Correlation function of a self–affine random medium

Luděk Klimeš

Department of Geophysics, Charles University, Ke Karlovu 3, 121 16 Praha 2, Czech Republic, Fax: +42-2-21911292, E-mail: klimes@seis.karlov.mff.cuni.cz

Summary

Self–affine random medium is assumed to be realized in terms of a white noise filtered by the power–law spectral filter of amplitude proportional to a reasonable power of the wavenumber. The power–law medium correlation function is then derived. Travel–time variances are thus proportional to a power of ray lengths. A method designed to estimate the medium correlation function using field travel–times is proposed.

Keywords

Travel times, self–affine random medium, correlation function, scaling geology, fractal geology.

1 Introduction

In order to estimate the relation of the seismic model to the geological medium and, in consequence, to estimate the relation of synthetic quantities calculated in the seismic model to the reality, it is important to have an estimation of the medium correlation function

In a self–affine random medium, the material parameters may be scaled simultaneously with scaling the spatial dimensions in such a way that the statistical properties are unchanged by scaling. Since also a geological medium contains heterogeneities of all sizes, very similar at various scales, a self–affine random medium may be very suitable mathematical model for the approximation of the statistics of a geological medium.

In Section 2, a particular class of self–affine random media is considered, and the corresponding medium correlation function is derived. The medium correlation function is of principal importance in refraction travel–time tomographic inversion, especially when estimating the accuracy of the seismic model, its relation to the geological medium, or the covariance matrix describing the statistics of synthetic travel times.

Section 3 is devoted to the dependence of the a priori covariance matrix of field travel times on the medium correlation function. The a priori covariance matrix of field travel times describes the deviations of travel times from the mean travel–time curve caused by (especially lateral) heterogeneities.

The medium correlation function derived in Section 2 is dependent on two parameters: a “fractal dimension” and the corresponding “standard deviation”. Section 4 is devoted to the estimation of these two parameters, essential for the travel–time tomography, from field travel times.

The proposed method is applied to field data from the Western Bohemia region in Section 5 to demonstrate the possibilities to estimate the medium correlation function.

The reader should be aware that the Einstein summation does not apply to the equations throughout this paper.

Seismic Waves in Complex 3-D Structures, Report 4 (Department of Geophysics, Charles University, Prague 1996)

2 Correlation function of a self–affine random medium

Realizations of a *self–affine random medium* uniformly scalable over all lengths may be obtained by multiplying the Fourier transform of the realizations of a *white noise* by spectral filter

$$\widehat{f}(\mathbf{k}) = \kappa k^{-\frac{\beta}{2}} \tag{1}$$

with

$$k = (\mathbf{k}^{\mathrm{T}}\mathbf{k})^{\frac{1}{2}} \quad , \tag{2}$$

and inversely Fourier transforming the products back into space domain. Here κ is a constant proportional to the standard deviation of the resulting self–affine random functions, and β is a constant related by

$$\beta = 2(d - D) + 3 \tag{3}$$

to the Euclidian dimension d of the space, and to the fractal dimension D of the self–affine random functions, as defined by Turcotte (1989).

Assuming that the white noise has the unit standard deviation, the *medium correlation function* is then

$$C(\mathbf{x}_1, \mathbf{x}_2) = c(\mathbf{x}_1 - \mathbf{x}_2) \tag{4}$$

with

$$c(\mathbf{x}) = (2\pi)^{-d} \int_{-\infty}^{+\infty} \mathrm{d}k_1 \cos(k_1 x_1) ... \int_{-\infty}^{+\infty} \mathrm{d}k_d \cos(k_d x_d) \; [\widehat{f}(\mathbf{k})]^2 \quad . \tag{5}$$

Since spectral filter (1) is rotationally symmetric, we may rotate, before integrating, the k_1 axis into the direction of vector $\mathbf{x}$ to arrive at

$$c(\mathbf{x}) = \kappa^2 (2\pi)^{-d} \int_{-\infty}^{+\infty} \mathrm{d}k_1 \cos(k_1 x) \int_{-\infty}^{+\infty} \mathrm{d}k_2 ... \int_{-\infty}^{+\infty} \mathrm{d}k_d \;\; k^{-\beta} \quad , \tag{6}$$

where

$$x = (\mathbf{x}^{\mathrm{T}}\mathbf{x})^{\frac{1}{2}} \quad . \tag{7}$$

For $d > 1$, we introduce the distance r from the k_1 axis,

$$r = [(k_2)^2 + ... + (k_d)^2]^{\frac{1}{2}} \quad , \tag{8}$$

and recall the equation

$$V_d(r) = \frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2} + 1)} \; r^d \tag{9}$$

for the volume of d–dimensional sphere of radius r. Differentiating the volume with respect to the radius, we get the surface of the sphere,

$$S_d(r) = \frac{d \; \pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2} + 1)} \; r^{d-1} \quad . \tag{10}$$

Integrating (6) over the surface of the $(d-1)$–dimensional sphere of radius r in the subspace $k_1 = constant$, and taking into account that the integrands are constant along the surface, we arrive at

$$c(\mathbf{x}) = \kappa^2 (2\pi)^{-d} \frac{(d-1)\ \pi^{\frac{d-1}{2}}}{\Gamma(\frac{d-1}{2}+1)} \int\limits_{-\infty}^{+\infty} \mathrm{d}k_1 \cos(k_1 x) \int\limits_{0}^{+\infty} \mathrm{d}r\ r^{d-2} [(r)^2 + (k_1)^2]^{-\frac{\beta}{2}} \quad . \tag{11}$$

For $0 < d-1 < \beta$, the integral with respect to r may be evaluated,

$$c(\mathbf{x}) = \kappa^2 \frac{(d-1)\ 2^{-d} \pi^{-\frac{d+1}{2}}}{\Gamma(\frac{d-1}{2}+1)}\ \frac{\Gamma(\frac{d-1}{2})\Gamma(\frac{\beta-d+1}{2})}{2\ \Gamma(\frac{\beta}{2})} \int\limits_{-\infty}^{+\infty} \mathrm{d}k_1 \cos(k_1 x)\ |k_1|^{d-1-\beta} \quad . \tag{12}$$

For $d = 1$, equation (6) reads

$$c(\mathbf{x}) = \kappa^2 (2\pi)^{-1} \int\limits_{-\infty}^{+\infty} \mathrm{d}k_1 \cos(k_1 x)\ |k_1|^{-\beta} \quad . \tag{13}$$

For $0 \leq d-1 < \beta$, equations (12) and (13) take the common form of

$$c(\mathbf{x}) = \kappa^2 2^{-d} \pi^{-\frac{d+1}{2}}\ \frac{\Gamma(\frac{\beta-d+1}{2})}{\Gamma(\frac{\beta}{2})} \int\limits_{-\infty}^{+\infty} \mathrm{d}k_1 \cos(k_1 x)\ |k_1|^{d-1-\beta} \quad . \tag{14}$$

For $d-1 < \beta < d$, the integral with respect to k_1 may be evaluated,

$$c(\mathbf{x}) = \kappa^2 2^{-d} \pi^{-\frac{d+1}{2}}\ \frac{\Gamma(\frac{\beta-d+1}{2})}{\Gamma(\frac{\beta}{2})}\ 2\ \Gamma(d-\beta) \sin(\pi \tfrac{\beta-d+1}{2})\ \ x^{\beta-d} \quad . \tag{15}$$

Defining constant

$$\sigma^2 = \kappa^2 L^{\beta-d} 2^{-d+1} \pi^{-\frac{d+1}{2}}\ \frac{\Gamma(\frac{\beta-d+1}{2})}{\Gamma(\frac{\beta}{2})}\ \Gamma(d-\beta) \cos(\pi \tfrac{d-\beta}{2}) \quad , \tag{16}$$

where L is an arbitrary reference length introduced here for the definition of σ reasonable with respect to physical units, the medium correlation function (4) may be expressed in the form of

$$\boxed{C(\mathbf{x}_1, \mathbf{x}_2) = \sigma^2 \ \left(\frac{|\mathbf{x}_1 - \mathbf{x}_2|}{L} \right)^{\beta-d} \quad ,} \tag{17}$$

where

$$-1 < \beta - d < 0 \quad . \tag{18}$$

In 3-D space, $d = 3$, which is of particular interest in the travel–time tomography, equation (16) reads

$$\sigma^2 = \kappa^2 L^{\beta-3} 2^{-1} \pi^{-2} \, \frac{\Gamma(3-\beta)}{\beta-2} \cos(\pi \tfrac{3-\beta}{2}) \quad . \tag{19}$$

Let us note that the limiting case of $\beta = d-1$ and the cases of $0 < \beta < d-1$ are somewhat unstable (see the above derivation) and their statistical properties resemble white noise produced by $\beta = 0$ in (1), with the medium correlation function of

$$C(\mathbf{x}_1, \mathbf{x}_2) = \kappa^2 \delta(\mathbf{x}_1 - \mathbf{x}_2) \quad . \tag{20}$$

The limiting case of $\beta = d$ is also somewhat unstable and its statistical properties resemble random homogeneous medium, with the medium correlation function of

$$C(\mathbf{x}_1, \mathbf{x}_2) = \kappa^2 \quad . \tag{21}$$

In 2-D space, $d = 2$, the limiting case of $\beta = d-1 = 1$ is called a *flicker noise* and the limiting case of $\beta = d = 2$ is called a *brown noise* by Crossley & Jensen (1989).

3 A priori covariance matrix of travel times

Assuming the linearization approach, field travel times may be expressed in the form of

$$T_I = \tau_I + \delta T_I \tag{22}$$

where

$$\tau_I = \int_0^{s_I} \mathrm{d}s \; u\,(\mathbf{x}(s)) \tag{23}$$

is the integral of the slowness $u(\mathbf{x})$ along the corresponding ray, and δT_I is the error in determination of field travel time T_I.

The a priori covariance

$$S_{KL} = \langle \tau_K \tau_L \rangle \quad , \tag{24}$$

of the K^{th} and L^{th} travel times is then given by

$$S_{KL} = \int_0^{s_K} \mathrm{d}s'_K \int_0^{s_L} \mathrm{d}s'_L \; C\,(\mathbf{x}(s'_K), \mathbf{x}(s'_L)) \quad , \tag{25}$$

where the integration is performed along the corresponding rays of lengths s_K and s_L. Let us notice that, for the medium correlation function (17), $\sigma^{-2} S_{KL}$ is independent of σ and is determined by a single medium parameter $\beta - d$.

The derivative of covariance S_{KL} with respect to β is then

$$\frac{\partial S_{KL}}{\partial \beta} = \int_0^{s_K} \mathrm{d}s'_K \int_0^{s_L} \mathrm{d}s'_L \; C\,(\mathbf{x}(s'_K), \mathbf{x}(s'_L)) \; \ln\left(\frac{|\mathbf{x}(s'_K) - \mathbf{x}(s'_L)|}{L} \right) \tag{26}$$

and $\sigma^{-2} \frac{\partial S_{KL}}{\partial \beta}$ is again independent of σ.

3.1 A priori variances of travel times

If we approximate the distance between ray points $\mathbf{x}(s_1)$ and $\mathbf{x}(s_2)$ of the same ray by

$$|\mathbf{x}(s_1) - \mathbf{x}(s_2)| \approx |s_1 - s_2| \quad , \tag{27}$$

equation (17) may be inserted into (25) to arrive at

$$S_{KK} \approx \sigma^2 L^{d-\beta} \int_0^{s_K} \mathrm{d}s_1 \int_0^{s_K} \mathrm{d}s_2 |s_1 - s_2|^{\beta-d} \quad . \tag{28}$$

For $-1 < \beta - d$, the integrals may be evaluated to read

$$\begin{aligned} S_{KK} &\approx \sigma^2 L^{d-\beta} \int_0^{s_K} \mathrm{d}s_1 \left[\int_0^{s_1} \mathrm{d}s_2 \; (s_2)^{\beta-d} + \int_0^{s_K - s_1} \mathrm{d}s_2 \; (s_2)^{\beta-d} \right] \\ &= \sigma^2 L^{d-\beta} \int_0^{s_K} \mathrm{d}s_1 \left[\frac{(s_1)^{\beta-d+1}}{\beta - d + 1} + \frac{(s_K - s_1)^{\beta-d+1}}{\beta - d + 1} \right] \quad , \end{aligned} \tag{29}$$

and finally

$$S_{KK} \approx \frac{2 \; \sigma^2 \; L^2}{(\beta - d + 1)(\beta - d + 2)} \left(\frac{s_K}{L} \right)^{\beta-d+2} \quad . \tag{30}$$

The derivative of variance S_{KK} with respect to β is then

$$\frac{\partial S_{KK}}{\partial \beta} \approx \frac{2 \; \sigma^2 \; L^2}{(\beta - d + 1)(\beta - d + 2)} \left(\frac{s_K}{L} \right)^{\beta-d+2} \ln \left(\frac{s_K}{L} \right) \quad . \tag{31}$$

Unfortunately, off–diagonal elements $S_{K \neq L}$ of the travel–time covariance matrix have to be evaluated numerically.

4 Determination of the medium correlation function from the field travel times

4.1 Differences of the relative field travel times and their statistical moments

Let us study the mutual differences of the field travel times. Since the travel times are strongly dependent on hypocentral distances, it is possible to compare only the travel times T_K and T_L along the rays of similar lengths s_K and s_L. This restriction may be, to some extent, removed if we relate the travel times to some reference travel–time curve

$$\tau^0 = \tau^0(s) \quad . \tag{32}$$

We may than compare the relative travel times T_K/τ^0_K and T_L/τ^0_L, where

$$\tau^0_I = \tau^0(s_I) \quad . \tag{33}$$

The differences of the relative travel times are then dependent on the local value of the reference travel–time curve in terms of a multiplication factor which has practically no influence on the statistics. The differences of the relative travel times are distorted especially by the error in the derivative of the reference travel–time curve multiplied by $|s_K - s_L|$. We assume that the local error in the derivative of the reference travel–time curve compared to the exact mean travel–time curve is locally negligible in intervals defined by

$$q\,T_L < T_K < T_L \tag{34}$$

with given parameter q, $0 \le q < 1$.

We now define the variances of the relative travel–time differences

$$\vartheta_{KL,KL} = \left\langle \left[\frac{T_K}{\tau^0_K} - \frac{T_L}{\tau^0_L} \right]^2 \right\rangle \quad , \tag{35}$$

and the fourth–order variances

$$\vartheta_{KL,KL,KL,KL} = \left\langle \left[\left(\frac{T_K}{\tau^0_K} - \frac{T_L}{\tau^0_L} \right)^2 - \vartheta_{KL,KL} \right]^2 \right\rangle = \left\langle \left(\frac{T_K}{\tau^0_K} - \frac{T_L}{\tau^0_L} \right)^4 \right\rangle - [\vartheta_{KL,KL}]^2 \quad , \tag{36}$$

describing the standard deviations of the squared differences of the relative field travel times from the variances (35).

We assume picking errors δT_I statistically independent of travel times τ_I. The variances

$$\vartheta_{KL,KL} = \left\langle \left[\frac{\tau_K + \delta T_K}{\tau^0_K} - \frac{\tau_L + \delta T_L}{\tau^0_L} \right]^2 \right\rangle \quad , \tag{37}$$

of the relative travel–time differences may then be expressed in terms of the first two statistical moments of the relative travel times τ_I/τ^0_I,

$$s_K = \left\langle \frac{\tau_K}{\tau^0_K} \right\rangle \quad , \qquad s_{KL} = \left\langle \frac{\tau_K}{\tau^0_K} \, \frac{\tau_L}{\tau^0_L} \right\rangle \quad , \tag{38}$$

and the first two statistical moments of the relative picking errors $\delta T_I / \tau_I^0$,

$$t_K = \left\langle \frac{\delta T_K}{\tau_K^0} \right\rangle \quad , \qquad t_{KL} = \left\langle \frac{\delta T_K}{\tau_K^0} \; \frac{\delta T_L}{\tau_L^0} \right\rangle \quad , \tag{39}$$

as

$$\vartheta_{KL,KL} = s_{KK} - 2s_{KL} + s_{LL} + t_{KK} - 2t_{KL} + t_{LL} + 2\left[s_K t_K - s_K t_L - s_L t_K + s_L t_L \right] \; . \tag{40}$$

We assume zero mean value of the picking errors δT_K,

$$\langle \delta T_K \rangle = 0 \quad . \tag{41}$$

Then

$$t_K = 0 \tag{42}$$

and variances (40) become independent of the mean values s_K of the reduced travel times,

$$\vartheta_{KL,KL} = s_{KK} - 2s_{KL} + s_{LL} + t_{KK} - 2t_{KL} + t_{LL} \quad . \tag{43}$$

We assume to know, at least approximately, data covariance matrix

$$\langle \delta T_K \; \delta T_L \rangle = T_{KL} \quad . \tag{44}$$

Inserting (38) with (24) for s_{MN} and (39) with (44) for t_{MN} into (43), we see that variances (43) are dependent on parameters σ and β of the medium correlation function (17) through

$$\vartheta_{KL,KL} = \vartheta^0_{KL,KL} + \sigma^2 \vartheta^1_{KL,KL}(\beta) \tag{45}$$

with

$$\vartheta^0_{KL,KL} = \frac{T_{KK}}{\tau_K^0 \tau_K^0} - 2\frac{T_{KL}}{\tau_K^0 \tau_L^0} + \frac{T_{LL}}{\tau_L^0 \tau_L^0} \tag{46}$$

and

$$\vartheta^1_{KL,KL}(\beta) = \frac{\sigma^{-2} S_{KK}}{\tau_K^0 \tau_K^0} - 2\frac{\sigma^{-2} S_{KL}}{\tau_K^0 \tau_L^0} + \frac{\sigma^{-2} S_{LL}}{\tau_L^0 \tau_L^0} \quad . \tag{47}$$

Values (46) are constants with respect to β and σ. Functions (47) of β are independent of σ.

To be able to approximate the fourth–order moments in (36) using the second–order moments, we assume the Gaussian probability distributions for both the the self–affine random medium and the picking errors. If all probability distributions in the problem are Gaussian, the marginal probability distribution describing the relative travel–time differences is Gaussian too. For the Gaussian probability distribution,

$$\left\langle \left(\frac{T_K}{\tau_K^0} - \frac{T_L}{\tau_L^0} \right)^4 \right\rangle = 3 \left\langle \left(\frac{T_K}{\tau_K^0} - \frac{T_L}{\tau_L^0} \right)^2 \right\rangle^2 \quad , \tag{48}$$

and equation (36) reads

$$\vartheta_{KL,KL,KL,KL} = 2\left[\vartheta_{KL,KL} \right]^2 \quad . \tag{49}$$

4.2 Objective function

We select the objective function in the form of

$$y = \left[\sum_{K,L} 1 \right]^{-1} \sum_{K,L} \left[\left(\frac{T_K}{\tau_K^0} - \frac{T_L}{\tau_L^0} \right) - \vartheta_{KL,KL} \right]^2 \left[\vartheta_{KL,KL,KL,KL} \right]^{-1} \quad , \tag{50}$$

and minimize it with respect to parameters σ and β of the medium correlation function (17). Let us emphasize that the minimum has to be searched at constant fourth–order variances $\vartheta_{KL,KL,KL,KL}$.

Inserting (45), and (49) with constant $\sigma = \sigma_0$ and $\beta = \beta_0$, objective function (50) reads

$$y(\sigma,\beta) = \tfrac{1}{2} \left[\sum_{K,L} 1 \right]^{-1}$$

$$\times \sum_{K,L} \left[\left(\frac{T_K}{\tau_K^0} - \frac{T_L}{\tau_L^0} \right)^2 - \vartheta^0_{KL,KL} - \sigma^2 \vartheta^1_{KL,KL}(\beta) \right]^2 \left[\vartheta^0_{KL,KL} + (\sigma_0)^2 \vartheta^1_{KL,KL}(\beta_0) \right]^{-2} \quad . \tag{51}$$

Here parameters σ_0 and β_0 are fixed during the minimization, but should be selected close to the final solution,

$$\sigma_0 \approx \sigma \quad , \qquad \beta_0 \approx \beta \quad . \tag{52}$$

Objective function (51) has minimum with respect to σ for

$$\sigma^2(\beta) = \frac{F_1(\beta)}{F_2(\beta)} \tag{53}$$

with

$$F_0(\beta) = \sum_{K,L} \left[\left(\frac{T_K}{\tau_K^0} - \frac{T_L}{\tau_L^0} \right)^2 - \vartheta^0_{KL,KL} \right]^2 \left[\vartheta^0_{KL,KL} + (\sigma_0)^2 \vartheta^1_{KL,KL}(\beta_0) \right]^{-2} \quad , \tag{54}$$

$$F_1(\beta) = \sum_{K,L} \left[\left(\frac{T_K}{\tau_K^0} - \frac{T_L}{\tau_L^0} \right)^2 - \vartheta^0_{KL,KL} \right] \vartheta^1_{KL,KL}(\beta) \left[\vartheta^0_{KL,KL} + (\sigma_0)^2 \vartheta^1_{KL,KL}(\beta_0) \right]^{-2} \ , \tag{55}$$

and

$$F_2(\beta) = \sum_{K,L} \left[\vartheta^1_{KL,KL}(\beta) \right]^2 \left[\vartheta^0_{KL,KL} + (\sigma_0)^2 \vartheta^1_{KL,KL}(\beta_0) \right]^{-2} \quad . \tag{56}$$

The minimum value of the objective function with respect to σ is

$$y(\beta) = \tfrac{1}{2} \left[\sum_{K,L} 1 \right]^{-1} \left[F_0(\beta) - \frac{[F_1(\beta)]^2}{F_2(\beta)} \right] \quad . \tag{57}$$

Since inaccurate field travel times may severely distort the estimated statistics of the geological medium, it is reasonable to restrict summation only to travel times satisfying inequality

$$T_{KK} \le (\sigma_{\rm err})^2 \ \sigma^{-2} S_{KK} \quad , \tag{58}$$

where $\sigma_{\rm err}$ is a given constant. The right–hand side of (58) has to be evaluated at fixed $\beta = \beta_0$ in order not to alter the data set during the minimization of the objective function.

4.3 Minimization of the objective function

First we select reasonable values of constants q and $\sigma_{\rm err}$. Then we select the value of β_0. The corresponding value of σ_0 may be found iteratively: for an initial estimate of σ_0 we evaluate new $\sigma_0 = \sigma(\beta_0)$ using (53) to get a better estimate, rapidly approaching the value of σ_0 consistent with the chosen value of β_0. For fixed β_0 and σ_0 we calculate the values of the objective function (57) at different values of β in order to find the minimum. If the values of β depart from β_0, we should select new β_0 and determine new σ_0.

The minimum of the objective function with respect to β is not very pronounced and is very sensitive to bad mistakes in data. It is also influenced by artificial numerical parameters like q or $\sigma_{\rm err}$. This behaviour is due to the sensitivity of β to the fourth statistical moment of the field travel times. That is why β cannot be determined very accurately. The accuracy of the order of ± 0.1 in β may be thought to be an excellent result, heavily reached in practice. However, the author hopes that some small uncertainty in β should not influence the travel–time inversion considerably.

On the other hand, for given β, "standard deviation" σ is dependent on the second statistical moment of the field travel times and may be determined very accurately.

For the Gaussian probability distribution, the resulting minimum objective function should be close to 1.

5 Example: Western Bohemia

An attempt is made to estimate the correlation function for the region of Western Bohemia and the surrounding part of Germany, using the travel times from the refraction measurements performed during the years 1989 to 1991 (Bucha et al. 1992), see Figure 1.

5.1 Reference travel–time curve

The mean dependence of the travel times on the hypocentral distance has been roughly approximated by a rational function of the form of

$$\tau^0(s) = \frac{a\,s + b\,s^2}{c + s} \quad . \tag{59}$$

At large distances s, the reference travel time (59) approaches the asymptotic line given by slowness b and travel–time delay of $a - b\,c$. Constants a, b, and c has been fitted using the least squares,

$$a = 0.50\,\mathrm{s} \quad , \qquad b = 0.17\,\mathrm{s\,km^{-1}} \quad , \qquad c = 1.25\,\mathrm{km} \quad . \tag{60}$$

The deviations $T_K - \tau(s_K)$ of field travel times T_K with respect to the reference travel–time curve (59) are shown in Figure 2. The area of the dots in Figure 2 is proportional to the weights w'_K used in the least squares. The weights of

$$w_K = \frac{1}{1. + \frac{T_{KK}}{(\delta_{\rm err} + \rho_{\rm err}\; T_K)^2}} \tag{61}$$

with

$$\delta_{\rm err} = 0.010\,\mathrm{s} \quad , \qquad \rho_{\rm err} = 0.005 \tag{62}$$

has been normalized separately in each interval of length

$$\Delta s = 1\text{km} \tag{63}$$

using formula

$$w'_K = w_K \left[1 + \sum_L w_L \right]^{-1} \tag{64}$$

to get a relatively even coverage of all hypocentral distances s. The squared travel–time deviations have then been weighted with the least–square weights of

$$w''_K = w'_K \; s^{-p} \qquad \text{with} \qquad p = 0.5 \quad . \tag{65}$$

The colours in Figure 2 distinguish the travel times according to the receiver profiles. Black dots correspond to solitary receivers.

5.2 Evaluation of covariances between travel times

For the estimation of the parameters of the medium correlation function we consider here straight rays, like in a homogeneous medium. For the used refraction travel times which rays do not penetrate very deep into the Earth compared with the epicentral distance, it should be a reasonable approximation. Especially if the rays of considerably different lengths are not compared, see condition (34).

Covariance matrix (25) of travel times has been evaluated numerically, dividing rectangular integration area of dimensions $s_K \times s_L$ into small rectangular cells and replacing the integrand by a bilinear function in each cell. Since the integrand may reach infinity at some points, the integrand has been limited from above at each gridpoint in such a way to get an exact values of the integral in all square cells touching the diagonal for the special case of variances S_{KK}.

Unfortunately, the first version of the code used for these tests is not sufficiently debugged, is slow and not sufficiently accurate. This may influence the reliability of the presented numerical results. However, the numerical tests will further be improved.

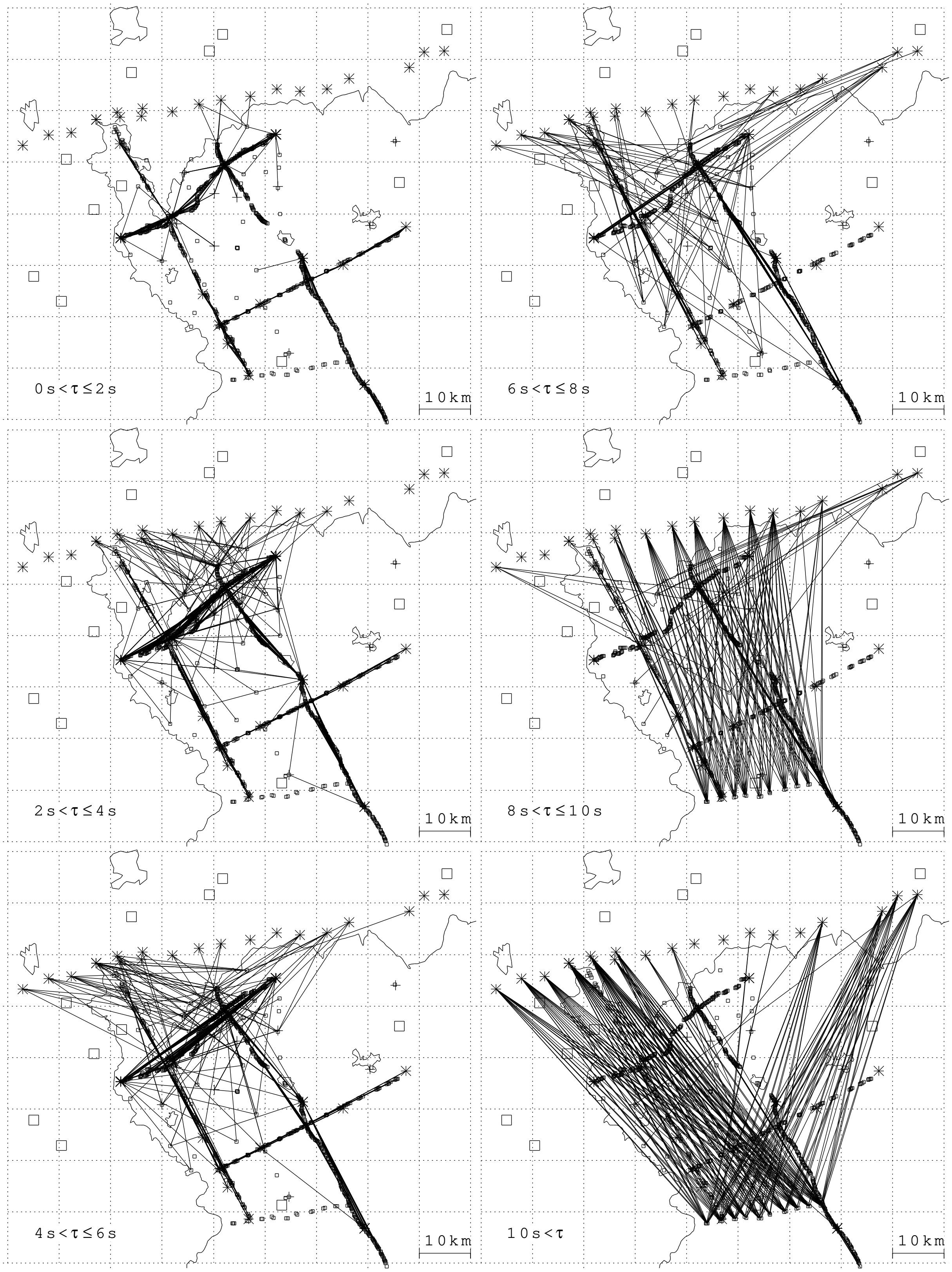

Figure 1. Travel times τ sorted according to their length.

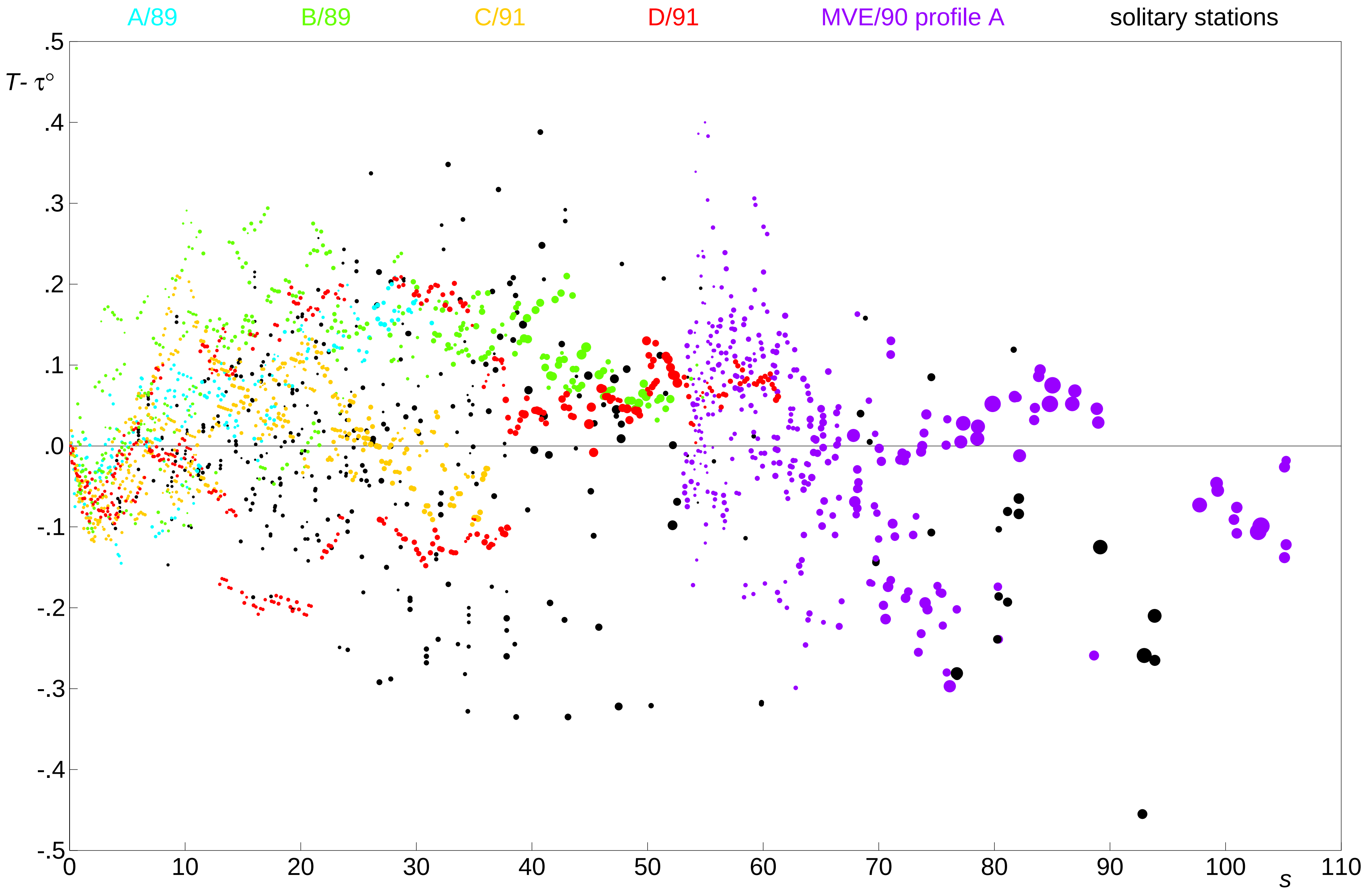

Figure 2. Deviations of field travel times from the reference travel time curve.

5.3 Medium correlation function

The reference length unit of $L = 1\mathrm{km}$ is used.

First we attempted to find a good value of numerical parameter σ_{err} selecting the set of field travel times measured with a sufficient accuracy, see (58), The following table shows the dependence of object function $y(\beta)$ and number $\sum_K 1$ of used field travel times on the selection of σ_{err}, for $q = 0.90$, $\beta_0 - d = -0.20$, and $\beta - d = -0.22$:

σ_{err}	$\sum_K 1$	$y(\beta)$
0.0008	1091	0.750
0.0009	1266	0.849
0.0010	1406	0.999
0.0011	1489	1.037
0.0012	1549	1.054
0.0013	1602	1.061
0.0014	1657	1.078
0.0015	1698	1.086
0.0016 $\leftarrow$	1728	1.084
0.0017	1759	1.095
0.0018	1792	1.138
0.0019	1854	1.222
0.0020	1881	1.274
0.0025	1980	1.719
0.0030	2054	2.007

Here the value of $\sigma_0 = 0.011330\,\mathrm{s\,km^{-1}}$ has been determined at $\sigma_{\mathrm{err}} = 0.001558\,\mathrm{s\,km^{-1}}$ and has been kept fixed when evaluating $y(\beta)$ for different σ_{err}. The value of the object function is relatively stable for $0.0010\,\mathrm{s\,km^{-1}} \le \sigma_{\mathrm{err}} \le 0.0017\,\mathrm{s\,km^{-1}}$ and increases considerably for larger σ_{err}. Such an increase probably indicates influence of bad travel–time data. The author has chosen $\sigma_{\mathrm{err}} = 0.001558\,\mathrm{s\,km^{-1}}$ for the subsequent calculations.

The next task is to select a reasonable value of the other numerical parameter q, selecting the couples of field travel times according to (34). Unfortunately, the position $\beta = \beta_{\mathrm{min}}$ of the minimum of the object function $y(\beta)$ is considerably influenced by the choice of numerical parameter q. Here is the table showing the dependence of the minimum of object function $y(\beta)$ on the selection of q, for $\sigma_{\mathrm{err}} = 0.001558\,\mathrm{s\,km^{-1}}$ and σ_0 corresponding to particular β_0.

q	$\beta_0 - d$	$\beta_{\mathrm{min}} - d$	$\sigma(\beta_{\mathrm{min}})$	$y(\beta_{\mathrm{min}})$
0.50	-0.48	-0.48	0.0092	0.805
0.60	-0.44	-0.46	0.0092	0.850
0.70	-0.36	-0.42	0.0093	0.905
0.75	-0.38	-0.40	0.0094	0.921
0.80	-0.28	-0.34	0.0097	1.006
0.85	-0.24	-0.30	0.0100	1.053
0.90 $\leftarrow$	-0.20	-0.24	0.0106	1.099
0.95	-0.20	-0.18	0.0117	1.121
0.98	-0.20	-0.16	0.0122	1.196

There are at least 3 different drawbacks of small values of q:
(a) For decreasing q, inaccurate reference travel–time curve $\tau^0(s)$ may begin to considerably influence the results.
(b) The number of differences $T_K/\tau^0_K - T_L/\tau^0_L$ of relative travel times is much greater then the number of field travel times T_K, whereas we treat the differences as independent data in objective function (50). Such a processing need not be correct from the statistical point of view and may get worse for smaller values of q.
(c) Here we substituted curved rays with straight ones. It is likely a reasonable approximation for rays of similar lengths, but for rays of different lengths the straight approximations of rays may be much closer together than the correct rays, separated in depth, are. For small q some of covariances (25) may thus be evaluated greater than correct which may result in the compensation by smaller values of $\beta - d$.

On the other hand, the drawback of q approaching to 1 consists in exclusion of field travel times not surrounded by other travel times, and consequently in considerable limitation of the amount of available information.

The author thus inclines to the values of $\beta - d$ close to the value obtained for $q = 0.90$. However, the tomographic inversion of the travel times should be performed with several different values of $\beta - d$, and the dependence of the resulting models on the uncertainty in $\beta - d$ should be studied.

E.g., for $q = 0.90$ we have got $\beta - d = -0.24$ [corresponding to the fractal dimension of $D = 3.12$, see (3)], and $\sigma = 0.0106\,\mathrm{s\,km}^{-1}$. The medium correlation function (17) then reads

$$C(\mathbf{x}_1, \mathbf{x}_2) \approx (0.0106\,\mathrm{s\,km}^{-1})^2 \left(\frac{|\mathbf{x}_1 - \mathbf{x}_2|}{\mathrm{km}}\right)^{-0.24} \quad , \tag{66}$$

and a priori standard deviations of travel times from the mean travel–time curve are

$$\sqrt{S_{KK}} = \sigma\ L\ \sqrt{\tfrac{2}{(1+\beta-d)(2+\beta-d)}}\ \left(\frac{|\mathbf{x}_1 - \mathbf{x}_2|}{L}\right)^{1+\frac{\beta-d}{2}} \approx 0.013\,\mathrm{s}\ \left(\frac{s_K}{\mathrm{km}}\right)^{0.88} \quad , \tag{67}$$

see (30). Let us emphasize that these standard deviations also describe the accuracy of the synthetic travel times in the hypothetical best 1-D model of the 3-D geological medium under Western Bohemia, and may, e.g., be used to estimate the accuracy of kinematic hypocentre determination in such a 1-D model.

References

Bucha, V., Klimeš, L., Dvořák, V. & Sýkorová, Z. (1992): Refraction 3-D seismic measurements in Western Bohemia. Proc. XXIII General Assembly Eur. Seismol. Comm., Geoph. Inst., Czechosl. Acad. Sci., Praha, pp. 155–158.

Crossley, D.J. & Jensen, O.G. (1989): Fractal velocity models in refraction seismology. Pageoph, **131**, 61–76.

Turcotte, D.L. (1989): Fractals in geology and geophysics. Pageoph, **131**, 171–196.