Relation between the propagator matrix of geodesic deviation and second-order derivatives of characteristic function

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Summary
The Hamiltonian formulation is used to derive the relation between the propagator matrix of geodesic deviation and the second–order spatial derivatives of characteristic function in the Finsler geometry.

Keywords
Finsler geometry, Riemann geometry, geodesic deviation, characteristic function.

1. Introduction
In the Riemann geometry, the metric tensor depends on coordinates. In the Finsler geometry, which is a generalization of the Riemann geometry, the metric tensor depends also on the direction of propagation. All equations applicable to the Finsler space are equally applicable to the Riemann space. Seismic, electromagnetic and other waves in high–frequency approximation generally propagate according to the Finsler geometry.

Basics of the Finsler geometry were formulated by Sir William Rowan Hamilton in 1832 (Hamilton, 1837). Hamilton's formulation is based on the Hamilton–Jacobi equations for the distance between points, called the characteristic function, and is summarized in Section 1.1. On the other hand, Finsler's (1918) formulation is based on the differential geometry. Relation between Finsler's formulation and Hamilton's formulation is well known, but the simplicity and power of Hamilton's formulation is too much underestimated. The characteristic function and geodesics in Finsler spaces can efficiently be calculated by Hamilton's method. Hamiltonian formulation simplifies many equations of the Riemann geometry. In Hamiltonian formulation, the Finsler geometry of geodesics is no more difficult than the Riemann geometry.

The non–linear ordinary differential equations of geodesics may serve for calculating geodesics, the characteristic function (two–point distance, two–point travel time) with its first–order spatial derivatives, or the action (general distance, general travel time) with its first–order spatial derivatives. The linear ordinary differential equations of geodesic deviation may serve for calculating geodesic deviations, amplitudes of waves, the second–order spatial derivatives of characteristic function, or the second–order spatial derivatives of the action. Geodesic perturbations, higher–order geodesic deviations, and the perturbation derivatives and higher–order spatial derivatives of characteristic function or action can be calculated by quadratures along geodesics.

In this paper, we use the Hamiltonian formulation to derive the relation between the propagator matrix of geodesic deviation and the second–order spatial derivatives of characteristic function in the Finsler geometry. The derived equations are, naturally, applicable to the Riemann spaces which are special cases of the Finsler spaces.

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1.1. Hamilton's (1837) formulation

Hamilton's (1837) formulation is based on the first–order partial differential equations for the distance between two points (*characteristic function*) or for a general distance (*action*). These equations for the distance are now called the *Hamilton–Jacobi equations.*

Hamilton (1837) defined the *figuratrix* (*surface of components of normal slowness*), and the *indicatrix* (*Fresnel wave surface*).

Hamilton (1837) derived the *equations of geodesics* (*general equations of rays*, now called the *Hamilton's equations*).

Hamilton (1837) introduced the covariant vector of the first–order partial derivatives of characteristic function with respect to spatial coordinates (*normal slowness*, now often called the *slowness vector*), which results from the solution of the equations of geodesics. He then derived the equations for transformation of this slowness vector at the surfaces of discontinuity of the metric (Snell's law for Finsler spaces).

Hamilton (1837) also introduced the second–order partial derivatives of characteristic function with respect to spatial coordinates, studied their properties, and derived the equations for transformation of these second–order derivatives at the surfaces of discontinuity of the metric.

The most important equations missing in the Hamilton's (1837) paper are the *equations of geodesic deviation.* The Hamiltonian equations of geodesic deviation were derived by Červený (1972).

2. Hamiltonian formulation of the Finsler geometry

2.1. Hamiltonian function and the metric tensor

We consider Hamiltonian function $H(x^i, y_j)$, which is a function of coordinates x^i and of covariant vector y_j from the cotangent space.

Hamilton (1837) assumed the Hamiltonian function to be homogeneous function $H^{(1)}(x^i, y_j)$ of the first degree with respect to y_j, which represents the quasi–norm of covariant vector y_j at point x^i,

$$H^{(1)}(x^i, y_j) = \|y_r\| \quad . \tag{1}$$

The standard Riemannian or Finslerian equations of geodesics and of geodesic deviation correspond to homogeneous Hamiltonian function $H^{(2)}(x^i, y_j)$ of the second degree with respect to y_j. In this paper, we shall thus concentrate on homogeneous Hamiltonian function $H^{(2)}(x^i, y_j)$ of the second degree with respect to y_j, which represents half the square of the quasi–norm of covariant vector y_j at point x^i,

$$H^{(2)}(x^i, y_j) = \tfrac{1}{2}\|y_r\|^2 \quad . \tag{2}$$

It is obvious that the quasi–norm need not be real–valued and need not be continuous for all covariant vectors y_j. For example, the quasi–norm is not continuous at the light cone in space–time.

In general, the Hamiltonian function need not be homogeneous and may be represented by any reasonably smooth function of x^i and y_j. Then the positive quasi–norm $\|y_r\|_C$ of covariant vector y_j at point x^i may be defined by equation

$$H(x^i, \frac{y_k}{\|y_r\|_C}) = C \quad . \tag{3}$$

Constant C is determined by the meaning of the Hamiltonian function. If the Hamiltonian function is homogeneous of degree N with respect to y_j, constant

$$C = \tfrac{1}{N} \tag{4}$$

is usually selected. Unlike the norm, this quasi–norm $\|y_r\|_C$ need not exist, need not be unique, need not be real–valued, need not be continuous for all covariant vectors y_j, need not be convex, and need not satisfy relation $\|-y_r\|_C = \|y_r\|_C$.

Note that the subset of the unit vectors y_j in the cotangent space at point x^i, defined by equation

$$H(x^i, y_j) = C \quad , \tag{5}$$

was called the *surface of components of normal slowness* by Hamilton (1837). Now it is often called *phase–slowness surface* or briefly *slowness surface*, sometimes *index surface*. In the Finsler geometry, it is referred to as the *figuratrix*.

Using the quasi–norm defined by equation (3), we can define homogeneous Hamiltonian function

$$H^{(2)}(x^i, y_j) = \tfrac{1}{2}\|y_r\|_C^2 \tag{6}$$

of the second degree with respect to y_j, which corresponds to general Hamiltonian function $H(x^i, y_j)$. Note that in the Riemann geometry, $H^{(2)}(x^m, y_n) = \frac{1}{2} y_i\, g^{ij}(x^m)\, y_j$.

Contravariant vector y^i corresponding to covariant vector y_j is defined by relation

$$y^i = \frac{\partial H^{(2)}}{\partial y_i}(x^m, y_n) \quad . \tag{7}$$

For example, if covariant vector y_i represents the slowness vector, corresponding contravariant vector y^i represents the velocity vector.

Contravariant components of the metric tensor are defined by equation

$$g^{ij}(x^m, y_n) = \frac{\partial^2 H^{(2)}}{\partial y_i \partial y_j}(x^m, y_n) \tag{8}$$

(Rund, 1959, eq. 1.5.4).

2.2. Action, characteristic function and the Hamilton–Jacobi equations

Hamilton–Jacobi equation for *action* $S(x^m)$ reads

$$H\big(x^i, \tfrac{\partial S}{\partial x^j}(x^m)\big) = C \quad . \tag{9}$$

Hamilton (1837) defined also the *characteristic function* (*two–point distance*, *two–point travel time*)

$$V(x^m, \tilde{x}^n) \tag{10}$$

from point $\tilde{x}^m$ to point x^n. The characteristic function satisfies Hamilton–Jacobi equations

$$H\big(x^m, \tfrac{\partial V}{\partial x^n}(x^a, \tilde{x}^b)\big) = C \tag{11}$$

and

$$H\big(\tilde{x}^m, -\tfrac{\partial V}{\partial \tilde{x}^n}(x^a, \tilde{x}^b)\big) = C \quad . \tag{12}$$

(Hamilton, 1837, eq. C). For homogeneous Hamiltonian function $H = H^{(2)}$ of the second degree, $C = \frac{1}{2}$. The Hamilton–Jacobi equations express the requirement that the gradient of action or of characteristic function is unit.

2.3. Equations of geodesics

Hamilton's equations (*equations of geodesic*) read

$$\frac{\mathrm{d}}{\mathrm{d}\tau} x^i = \frac{\partial H}{\partial y_i}(x^m, y_n) \quad , \tag{13}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau} y_i = -\frac{\partial H}{\partial x^i}(x^m, y_n) \quad . \tag{14}$$

These equations can simply be derived by differentiating Hamilton–Jacobi equation (9) or (11) with respect to coordinate x^j. The meaning of independent parameter τ along the geodesic and the sensitivity to initial conditions depend on the form of the Hamiltonian function.

For the homogeneous Hamiltonian function $H = H^{(2)}$ of the second degree, Hamilton's equations (13)–(14) are equivalent to the Riemannian or Finslerian equations

$$\frac{\mathrm{d}}{\mathrm{d}\tau} x^i = g^{ij}\, y_j \quad , \tag{15}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau} y_i = \Gamma^j_{ik}\, y_j\, g^{kl}\, y_l \tag{16}$$

of geodesic (Rund, 1959, eqs. 1.5.8, 2.2.5). Here Γ^j_{ik} represents the Christoffel symbols of the first kind (Rund, 1959, eqs. 2.2.3, 2.2.7). For a general Hamiltonian function, equations (13)–(14) and (15)–(16) differ just by independent parameter τ along the geodesic and by the sensitivity to initial conditions.

2.4. Equations of geodesic deviation

We define vectors

$$X^i = \frac{\partial x^i}{\partial \gamma} \tag{17}$$

and

$$Y_i = \frac{\partial y_i}{\partial \gamma} \tag{18}$$

representing the geodesic deviation corresponding to some parameter γ parametrizing the initial conditions. Since derivatives $\frac{\mathrm{d}}{\mathrm{d}\tau}$ and $\frac{\partial}{\partial\gamma}$ commute, the equations for X^i and Y_i are obtained by differentiating Hamilton's equations (13)–(14) with respect to γ. Resulting *Hamiltonian equations of geodesic deviation* (*paraxial ray equations, dynamic ray tracing equations*) read (Červený, 1972)

$$\frac{\mathrm{d}}{\mathrm{d}\tau} X^i = \quad H^{,i}_{,j}\, X^j + H^{,ij}\, Y_j \quad , \tag{19}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau} Y_i = -H_{,ij}\, X^j - H^{,j}_{,i}\, Y_j \quad , \tag{20}$$

where

$$H_{,ij} = \frac{\partial^2 H}{\partial x^i \partial x^j}(x^m, y_n) \quad , \tag{21}$$

$$H^{,i}_{,j} = \frac{\partial^2 H}{\partial y_i \partial x^j}(x^m, y_n) \quad , \tag{22}$$

$$H^{,ij} = \frac{\partial^2 H}{\partial y_i \partial y_j}(x^m, y_n) \quad . \tag{23}$$

For the homogeneous Hamiltonian function $H = H^{(2)}$ of the second degree, Hamiltonian equations (19)–(20) of geodesic deviation are equivalent to the Riemannian or Finslerian equations

$$\frac{\delta}{\delta\tau} X^i = g^{ij}\, \tilde{Y}_j \quad , \tag{24}$$

$$\frac{\delta}{\delta\tau} \tilde{Y}_i = -R_{irsj}\, y^r y^s X^j \tag{25}$$

of geodesic deviation (Rund, 1959, eq. 4.4.16). Here R_{isrj} represents one of the relevant curvature tensors (Rund, 1959, eqs. 4.2.12 or 4.2.15), and

$$\tilde{Y}_i = Y_i - \Gamma^s_{ij}\, y_s\, X^j \quad , \tag{26}$$

$$\frac{\delta}{\delta\tau} X^i = \frac{\mathrm{d}}{\mathrm{d}\tau} X^i + \Gamma^i_{js}\, y^s X^j \quad , \tag{27}$$

$$\frac{\delta}{\delta\tau} \tilde{Y}_i = \frac{\mathrm{d}}{\mathrm{d}\tau} \tilde{Y}_i - \Gamma^j_{is}\, y^s\, \tilde{Y}_j \quad . \tag{28}$$

For a general Hamiltonian function, equations (19)–(20) and (24)–(25) differ.

2.5. Propagator matrix of geodesic deviation

The propagator matrix of geodesic deviation from point $\tilde{x}^a$ to point x^a is defined by equation

$$\mathbf{\Pi}(x^a, \tilde{x}^b) = \begin{pmatrix} \frac{\partial x^i}{\partial \tilde{x}^j} & \frac{\partial x^i}{\partial \tilde{y}_j} \\ \frac{\partial y_i}{\partial \tilde{x}^j} & \frac{\partial y_i}{\partial \tilde{y}_j} \end{pmatrix} \quad , \tag{29}$$

where the derivatives are taken at fixed parameter τ along geodesics. The propagator matrix of geodesic deviation is symplectic and obeys the chain rule,

$$\mathbf{\Pi}(x^a, \tilde{x}^c) = \mathbf{\Pi}(x^a, \overline{x}^b)\, \mathbf{\Pi}(\overline{x}^c, \tilde{x}^d) \quad . \tag{30}$$

The propagator matrix of geodesic deviation contains all linearly independent solutions of the equations of geodesic deviation, and may thus be used for calculating the geodesic deviation for any initial conditions. The amplitude of the Green function is inversely proportional to the square root of the determinant of submatrix $\frac{\partial x^i}{\partial \tilde{y}_j}$, and the whole propagator matrix of geodesic deviation may be used to calculate the amplitude for any initial conditions. The relation between the propagator matrix of geodesic deviation and the second–order spatial derivatives of characteristic function is proposed in Section 3.2.

Hamiltonian equations (19)–(20) of geodesic deviation for the propagator matrix read

$$\tfrac{\mathrm{d}}{\mathrm{d}\tau} \mathbf{\Pi}(x^a, \tilde{x}^b) = \begin{pmatrix} H^{,i}_{,j} & H^{,ij} \\ -H_{,ij} & -H^{,j}_{,i} \end{pmatrix} \mathbf{\Pi}(x^a, \tilde{x}^b) \quad , \tag{31}$$

with unit initial conditions. Note that it is also possible to define the covariant version of the propagator matrix of geodesic deviation, but we do not need it here.

3. Derivatives of characteristic function

3.1. First–order spatial derivatives of characteristic function

The first–order spatial derivatives

$$\frac{\partial V}{\partial x^i} = y_i \quad , \tag{32}$$

$$\frac{\partial V}{\partial \tilde{x}^i} = -\tilde{y}_i \tag{33}$$

of characteristic function result from the solution of Hamilton's equations (13)–(14) (Hamilton, 1837).

3.2. Relation between the propagator matrix of geodesic deviation and the second–order spatial derivatives of characteristic function

Linear ordinary differential Hamiltonian equations (31) of geodesic deviation can be used to calculate the propagator matrix (29) of geodesic deviation. Then the second–order spatial derivatives of characteristic function can be obtained from the propagator matrix (29) of geodesic deviation.

The relations between the second–order spatial derivatives of characteristic function and the propagator matrix of geodesic deviation can be derived in the following way: The relations between the second–order spatial derivatives of characteristic function and the propagator matrix of geodesic deviation in ray–centred coordinates in isotropic media by Červený, Klimeš & Pšenčík (1984) can simply be generalized to anisotropic media, i.e., to the Finsler geometry. Then these relations can be transformed to general coordinates using the transformation equations by Klimeš (1994).

Finslerian equations of geodesic deviation correspond to a homogeneous Hamiltonian function of degree 2. For the homogeneous Hamiltonian function $H = H^{(2)}$ of the second degree, the derived unique relations between the second–order spatial derivatives of characteristic function (10) and the propagator matrix (29) of geodesic deviation read

$$\left(\frac{\partial^2 V}{\partial x^i \partial x^j} + \frac{1}{V}\frac{\partial V}{\partial x^i}\frac{\partial V}{\partial x^j}\right)\frac{\partial x^j}{\partial \tilde{y}_k} = \frac{\partial y_i}{\partial \tilde{y}_k} \quad , \tag{34}$$

$$\left(\frac{\partial^2 V}{\partial \tilde{x}^i \partial x^j} + \frac{1}{V}\frac{\partial V}{\partial \tilde{x}^i}\frac{\partial V}{\partial x^j}\right)\frac{\partial x^j}{\partial \tilde{y}_k} = -\delta_i^k \quad , \tag{35}$$

$$\frac{\partial x^i}{\partial \tilde{y}_j}\left(\frac{\partial^2 V}{\partial \tilde{x}^j \partial \tilde{x}^k} + \frac{1}{V}\frac{\partial V}{\partial \tilde{x}^j}\frac{\partial V}{\partial \tilde{x}^k}\right) = \frac{\partial x^i}{\partial \tilde{x}^k} \quad , \tag{36}$$

where Kronecker delta δ_i^k represents the components of the identity matrix. These relations are proved in Section 4.

Only three submatrices of propagator matrix (29) of geodesic deviation are used in equations (34)–(36). Note that the fourth submatrix of matrix (29) carry no additional information, it can be calculated from the three submatrices used in equations (34)–(36) thanks to the symplectic property of propagator matrix (29) of geodesic deviation.

3.3. Perturbation derivatives or third–order and higher–order spatial derivatives of characteristic function

The third–order and higher–order spatial derivatives and all perturbation derivatives of the action in Finsler or Riemann spaces can be calculated by quadratures along geodesics (Klimeš, 2002).

The equations for the analogous derivatives of characteristic function in Finsler spaces and the equations for transformation of these derivatives at the surfaces of discontinuity of the metric are still to be derived.

4. Proof of the relation between the propagator matrix of geodesic deviation and the second–order spatial derivatives of characteristic function

We first calculate the limits of equations (34)–(36) at the initial point of a geodesic, i.e., for point x^a approaching the initial point $\tilde{x}^a$ of the geodesic, and show that equations (34)–(36) hold at the initial point.

We then assume that equations (34)–(36) are satisfied at point x^a of the geodesic, calculate the derivatives of equations (34)–(36) along the geodesic at point x^a, and show that these derivatives hold. In this way, we prove that equations (34)–(36) hold along the whole geodesic.

Equations (34)–(36) hold for homogeneous Hamiltonian function $H^{(2)}(x^i, y_j)$ of the second degree with respect to y_j only. The homogeneous Hamiltonian function of the second degree is thus used hereinafter. In order to simplify the equations, we shall denote it by abbreviation $H(x^i, y_j)$.

4.1. Limit of the relations at the initial point of a geodesic

For small distances between $\tilde{x}^a$ and x^a, the characteristic function can be approximated using the fundamental function $F(x^i, y^j)$ as

$$V(x^b, \tilde{x}^a) = F(\tilde{x}^m, x^n - \tilde{x}^n) + O(V^2) \tag{37}$$

(Rund, 1959, eq. 1.1.14). Note that the fundamental function $F(x^i, y^j)$ specifies the length of infinitesimally small vectorial distance y^j, and is thus homogeneous of the first degree with respect to y^j. The approximate velocity vector along the geodesic from $\tilde{x}^a$ to x^a reads

$$y^i = \frac{x^i - \tilde{x}^i}{F(\tilde{x}^m, x^n - \tilde{x}^n)} \quad . \tag{38}$$

The slowness vector corresponding to this velocity vector is

$$y_i(\tilde{x}^m, y^n) = \frac{1}{2}\frac{\partial F^2}{\partial y^i}(\tilde{x}^m, y^n) \tag{39}$$

(Rund, 1959, eq. 1.5.6), and the corresponding covariant metric tensor is defined by relation

$$g_{ij}(\tilde{x}^m, y^n) = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}(\tilde{x}^m, y^n) \tag{40}$$

(Rund, 1959, eq. 1.3.1). Inserting (38) into the right–hand side of (40), and considering the homogeneity of the fundamental function $F(x^i, y^j)$ and of its derivatives with respect to y^j, we obtain

$$g_{ij}(\tilde{x}^m, y^n) = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}(\tilde{x}^m, x^n - \tilde{x}^n) \quad . \tag{41}$$

Since the fundamental function F appears in (41) in its square, we express approximation (37) in form

$$V(x^b,\tilde{x}^a) = \sqrt{F^2(\tilde{x}^m, x^n-\tilde{x}^n)} + O(V^2) \quad . \tag{42}$$

We now differentiate approximation (42) once,

$$\frac{\partial V}{\partial x^i}(x^b,\tilde{x}^a) = \left[\frac{1}{2\sqrt{F^2}}\frac{\partial F^2}{\partial y^i}\right](\tilde{x}^m, x^n-\tilde{x}^n) + O(V^1) \quad , \tag{43}$$

$$\frac{\partial V}{\partial \tilde{x}^i}(x^b,\tilde{x}^a) = -\left[\frac{1}{2\sqrt{F^2}}\frac{\partial F^2}{\partial y^i}\right](\tilde{x}^m, x^n-\tilde{x}^n) + O(V^1) \quad , \tag{44}$$

and twice,

$$\frac{\partial^2 V}{\partial x^i \partial x^j}(x^b,\tilde{x}^a) = \left[\frac{1}{2\sqrt{F^2}}\frac{\partial^2 F^2}{\partial y^i \partial y^j} - \frac{1}{4\left[\sqrt{F^2}\right]^3}\frac{\partial F^2}{\partial y^i}\frac{\partial F^2}{\partial y^j}\right](\tilde{x}^m, x^n-\tilde{x}^n) + O(V^0) \quad , \tag{45}$$

$$\frac{\partial^2 V}{\partial \tilde{x}^i \partial x^j}(x^b,\tilde{x}^a) = -\left[\frac{1}{2\sqrt{F^2}}\frac{\partial^2 F^2}{\partial y^i \partial y^j} - \frac{1}{4\left[\sqrt{F^2}\right]^3}\frac{\partial F^2}{\partial y^i}\frac{\partial F^2}{\partial y^j}\right](\tilde{x}^m, x^n-\tilde{x}^n) + O(V^0) \quad , \tag{46}$$

$$\frac{\partial^2 V}{\partial \tilde{x}^i \partial \tilde{x}^j}(x^b,\tilde{x}^a) = \left[\frac{1}{2\sqrt{F^2}}\frac{\partial^2 F^2}{\partial y^i \partial y^j} - \frac{1}{4\left[\sqrt{F^2}\right]^3}\frac{\partial F^2}{\partial y^i}\frac{\partial F^2}{\partial y^j}\right](\tilde{x}^m, x^n-\tilde{x}^n) + O(V^0) \quad . \tag{47}$$

Inserting relations (41), (42), (43) and (44) into (45)–(47), we arrive at

$$\frac{\partial^2 V}{\partial x^i \partial x^j}(x^b,\tilde{x}^a) = \frac{1}{V(x^b,\tilde{x}^a)}\left[g_{ij}(\tilde{x}^m,y^n) - \frac{\partial V}{\partial x^i}(x^b,\tilde{x}^a)\frac{\partial V}{\partial x^j}(x^d,\tilde{x}^c)\right] + O(V^0) \ , \tag{48}$$

$$\frac{\partial^2 V}{\partial \tilde{x}^i \partial x^j}(x^b,\tilde{x}^a) = \frac{1}{V(x^b,\tilde{x}^a)}\left[-g_{ij}(\tilde{x}^m,y^n) - \frac{\partial V}{\partial \tilde{x}^i}(x^b,\tilde{x}^a)\frac{\partial V}{\partial x^j}(x^d,\tilde{x}^c)\right] + O(V^0) \ , \tag{49}$$

$$\frac{\partial^2 V}{\partial \tilde{x}^i \partial \tilde{x}^j}(x^b,\tilde{x}^a) = \frac{1}{V(x^b,\tilde{x}^a)}\left[g_{ij}(\tilde{x}^m,y^n) - \frac{\partial V}{\partial \tilde{x}^i}(x^b,\tilde{x}^a)\frac{\partial V}{\partial \tilde{x}^j}(x^d,\tilde{x}^c)\right] + O(V^0) \ . \tag{50}$$

From the Hamiltonian equations (31) of geodesic deviation, we see that

$$\begin{pmatrix} \frac{\partial x^i}{\partial \tilde{x}^j} & \frac{\partial x^i}{\partial \tilde{y}_j} \\ \frac{\partial y_i}{\partial \tilde{x}^j} & \frac{\partial y_i}{\partial \tilde{y}_j} \end{pmatrix} = \begin{pmatrix} \delta^i_j + \frac{\partial^2 H}{\partial y_i \partial x^j}V & \frac{\partial^2 H}{\partial y_i \partial y_j}V \\ -\frac{\partial^2 H}{\partial x^i \partial x^j}V & \delta^j_i - \frac{\partial^2 H}{\partial x^i \partial y_j}V \end{pmatrix} + O(V^2) \quad . \tag{51}$$

Inserting (8) into (51), we express

$$\frac{\partial x^i}{\partial \tilde{y}_j} = V\, g^{ij} + O(V^2) \quad . \tag{52}$$

We multiply (48)–(50) with (52) and obtain approximations

$$\left(\frac{\partial^2 V}{\partial x^i \partial x^j} + \frac{1}{V}\frac{\partial V}{\partial x^i}\frac{\partial V}{\partial x^j}\right)\frac{\partial x^j}{\partial \tilde{y}_k} = \delta^k_i + O(V^1) \quad , \tag{53}$$

$$\left(\frac{\partial^2 V}{\partial \tilde{x}^i \partial x^j} + \frac{1}{V}\frac{\partial V}{\partial \tilde{x}^i}\frac{\partial V}{\partial x^j}\right)\frac{\partial x^j}{\partial \tilde{y}_k} = -\delta^k_i + O(V^1) \quad , \tag{54}$$

$$\left(\frac{\partial^2 V}{\partial \tilde{x}^i \partial \tilde{x}^j} + \frac{1}{V}\frac{\partial V}{\partial \tilde{x}^i}\frac{\partial V}{\partial \tilde{x}^j}\right)\frac{\partial x^k}{\partial \tilde{y}_j} = \delta^k_i + O(V^1) \quad . \tag{55}$$

From (51) and (53)–(55), we see that equations (34)–(36) are satisfied for small distances between $\tilde{x}^a$ and x^a with the accuracy of $O(V^1)$, and are thus satisfied for $x^a \to \tilde{x}^a$.

4.2. Differentiating the Hamilton–Jacobi equations

Hereinafter, a subscript following a comma denotes the partial derivative with respect to coordinate x^i, e.g., $H_{,i} = \frac{\partial H}{\partial x^i}$ or $V_{,i} = \frac{\partial V}{\partial x^i}$. A subscript with a tilde following a comma denotes the partial derivative with respect to initial coordinate $\tilde{x}^a$, e.g., $V_{,\tilde{a}} = \frac{\partial V}{\partial \tilde{x}^a}$. A superscript following a comma denotes the partial derivative with respect to slowness vector component y_i, e.g., $H^{,i} = \frac{\partial H}{\partial y_i}$.

We differentiate Hamilton–Jacobi equation (11) with respect to x^i and obtain equation

$$H_{,i} + H^{,k} V_{,ki} = 0 \tag{56}$$

(Hamilton, 1837, eqs. Q, I, K). We differentiate Hamilton–Jacobi equation (11) with respect to $\tilde{x}^a$ and obtain equation

$$H^{,k} V_{,k\tilde{a}} = 0 \tag{57}$$

(Hamilton, 1837, eqs. U, I).

We differentiate equation (56) with respect to x^j and obtain equation

$$H_{,ij} + H_{,i}^{,m} V_{,mj} + V_{,im} H_{,j}^{,m} + V_{,im} H^{,mn} V_{,nj} + H^{,k} V_{,kij} = 0 \quad . \tag{58}$$

We differentiate equation (56) with respect to $\tilde{x}^a$ or equation (57) with respect to x^j and obtain equation

$$V_{,\tilde{a}m} H_{,j}^{,m} + V_{,\tilde{a}m} H^{,mn} V_{,nj} + H^{,k} V_{,k\tilde{b}j} = 0 \quad . \tag{59}$$

We differentiate equation (57) with respect to $\tilde{x}^b$ and obtain equation

$$V_{,\tilde{a}m} H^{,mn} V_{,n\tilde{b}} + H^{,k} V_{,k\tilde{a}\tilde{b}} = 0 \quad . \tag{60}$$

4.3. Derivatives along geodesics

Equation (13) yields

$$\frac{\mathrm{d}}{\mathrm{d}\tau} = H^{,k} \frac{\partial}{\partial x^k} \quad . \tag{61}$$

We apply equation (61) to V and obtain

$$\frac{\mathrm{d}}{\mathrm{d}\tau} V = H^{,k} V_{,k} \quad . \tag{62}$$

Because of homogeneity of Hamiltonian function H, equation (62) reads

$$\frac{\mathrm{d}}{\mathrm{d}\tau} V = 1 \quad . \tag{63}$$

Equation (56) with (61) yields

$$\frac{\mathrm{d}}{\mathrm{d}\tau} V_{,i} = -H_{,i} \quad . \tag{64}$$

Equation (57) with (61) yields

$$\frac{\mathrm{d}}{\mathrm{d}\tau} V_{,\tilde{a}} = 0 \quad . \tag{65}$$

Equation (58) with (61) yields Riccati equation

$$\frac{\mathrm{d}}{\mathrm{d}\tau} V_{,ij} = -H_{,ij} - H_{,i}^{,m} V_{,mj} - V_{,im} H_{,j}^{,m} - V_{,im} H^{,mn} V_{,nj} \tag{66}$$

for $V_{,ij}$. Equation (59) with (61) yields equation

$$\frac{\mathrm{d}}{\mathrm{d}\tau} V_{,\tilde{a}j} = -V_{,\tilde{a}m} H_{,j}^{,m} - V_{,\tilde{a}m} H^{,mn} V_{,nj} \quad , \tag{67}$$

which represents, for given $V_{,ij}$, linear ordinary differential equation for $V_{,\tilde{a}j}$. Equation (60) with (61) yields expression

$$\frac{\mathrm{d}}{\mathrm{d}\tau} V_{,\tilde{a}\tilde{b}} = -V_{,\tilde{a}m} H^{,mn} V_{,n\tilde{b}} \tag{68}$$

for the derivative of $V_{,\tilde{a}\tilde{b}}$ along the geodesic in terms of $V_{,\tilde{a}j}$. Equations (31) read

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial x^i}{\partial \tilde{x}^{\tilde{c}}} = H_{,k}^{,i}\frac{\partial x^k}{\partial \tilde{x}^{\tilde{c}}} + H^{,ik}\frac{\partial y_k}{\partial \tilde{x}^{\tilde{c}}} \quad , \tag{69}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial x^i}{\partial \tilde{y}_{\tilde{c}}} = H_{,k}^{,i}\frac{\partial x^k}{\partial \tilde{y}_{\tilde{c}}} + H^{,ik}\frac{\partial y_k}{\partial \tilde{y}_{\tilde{c}}} \quad , \tag{70}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial y_i}{\partial \tilde{x}^{\tilde{c}}} = -H_{,ik}\frac{\partial x^k}{\partial \tilde{x}^{\tilde{c}}} - H_{,i}^{,k}\frac{\partial y_k}{\partial \tilde{x}^{\tilde{c}}} \quad , \tag{71}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial y_i}{\partial \tilde{y}_{\tilde{c}}} = -H_{,ik}\frac{\partial x^k}{\partial \tilde{y}_{\tilde{c}}} - H_{,i}^{,k}\frac{\partial y_k}{\partial \tilde{y}_{\tilde{c}}} \quad . \tag{72}$$

4.4. Derivative of (34)

We express the derivative of equation (34) along the geodesic using equations (63), (64), (66), (70) and (72),

$$\begin{aligned}\big[-H_{,ij} - H_{,i}^{,m}V_{,mj} - V_{,im}H_{,j}^{,m} - V_{,im}H^{,mn}V_{,nj} - V^{-1}(H_{,i}V_{,j}+V_{,i}H_{,j}) - V^{-2}V_{,i}V_{,j}\big]\frac{\partial x^j}{\partial \tilde{y}_{\tilde{c}}}&\\ +(V_{,ij}+V^{-1}V_{,i}V_{,j})(H_{,k}^{,j}\frac{\partial x^k}{\partial \tilde{y}_{\tilde{c}}} + H^{,jk}\frac{\partial y_k}{\partial \tilde{y}_{\tilde{c}}}) = -H_{,ik}\frac{\partial x^k}{\partial \tilde{y}_{\tilde{c}}} - H_{,i}^{,k}\frac{\partial y_k}{\partial \tilde{y}_{\tilde{c}}} \quad .&\end{aligned} \tag{73}$$

We consider the homogeneity of Hamiltonian function H,

$$\begin{aligned}\big[-H_{,ij} - H_{,i}^{,m}V_{,mj} - V_{,im}H_{,j}^{,m} - V_{,im}H^{,mn}V_{,nj} - V^{-1}(H_{,i}V_{,j}+V_{,i}H_{,j}) - V^{-2}V_{,i}V_{,j}\big]\frac{\partial x^j}{\partial \tilde{y}_{\tilde{c}}}&\\ +(V_{,ij}H_{,k}^{,j} + 2V^{-1}V_{,i}H_{,k})\frac{\partial x^k}{\partial \tilde{y}_{\tilde{c}}} + (V_{,ij}H^{,jk}+V^{-1}V_{,i}H^{,k})\frac{\partial y_k}{\partial \tilde{y}_{\tilde{c}}} = -H_{,ik}\frac{\partial x^k}{\partial \tilde{y}_{\tilde{c}}} - H_{,i}^{,k}\frac{\partial y_k}{\partial \tilde{y}_{\tilde{c}}} \quad .&\end{aligned} \tag{74}$$

We collect the terms containing $\frac{\partial x^j}{\partial \tilde{y}_{\tilde{c}}}$ and the terms containing $\frac{\partial y_l}{\partial \tilde{y}_{\tilde{c}}}$,

$$\begin{aligned}\big[-H_{,i}^{,m}V_{,mj} - V_{,im}H^{,mn}V_{,nj} - V^{-1}H_{,i}V_{,j} + V^{-1}V_{,i}H_{,j} - V^{-2}V_{,i}V_{,j}\big]\frac{\partial x^j}{\partial \tilde{y}_{\tilde{c}}}&\\ +\big[V_{,im}H^{,mn} + V^{-1}V_{,i}H^{,n} + H_{,i}^{,n}\big]\frac{\partial y_n}{\partial \tilde{y}_{\tilde{c}}} = 0 \quad .&\end{aligned} \tag{75}$$

We insert (34) for $\frac{\partial y_l}{\partial \tilde{y}_{\tilde{c}}}$,

$$\begin{aligned}\big[-H_{,i}^{,m}V_{,mj} - V_{,im}H^{,ml}V_{,lj} - V^{-1}H_{,i}V_{,j} + V^{-1}V_{,i}H_{,j} - V^{-2}V_{,i}V_{,j}&\\ (V_{,im}H^{,mn} + V^{-1}V_{,i}H^{,n} + H_{,i}^{,n})(V_{,nj}+V^{-1}V_{,n}V_{,j})\big]\frac{\partial x^j}{\partial \tilde{y}_{\tilde{c}}} = 0 \quad .&\end{aligned} \tag{76}$$

We consider the homogeneity of Hamiltonian function H,

$$\begin{aligned}\big[-H_{,i}^{,m}V_{,mj} - V_{,im}H^{,mn}V_{,nj} - V^{-1}H_{,i}V_{,j} + V^{-1}V_{,i}H_{,j} - V^{-2}V_{,i}V_{,j}&\\ +V_{,im}H^{,mn}V_{,nj} + V^{-1}V_{,i}H^{,m}V_{,mj} + H_{,i}^{,m}V_{,mj}&\\ +V^{-1}V_{,im}H^{,m}V_{,j} + V^{-2}V_{,i}V_{,j} + 2V^{-1}H_{,i}V_{,j}\big]\frac{\partial x^j}{\partial \tilde{y}_{\tilde{c}}} = 0 \quad .&\end{aligned} \tag{77}$$

We now execute summation in (77) and arrive at

$$\big[V^{-1}H_{,i}V_{,j} + V^{-1}V_{,i}H_{,j} + V^{-1}V_{,i}H^{,m}V_{,mj} + V^{-1}V_{,im}H^{,m}V_{,j}\big]\frac{\partial x^j}{\partial \tilde{y}_{\tilde{c}}} = 0 \quad . \tag{78}$$

This equation holds in consequence of equation (56).

4.5. Derivative of (35)

We express the derivative of equation (35) along the geodesic using equations (63), (64), (65), (67) and (70),

$$\begin{aligned}\big(-V_{,\tilde{a}m}H^{;m}_{,j} - V_{,\tilde{a}m}H^{,mn}V_{,nj} - V^{-1}V_{,\tilde{a}}H_{,j} - V^{-2}V_{,\tilde{a}}V_{,j}\big)\tfrac{\partial x^j}{\partial \tilde{y}_{\tilde{c}}}\\ +\big(V_{,\tilde{a}j} + V^{-1}V_{,\tilde{a}}V_{,j}\big)\big(H^{;j}_{,l}\tfrac{\partial x^l}{\partial \tilde{y}_{\tilde{c}}} + H^{,jl}\tfrac{\partial y_l}{\partial \tilde{y}_{\tilde{c}}}\big) = 0 \quad .\end{aligned} \tag{79}$$

We consider the homogeneity of Hamiltonian function H,

$$\begin{aligned}\big(-V_{,\tilde{a}m}H^{;m}_{,j} - V_{,\tilde{a}m}H^{,mn}V_{,nj} - V^{-1}V_{,\tilde{a}}H_{,j} - V^{-2}V_{,\tilde{a}}V_{,j}\big)\tfrac{\partial x^j}{\partial \tilde{y}_{\tilde{c}}}\\ +\big(V_{,\tilde{a}j}H^{;j}_{,l} + 2V^{-1}V_{,\tilde{a}}H_{,l}\big)\tfrac{\partial x^l}{\partial \tilde{y}_{\tilde{c}}} + \big(V_{,\tilde{a}j}H^{,jl} + V^{-1}V_{,\tilde{a}}H^{,l}\big)\tfrac{\partial y_l}{\partial \tilde{y}_{\tilde{c}}} = 0 \quad .\end{aligned} \tag{80}$$

We collect the terms containing $\frac{\partial x^j}{\partial \tilde{y}_{\tilde{c}}}$ and the terms containing $\frac{\partial y_l}{\partial \tilde{y}_{\tilde{c}}}$,

$$\begin{aligned}\big(-V_{,\tilde{a}m}H^{,mn}V_{,nj} + V^{-1}V_{,\tilde{a}}H_{,j} - V^{-2}V_{,\tilde{a}}V_{,j}\big)\tfrac{\partial x^j}{\partial \tilde{y}_{\tilde{c}}}\\ +\big(V_{,\tilde{a}m}H^{,mn} + V^{-1}V_{,\tilde{a}}H^{,n}\big)\tfrac{\partial y_n}{\partial \tilde{y}_{\tilde{c}}} = 0 \quad .\end{aligned} \tag{81}$$

We insert (34) for $\frac{\partial y_n}{\partial \tilde{y}_{\tilde{c}}}$,

$$\begin{aligned}\big[-V_{,\tilde{a}m}H^{,mn}V_{,nj} + V^{-1}V_{,\tilde{a}}H_{,j} - V^{-2}V_{,\tilde{a}}V_{,j}\\ +\left(V_{,\tilde{a}m}H^{,mn} + V^{-1}V_{,\tilde{a}}H^{,n}\right)\left(V_{,nj} + V^{-1}V_{,n}V_{,j}\right)\big]\tfrac{\partial x^j}{\partial \tilde{y}_{\tilde{c}}} = 0 \quad .\end{aligned} \tag{82}$$

We consider the homogeneity of Hamiltonian function H,

$$\begin{aligned}\big(-V_{,\tilde{a}m}H^{,mn}V_{,nj} + V^{-1}V_{,\tilde{a}}H_{,j} - V^{-2}V_{,\tilde{a}}V_{,j}\\ +V_{,\tilde{a}m}H^{,mn}V_{,nj} + V^{-1}V_{,\tilde{a}}H^{,n}V_{,nj} + V^{-1}V_{,\tilde{a}m}H^{,m}V_{,j} + V^{-2}V_{,\tilde{a}}V_{,j}\big)\tfrac{\partial x^j}{\partial \tilde{y}_{\tilde{c}}} = 0 \quad .\end{aligned} \tag{83}$$

We now execute summation in (83) and arrive at

$$\big(V^{-1}V_{,\tilde{a}}H_{,j} + V^{-1}V_{,\tilde{a}}H^{,n}V_{,nj} + V^{-1}V_{,\tilde{a}m}H^{,m}V_{,j}\big)\tfrac{\partial x^j}{\partial \tilde{y}_{\tilde{c}}} = 0 \quad . \tag{84}$$

This equation holds in consequence of equations (56) and (57).

4.6. Derivative of (36)

We express the derivative of equation (36) along the geodesic using equations (63), (65), (68), (69) and (70),

$$\begin{aligned}&\tfrac{\partial x^i}{\partial \tilde{y}_{\tilde{a}}}\big(-V_{,\tilde{a}m}H^{,mn}V_{,n\tilde{b}} - V^{-2}V_{,\tilde{a}}V_{,\tilde{b}}\big)\\ &+\big(H^{;i}_{,m}\tfrac{\partial x^m}{\partial \tilde{y}_{\tilde{a}}} + H^{,im}\tfrac{\partial y_m}{\partial \tilde{y}_{\tilde{a}}}\big)\big(V_{,\tilde{a}\tilde{b}} + V^{-1}V_{,\tilde{a}}V_{,\tilde{b}}\big) = H^{;i}_{,m}\tfrac{\partial x^m}{\partial \tilde{x}^{\tilde{b}}} + H^{,im}\tfrac{\partial y_m}{\partial \tilde{x}^{\tilde{b}}} \quad .\end{aligned} \tag{85}$$

Considering equation (36), the terms containing $H^{;i}_{,m}$ cancel,

$$\tfrac{\partial x^i}{\partial \tilde{y}_{\tilde{a}}}\big(-V_{,\tilde{a}m}H^{,mn}V_{,n\tilde{b}} - V^{-2}V_{,\tilde{a}}V_{,\tilde{b}}\big) + H^{,im}\tfrac{\partial y_m}{\partial \tilde{y}_{\tilde{a}}}\big(V_{,\tilde{a}\tilde{b}} + V^{-1}V_{,\tilde{a}}V_{,\tilde{b}}\big) = H^{,im}\tfrac{\partial y_m}{\partial \tilde{x}^{\tilde{b}}} \quad . \tag{86}$$

Considering (57) and the second–degree homogeneity of H, we may express equation (86) in form

$$\begin{aligned}&-\tfrac{\partial x^i}{\partial \tilde{y}_{\tilde{a}}}\big(V_{,\tilde{a}m} + V^{-1}V_{,\tilde{a}}V_{,m}\big)H^{,mn}\big(V_{,n\tilde{b}} + V^{-1}V_{,n}V_{,\tilde{b}}\big)\\ &+H^{,im}\tfrac{\partial y_m}{\partial \tilde{y}_{\tilde{a}}}\big(V_{,\tilde{a}\tilde{b}} + V^{-1}V_{,\tilde{a}}V_{,\tilde{b}}\big) = H^{,im}\tfrac{\partial y_m}{\partial \tilde{x}^{\tilde{b}}} \quad .\end{aligned} \tag{87}$$

We multiply equation (87) from the right–hand side by matrix $\frac{\partial x^k}{\partial \tilde{y}_{\tilde{b}}}$,

$$- \frac{\partial x^i}{\partial \tilde{y}_{\tilde{a}}} \big(V_{,\tilde{a}m} + V^{-1} V_{,\tilde{a}} V_{,m} \big) H^{,mn} \big(V_{,n\tilde{b}} + V^{-1} V_{,n} V_{,\tilde{b}} \big) \frac{\partial x^k}{\partial \tilde{y}_{\tilde{b}}}$$
$$+ H^{,im} \frac{\partial y_m}{\partial \tilde{y}_{\tilde{a}}} \big(V_{,\tilde{a}\tilde{b}} + V^{-1} V_{,\tilde{a}} V_{,\tilde{b}} \big) \frac{\partial x^k}{\partial \tilde{y}_{\tilde{b}}} = H^{,im} \frac{\partial y_m}{\partial \tilde{x}^{\tilde{b}}} \frac{\partial x^k}{\partial \tilde{y}_{\tilde{b}}} \quad . \tag{88}$$

Equation (35) implies

$$\frac{\partial x^i}{\partial \tilde{y}_{\tilde{a}}} \left(V_{,\tilde{a}j} + V^{-1} V_{,\tilde{a}} V_{,j} \right) = -\delta^i_j \quad . \tag{89}$$

We insert (36) and (89) into equation (88) and obtain

$$-H^{,ik} + H^{,im} \frac{\partial y_m}{\partial \tilde{y}_{\tilde{a}}} \frac{\partial x^k}{\partial \tilde{x}^{\tilde{a}}} = H^{,im} \frac{\partial y_m}{\partial \tilde{x}^{\tilde{b}}} \frac{\partial x^k}{\partial \tilde{y}_{\tilde{b}}} \quad . \tag{90}$$

Equation (90) holds thanks to the consequence

$$\frac{\partial y_i}{\partial \tilde{y}_{\tilde{c}}} \frac{\partial x^k}{\partial \tilde{x}^{\tilde{c}}} - \frac{\partial y_i}{\partial \tilde{x}^{\tilde{c}}} \frac{\partial x^k}{\partial \tilde{y}_{\tilde{c}}} = \delta^k_i \quad . \tag{91}$$

of the symplectic property of matrix (29).

5. Conclusions

Derived relations (34)–(36) between the propagator matrix (29) of geodesic deviation and the second–order spatial derivatives of characteristic function are applicable to Finsler geometry, to Riemann geometry, and to their various applications like the general relativity or the high–frequency approximations of wave propagation.

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References

Červený, V. (1972): Seismic rays and ray intensities in inhomogeneous anisotropic media. *Geophys. J. R. astr. Soc.*, **29**, 1–13.

Červený, V., Klimeš, L. & Pšenčík, I. (1984): Paraxial ray approximations in the computation of seismic wavefields in inhomogeneous media. *Geophys. J. R. astr. Soc.*, **79**, 89–104.

Finsler, P. (1918): *Über Kurven und Flächen in Allgemeinen Räumen.* PhD Thesis, Göttingen, Reedition: Birkhäuser, Basel, 1951.

Hamilton, W.R. (1837): Third supplement to an essay on the theory of systems of rays. *Trans. Roy. Irish Acad.*, **17**, 1–144.

Klimeš, L. (1994): Transformations for dynamic ray tracing in anisotropic media. *Wave Motion*, **20**, 261–272.

Klimeš, L. (2002): Second–order and higher–order perturbations of travel time in isotropic and anisotropic media. *Stud. geophys. geod.*, **46**, 213–248.

Rund, H. (1959): *The Differential Geometry of Finsler Spaces.* Springer, Berlin–Göttingen–Heilderberg.