Orlov (1980), see also Naida (1977). If the vectors $f_i^{(I)}$ are the zero-order QI approximation of the polarization vectors $g_i^{(I)}$ of the qS waves, then the system (36) leads to the equations of Coates & Chapman (1990), see Sec.4.2 below. If we consider the vectors $f_i^{(I)}$ to be the basis vectors of the ray centered coordinate system, i.e., $f_i^{(I)} = e_i^{(I)}$, then we get the system of Zillmer et al. (1997). In the latter case, the coupling terms containing the scalar products $\frac{df_j^{(2)}}{d\tau}f_j^{(1)}$ and $\frac{df_j^{(1)}}{d\tau}f_j^{(2)}$ vanish due to Eq.(15). In all the mentioned cases, in order to evaluate the scalar ray amplitudes $\mathcal{B}$ and $\mathcal{C}$, we have to solve the two coupled ordinary differential equations (36) numerically along the ray in the background isotropic medium, successively for all the considered frequencies.

In the following, we investigate first Eqs.(36) expressed with respect to the ray-centered vectorial basis $e_i^{(I)}$ (Sec.4.1) and then with respect to the basis formed by the polarization vectors $g_i^{(I)}$ of the qS waves (Sec.4.2).

4.1 Vectorial basis $f_i^{(I)} = e_i^{(I)}$

We consider $f_i^{(I)} = e_i^{(I)}$ and, therefore, we have $\mathcal{B} = U_1^{e(0)}$ and $\mathcal{C} = U_2^{e(0)}$. The coupling terms in Eqs.(36) containing the derivatives of the vectors $f_i^{(I)}$ vanish due to Eq.(15). In analogy with derivation of formulae for the qP waves, we look for the solution of Eqs.(36) in the form

$$\mathcal{B}(\tau) = \frac{\mathcal{B}_0(\tau)}{\sqrt{\rho(\tau)\beta(\tau)\Omega_M(\tau)}}, \qquad \mathcal{C}(\tau) = \frac{\mathcal{C}_0(\tau)}{\sqrt{\rho(\tau)\beta(\tau)\Omega_M(\tau)}}. \tag{37}$$

The quantity $\Omega_M(\tau)$ has the same meaning as in Eq.(26), here, of course, for qS waves. Inserting (37) into (36) yields

$$\frac{d\mathcal{B}_0}{d\tau} = -\frac{1}{2} i\omega\beta^{-2}(B_{11}^e \mathcal{B}_0 + B_{12}^e \mathcal{C}_0), \qquad \frac{d\mathcal{C}_0}{d\tau} = -\frac{1}{2} i\omega\beta^{-2}(B_{12}^e \mathcal{B}_0 + B_{22}^e \mathcal{C}_0). \tag{38}$$

The superscipts "e" in B_{mn}^e indicate that the vectors $f_i^{(I)}$ are substituted by $e_i^{(I)}$ in (21). Eqs.(38) represent two coupled linear differential equations for $\mathcal{B}_0$ and $\mathcal{C}_0$. If the wavefield is generated by a point force f_k, then comparison of Eqs.(38) with the corresponding expressions of Pšenčík & Teles (1996) yields initial conditions for (38) in the form

$$\mathcal{B}_0(\tau_0) = \frac{e_k^{(1)}(\tau_0) f_k}{4\pi\sqrt{\rho(\tau_0)\beta(\tau_0)}}, \qquad \mathcal{C}_0(\tau_0) = \frac{e_k^{(2)}(\tau_0) f_k}{4\pi\sqrt{\rho(\tau_0)\beta(\tau_0)}}. \tag{39}$$

If we specify the force f_k as a unit vector oriented along an arbitrary coordinate axis x_n, i.e. $f_k = \delta_{kn}$, the zero-order QI approximation of the qS wave Green function in unbounded inhomogeneous weakly anisotropic media reads

$$G_{in}(\tau, \tau_0, \omega) = \Big(\mathcal{B}(\tau) e_i^{(1)}(\tau) + \mathcal{C}(\tau) e_i^{(2)}(\tau)\Big) e^{i\omega\tau}. \tag{40}$$

The functions $\mathcal{B}(\tau)$ and $\mathcal{C}(\tau)$ are given by the relations (37), the functions $\mathcal{B}_0(\tau)$ and $\mathcal{C}_0(\tau)$ are obtained by solving Eqs.(38) with initial conditions (39).

Eqs.(38) must be solved, in general, numerically along the ray of an S wave in the background isotropic medium. In certain situations, explicit solutions of Eqs.(38) can be found. In such cases, we can seek the solutions in the form

$$\mathcal{B}_0(\tau) = \bar{\mathcal{B}}_0 \exp(i\omega\Delta\tau), \qquad \mathcal{C}_0(\tau) = \bar{\mathcal{C}}_0 \exp(i\omega\Delta\tau), \tag{41}$$

Inserting (41) into (38) yields

$$\frac{d\bar{\mathcal{B}}_0}{d\tau} + i\omega\frac{d\Delta\tau}{d\tau}\bar{\mathcal{B}}_0 + \frac{1}{2} i\omega\beta^{-2}(B_{11}^e \bar{\mathcal{B}}_0 + B_{12}^e \bar{\mathcal{C}}_0) = 0,$$

$$\frac{d\bar{\mathcal{C}}_0}{d\tau} + i\omega\frac{d\Delta\tau}{d\tau}\bar{\mathcal{C}}_0 + \frac{1}{2}i\omega\beta^{-2}(B^e_{12}\bar{\mathcal{B}}_0 + B^e_{22}\bar{\mathcal{C}}_0) = 0. \tag{42}$$

In homogeneous media or in media, in which $B^e_{12}(\tau) = 0$ along the ray in the isotropic background, the quantities $\bar{\mathcal{B}}_0$ and $\bar{\mathcal{C}}_0$ are constant and Eqs.(42) reduce to

$$\frac{d\Delta\tau}{d\tau}\bar{\mathcal{B}}_0 + \frac{1}{2}\beta^{-2}(B^e_{11}\bar{\mathcal{B}}_0 + B^e_{12}\bar{\mathcal{C}}_0) = 0, \quad \frac{d\Delta\tau}{d\tau}\bar{\mathcal{C}}_0 + \frac{1}{2}\beta^{-2}(B^e_{12}\bar{\mathcal{B}}_0 + B^e_{22}\bar{\mathcal{C}}_0) = 0. \tag{43}$$

Eqs.(43) can be also obtained approximately if variations of parameters of the medium are moderate in comparison with the strength of anisotropy and studied waves do not propagate on long distances. This happens when the terms from Eqs.(42), which do not appear in (43) are substantially smaller than all the remaining terms. The first terms are proportional to c/L, the remaining terms to $\omega\frac{\Delta c}{c} \sim \omega\frac{|\Delta c_{ijkl}|}{||c^0_{ijkl}||}$. Here c denotes a phase velocity and Δc the difference of phase velocities. The quantity L is the characteristic length. If we recall the definitions of the small quantities ϵ_1 and ϵ_2, the condition for finding an approximate solution of Eqs.(38) reads

$$\epsilon_1 \ll \epsilon_2 \tag{44a}$$

or, equivalently

$$\omega^{-1} \ll \frac{\Delta c}{c|\nabla c|}, \tag{44b}$$

see Kravtsov & Orlov (1980), Gajewski & Pšenčík (1990).

Eqs.(42) represent a system of equations for two eigenvalues $-\frac{d\Delta\tau}{d\tau}$ and corresponding eigenvectors $g_i^{(1)}, g_i^{(2)}$ of the matrix $\frac{1}{2}\beta^{-2}B^e_{MN}$. According to Jech & Pšenčík (1989), the vectors $g_i^{(I)}$ are the polarization vectors of the two qS waves in the zero-order QI approximation. They are unit, mutually orthogonal and are situated in the plane specified by the vectors $e_i^{(I)}$, i.e. in the plane perpendicular to the ray of the S wave in the background isotropic medium

$$g_i^{(1)}(\tau) = e_i^{(1)}(\tau)\cos\phi + e_i^{(2)}(\tau)\sin\phi, \quad g_i^{(2)}(\tau) = -e_i^{(1)}(\tau)\sin\phi + e_i^{(2)}(\tau)\cos\phi. \tag{45}$$

In the approximation (44), the angle ϕ is constant along the ray in the background. It can be found from Eqs.(43) in the form of Eqs.(20) of Jech & Pšenčík (1989). Here we present a corrected form of these equations (see also Thomson, Kendall & Guest, 1992, Eq.(A13)):

$$\cos\phi = [\frac{1}{2}(1 + \frac{B^e_{11} - B^e_{22}}{D})]^{1/2}, \qquad \sin\phi = \mathrm{sgn}(B^e_{12})[\frac{1}{2}(1 - \frac{B^e_{11} - B^e_{22}}{D})]^{1/2}. \tag{46}$$

The determination of the angle ϕ from Eqs.(46) is straightforward in anisotropic media. In isotropic media, however, the RHS of Eqs.(46) become indefinite since $B^e_{mn} = 0$ there. This reflects the fact that in isotropic media the polarization vectors of an S wave cannot be uniquely determined. Any two mutually perpendicular unit vectors situated in the plane perpendicular to a ray can serve as polarization vectors. The most natural choice in this case is $\phi = 0$. In media, in which $B^e_{12}(\tau) = 0$, $\phi = 0$ for $B^e_{11} > B^e_{22}$ and $\phi = \frac{\pi}{2}$ for $B^e_{11} < B^e_{22}$.

The eigenvalues of the matrix $\beta^{-2}B_{mn}$ yield

$$\Delta\tau(\tau) = -\frac{1}{4}\int_{\tau_0}^{\tau} \beta^{-2}(\xi)[(B^e_{11}(\xi) + B^e_{22}(\xi)) \pm D(\xi)]d\xi, \tag{47}$$

where

$$D = [(B^e_{11} - B^e_{22})^2 + 4(B^e_{12})^2]^{1/2}. \tag{48}$$

Eq.(47) gives the travel time (phase) shifts of the two qS waves with respect to the S wave in the isotropic background due to the deviation of the considered weakly anisotropic medium from the background isotropic medium. The angle ϕ in (45), (46) was chosen so that the vector $g_i^{(1)}$ corresponds to the time shift with the positive sign in front of $D(\xi)$ in (47), i.e. to the faster of the qS waves. We call it the $qS1$ wave and denote its time shift by $\Delta\tau_{qS1}$. The vector $g_i^{(2)}$ corresponds to the slower qS wave, which we call the $qS2$ wave. It is related to the time shift with the negative sign in front of $D(\xi)$ in (47); we denote it by $\Delta\tau_{qS2}$.

We can thus write for the principal components of the qS waves generated by a point force f_k two independent expressions

$$U_i^{qS1(0)} = \frac{g_k^{(1)}(\tau_0) f_k g_i^{(1)}(\tau)}{4\pi\sqrt{\rho(\tau_0)\beta(\tau_0)\rho(\tau)\beta(\tau)\Omega_M(\tau)}} \exp(i\omega\Delta\tau_{qS1}),$$

$$U_i^{qS2(0)} = \frac{g_k^{(2)}(\tau_0) f_k g_i^{(2)}(\tau)}{4\pi\sqrt{\rho(\tau_0)\beta(\tau_0)\rho(\tau)\beta(\tau)\Omega_M(\tau)}} \exp(i\omega\Delta\tau_{qS2}). \tag{49}$$

Thus in media satisfying the condition (44), Eqs.(36) yield two *uncoupled* qS waves. In case of $B^e_{12} = 0$, the qS waves are uncoupled even if condition (44) is not satisfied.

The derived formulae are regular everywhere except singular regions of the ray method for isotropic media. The formulae are valid both in weakly anisotropic media as well as isotropic media. In the limit of infinitely weak anisotropy, the formulae reduce to the formulae for the zero-order ray approximation of the S wave Green function in the isotropic medium. The formulae are applicable also to media containing smooth transitions from isotropy to weak anisotropy and vice versa. See discussion of this topic in Section 5. When condition (44) is satisfied, the qS waves are decoupled and can be described by explicit formulae.

From the form of the zero-order QI approximation of the qS wave Green function in Eq.(40), we can immediatelly see that its polarization is generally *elliptical*. By the same reasoning as in the case of the qP wave, we can show that the qS wave Green function is *reciprocal*.

Similarly as in the case of qP waves, we can also try to evaluate the leading vectorial term $G_{in}^{(L)}$ of the qS wave Green function

$$G_{in}^{(L)}(\tau, \tau_0, \omega) = e^{i\omega\tau}(\mathcal{B}e_i^{(1)} + \mathcal{C}e_i^{(2)} + \frac{\mathcal{A}^1 e_i^{(3)}}{(-i\omega)}), \tag{50}$$

where $\mathcal{A}^1$ denotes the first-order additional component, $\mathcal{A}^1 = U_3^{e(1)}$. Inserting (10b) into (20b), after some algebra, we get for $\mathcal{A}^1$

$$\mathcal{A}^1 = [\beta\frac{\partial\mathcal{B}}{\partial q_1} + \frac{\mathcal{B}e_i^{(1)}}{\alpha^2-\beta^2}Y_i + \frac{i\omega\mathcal{B}}{\alpha^2-\beta^2}B_{13}^e] + [\beta\frac{\partial\mathcal{C}}{\partial q_2} + \frac{\mathcal{C}e_i^{(2)}}{\alpha^2-\beta^2}Y_i + \frac{i\omega\mathcal{C}}{\alpha^2-\beta^2}B_{23}^e]. \qquad (51)$$

In (51), the symbol Y_i denotes

$$Y_i = (\alpha^2 + 3\beta^2)\beta_{,i} + \beta^2\rho^{-1}\rho_{,i}. \qquad (52)$$

As in the case of the qP wave, we can see that weak anisotropy is responsible for the corrections of directions of the polarization vectors $g_i^{(I)}$. One of these terms has the form

$$\frac{B_{I3}^e e_i^{(3)}}{\beta^2-\alpha^2}. \qquad (53)$$

It corresponds to the correction predicted by the use of the perturbation theory, see Jech & Pšenčík (1989). Due to the corrections of the direction, the qS wave is polarized in a plane, which is not perpendicular to the ray in the background isotropic medium. The effects of the terms remain nonzero even in homogeneous media.

4.2 Vectorial basis $f_i^{(I)} = g_i^{(I)}$

Let us specify Eqs.(36) in the frame formed by the first-order approximation of the polarization vectors $g_i^{(I)}$ of the qS waves, i.e. $f_i^{(I)} = g_i^{(I)}$. The quantities $\mathcal{B}$ and $\mathcal{C}$ in Eq.(35) are now defined as $\mathcal{B} = U_1^{g(0)}$ and $\mathcal{C} = U_2^{g(0)}$. As in the previous section, we define $\mathcal{B}$ and $\mathcal{C}$ as in (37). Inserting this into Eq.(36), we get the following two coupled linear differential equations:

$$\begin{aligned}
&\rho\frac{d\mathcal{B}_0}{d\tau} + \rho\frac{dg_j^{(2)}}{d\tau}g_j^{(1)}\mathcal{C}_0 + \frac{1}{2}i\omega\rho\beta^{-2}B_{11}^g\mathcal{B}_0 = 0,\\
&\rho\frac{d\mathcal{C}_0}{d\tau} + \rho\frac{dg_j^{(1)}}{d\tau}g_j^{(2)}\mathcal{B}_0 + \frac{1}{2}i\omega\rho\beta^{-2}B_{22}^g\mathcal{B}_0 = 0.
\end{aligned} \qquad (54)$$

In deriving Eqs.(54), we took into account an important identity:

$$B_{12}^g = 0. \qquad (55)$$

The superscipts "g" in B_{mn}^g indicate that the vectors $f_i^{(I)}$ are substituted by $g_i^{(I)}$ in (21). For the proof of Eq.(55), see, for example, Eqs.(19) of Jech & Pšenčík (1989).

We can simplify the equations (54) by making use of the results obtained in the previous section. Let us express $\mathcal{B}_0$ and $\mathcal{C}_0$ as follows

$$\mathcal{B}_0(\tau) = r_1(\tau)\exp(i\omega\Delta\tau_1), \quad \mathcal{C}_0(\tau) = r_2(\tau)\exp(i\omega\Delta\tau_2). \qquad (56)$$

The travel time shifts are given by

$$\Delta\tau_1(\tau) = -\frac{1}{2}\int_{\tau_0}^{\tau}\beta^{-2}(\xi)B_{11}^g(\xi)d\xi, \quad \Delta\tau_2(\tau) = -\frac{1}{2}\int_{\tau_0}^{\tau}\beta^{-2}(\xi)B_{22}^g(\xi)d\xi, \qquad (57)$$

which follows from (47) taking into account Eq.(55). By inspecting the formula (47), we can easily find that the time shifts, which it yields, are independent of the choice of the vector frame along the ray (this is a consequence of the fact that $-2\frac{d\Delta\tau}{d\tau}$ are eigenvalues of the matrix B_{mn}). This means that the quantities $\Delta\tau_1(\tau)$ and $\Delta\tau_2(\tau)$ in Eq.(60) are equal to $\Delta\tau_{qS1}(\tau)$ and $\Delta\tau_{qS2}(\tau)$ from (47). If $B^g_{11} > B^g_{22}$, then $\Delta\tau_1(\tau) = \Delta\tau_{qS1}(\tau)$ and $\Delta\tau_2(\tau) = \Delta\tau_{qS2}(\tau)$ and vice versa for $B^g_{11} < B^g_{22}$.

Inserting (56) into (54), we get the following two coupled equations:

$$\frac{dr_1(\tau)}{d\tau}e^{i\omega\Delta\tau_1} + r_2(\tau)\frac{dg_j^{(2)}}{d\tau}g_j^{(1)}e^{i\omega\Delta\tau_2} = 0, \qquad \frac{dr_2(\tau)}{d\tau}e^{i\omega\Delta\tau_2} + r_2(\tau)\frac{dg_j^{(1)}}{d\tau}g_j^{(2)}e^{i\omega\Delta\tau_1} = 0. \quad (58)$$

Due to the perpendicularity of the vectors $g_j^{(1)}$ and $g_j^{(2)}$, we have

$$\frac{dg_j^{(1)}}{d\tau}g_j^{(2)} = -\frac{dg_j^{(2)}}{d\tau}g_j^{(1)} = \gamma_{12}. \quad (59)$$

If we insert the vectors $g_i^{(I)}$ specified by (45) and (46) into the above relation and take into account Eq.(15), we find, see also Coates & Chapman (1990), that

$$\gamma_{12} = \frac{d\phi}{d\tau}, \quad (60)$$

where ϕ is the angle introduced in (45). Thus the quantity γ_{12} describes the rate of the change of the angle ϕ, by which the polarization vectors of qS waves rotate with respect to the vectorial base of the ray centered coordinate system. If $\gamma_{12} = 0$, the polarization vectors of qS waves behave like the vectors $e_i^{(I)}$.

Let us further denote

$$\delta\tau_{12} = \Delta\tau_1 - \Delta\tau_2 \sim \tau_1 - \tau_2, \qquad \delta\tau_{21} \sim \tau_2 - \tau_1. \quad (61)$$

Here τ_1 and τ_2 are the approximate travel times of the qS waves related to $\Delta\tau_1$ and $\Delta\tau_2$, respectively. With the above notation, we can rewrite Eqs.(58) into the final form

$$\frac{dr_1(\tau)}{d\tau} = \gamma_{12}r_2(\tau)e^{-i\omega\delta\tau_{12}}, \qquad \frac{dr_2(\tau)}{d\tau} = -\gamma_{12}r_1(\tau)e^{-i\omega\delta\tau_{21}}. \quad (62)$$

Quantities r_I are obviously functions of frequency ω. If the initial conditions for solving (62) have the form

$$r_1(\tau_0) = \frac{g_n^{(1)}(\tau_0)}{4\pi\sqrt{\rho(\tau_0)\beta(\tau_0)}}, \qquad r_2(\tau_0) = \frac{g_n^{(2)}(\tau_0)}{4\pi\sqrt{\rho(\tau_0)\beta(\tau_0)}}, \quad (63)$$

we can construct from the above formulae the Green function similar to that in Eqs.(40) and (50).

Eqs.(62) are equivalent to Eqs.(30) of Coates & Chapman (1990). Coates & Chapman obtained the above equations by a completely independent approach, based on the use of a coupling volume integral. Let us mention that the equations (30) of Coates & Chapman are a specification of a more general system whose applicability is much broader than that of the equations presented in this paper. On the other hand, Coates & Chapman (1990) observe that the coupling coefficient (60) often changes rapidly through a large value (near singularities), which makes numerical evaluation difficult. From what was shown in the preceding section, we can see that these complications can be avoided if instead of vectors $g_i^{(I)}$ the vectors $e_i^{(I)}$ are used as a vectorial frame along the ray. In addition to the removal of the mentioned problems, the procedure described in the previous section requires neither the polarization vectors $g_i^{(I)}$ nor the time shifts $\Delta\tau_I$ along the ray of the S wave in the background medium.

5 Discussion and conclusions

Formulae for the calculation of the $(0+1A)$ terms of the Green functions for qP and qS waves in the quasi-isotropic approximation were derived. The formulae are applicable to laterally inhomogeneous media of arbitrary but weak anisotropy. In the limit of infinitely weak anisotropy, the formulae reduce to the formulae for the leading vectorial terms of the P and S wave ray Green functions for inhomogeneous isotropic media. When anisotropy becomes stronger and inhomogeneity is moderate, i.e. when the condition (44) is fulfilled, the results of the QI approximation are approximately decoupled waves, which can be further studied separately by the standard ray method for anisotropic media. Except singular regions of the ray method for isotropic media, such as, for example, vicinity of a caustic, the zero-order QI approximation is regular everywhere. It yields regular results not only in regions of weak anisotropy but also in regions of qS wave singularities such as intersection or kiss singularities, for example.

The procedure of constructing the Green functions in the QI approximation can be divided into two steps: (i) calculation of rays, travel times, the geometrical spreading and polarization vectors in the background isotropic medium; (ii) calculation of corrections of travel times, amplitudes and the polarization vectors due to the deviation of the weakly anisotropic medium from the isotropic background at termination points of rays of the considered wave. The corrections for qP waves are given by explicit frequency dependent expressions. The corrections for qS waves can be obtained either by solving numerically a system of two simple linear ordinary differential equations for each considered frequency or from explicit frequency dependent formulae. In the latter case, the formulae describe uncoupled qS waves.

The QI formulae were presented here in the frequency domain. Their transformation into the time domain is straightforward. The formulae can be easily implemented into the programs for the computation of seismic wave fields in laterally varying layered isotropic media like "Complete ray tracing" package (Červený, Klimeš & Pšenčík, 1988) or ANRAY package (Gajewski & Pšenčík, 1990). The coupled differential equations for qS waves must then be solved along rays in the isotropic background repeatedly for all the considered frequencies.

In the zero-order QI approximation, the presented formulae for the qS waves are equivalent to the coupling ray theory (CRT) formulae of Coates & Chapman (1990). This means that the QI approximation is applicable everywhere where the CRT works. The QI approximation is expected to give results of comparable or even better quality (when, in addition to the zero-order terms, the first-order additional terms of the QI approximation are also considered). The equivalence with the CRT guarantees that the QI approximation yields acceptable results in inhomogeneous weakly anisotropic media with anisotropy upto about 10% as well as in qS wave singular regions in strongly anisotropic media, see Coates & Chapman (1990).

As mentioned above, in regions of stronger anisotropy, the QI approximation yields results, which can be matched with the results of the standard ray method for anisotropic media. The combination of the two methods should thus cause no difficulties. It is interesting to note that the use of such a combination in a vicinity of a singularity with the QI formulae applied across the singular region in an anisotropic medium of arbitrary strength can lead to a generation of the other qS wave behind the singularity than the incident qS wave.

Without any problem, the formulae can also be applied in regions of a smooth transition from an isotropic to an anisotropic medium and vice versa. This problem was investigated recently by Thomson et al.(1992) using a generalization of the standard ray method. A single S wave propagating from an isotropic medium to an anisotropic medium will split into two waves calculated along a common ray specified in the background isotropic medium. When anisotropy becomes stronger and the two qS waves separate, the formulae of the QI approximation must be matched with the standard formulae of the ray method for anisotropic media, see e.g. Červený (1972), also Gajewski & Pšenčík (1990) or the perturbation formulae proposed by Farra (1989). From the point of matching, two separate rays will be calculated. The initial values of ray amplitudes along them will be determined from the matching. On the transition from a weakly anisotropic medium to an isotropic medium, two separate waves with a time delay developed in the anisotropic region will propagate along a common ray.

The presented procedure has, of course, also certain limitations and its application opens new questions. The procedure has only limited region of applicability. It is evident that the region of applicability of the quasi-isotropic approximation differs from the region of applicability of the standard ray method. This is a consequence of appearance of the new small parameter $\epsilon_2 = \frac{|\Delta c_{ijkl}|}{||c^0_{ijkl}||}$ in the QI approximation (in addition to the small parameter $\epsilon_1 = \frac{c}{\omega L}$. Consequences of this fact can be seen in the calculation of the travel time. In the standard ray method, the travel time calculated along the ray is exact and does not need any additional corrections. It is used in all the higher-order approximations. In the quasi-isotropic approximation, the travel time is evaluated only approximatelly as a sum of the "isotropic" travel time along the ray in the background isotropic medium and a correction $\Delta\tau$ due to the deviation Δc_{ijkl} from isotropy. The exact ray travel time is thus expanded into a series in powers of ϵ_2, from which only the linear term is kept. An upper estimate of the error of the QI solutions is given by Naida (1977), who also proposes a generalization of the QI approximation extending the region of its applicability. Neglecting the second-order and higher-order terms in the travel time expansion in the QI approximation leads to the limitation of applicability of the QI formulae for smaller distances from the source. This fact probably explains a lower accuracy of the perturbation formulae for the calculation of travel times of qP and qS waves, reported by various authors. Further investigation of accuracy of the presented formulae either by studying behaviour of the higher-order terms of the QI approximation or by a comparison with other methods for modelling seismic wave fields in inhomogeneous anisotropic media is, therefore, necessary. A forthcoming paper will be devoted to these problems.

The described procedure suffers from all the limitations of the standard ray method: it is approximate, fails at caustics regions and transitions from the illuminated to shadow regions.

The formulae offer also extensions and generalizations. The most straightforward extension would be the extension to slightly absorbing, weakly anisotropic media in the way similar to that used by Gajewski & Pšenčík (1992).

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