Stochastic wavefield inversion using the sensitivity Gaussian packets

Luděk Klimeš

Department of Geophysics, Faculty of Mathematics and Physics, Charles University, Ke Karlovu 3, 121 16 Praha 2, Czech Republic, http://sw3d.cz/staff/klimes.htm

Summary

Perturbations of the elastic medium can be decomposed into the structural Gabor functions. The wavefield scattered by one structural Gabor function is approximately composed of a few sensitivity Gaussian packets only. We derive the equations for the linearized stochastic wavefield inversion based on the relation between the structural Gabor functions and the scattered sensitivity Gaussian packets. The resulting algorithm allows for the true inversion of the complete set of seismograms recorded for all shots.

Keywords

Wavefield inversion, covariance functions and matrices, elastic waves, elastic moduli, seismic anisotropy, heterogeneous media, perturbation.

1. Introduction

Klimeš (2007) studied how the perturbations of a generally heterogeneous isotropic or anisotropic structure manifest themselves in the wavefield, and which perturbations can be detected within a limited aperture and a limited frequency band. He decomposed infinitesimally small perturbations of elastic moduli and density into the *structural Gabor functions* and demonstrated that, for each source generating a short–duration broad–band incident wave, the wavefield scattered by each structural Gabor function is composed of a few *sensitivity Gaussian packets* propagating in specific directions and sensitive to specific linear combinations of perturbations of material parameters. The derived correspondence between the perturbations of the structure and the recorded wavefield may play a decisive role in understanding the information on geological structures carried by seismic wavefields and recorded by a particular source–receiver configuration.

This paper is devoted to the algorithm of true linearized inversion of the complete set of seismograms recorded for all shots during seismic reflection survey, based on the above mentioned approximate solution of the forward scattering problem by Klimeš (2007).

We assume that the spatial distribution of material parameters describing the elastic medium (e.g., of density, density–reduced elastic moduli in the anisotropic medium, P–wave and S–wave propagation velocities in the isotropic medium, or their combinations) is composed of the known smoothly varying velocity model and small structural perturbations.

In the stochastic inversion, these structural perturbations are assumed to be representations of random quantities (Tarantola, 1987; Klimeš, 2002). For the correct

In: Seismic Waves in Complex 3–D Structures, Report 18 (Department of Geophysics, Charles University, Prague, 2008), pp. 71–85

weighting of measured seismograms, which represent the data in the stochastic wavefield inversion, we need to know the model covariance function describing the properties of these random structural perturbations and the data covariance matrix describing the inaccuracy of the measured seismograms. Since we calculate the wavefield scattered by structural perturbations only approximately, the data covariance matrix should account not only for the seismic noise, but also for the inaccuracy of the wavefield approximation.

In this paper, we assume that the structural perturbations are composed of the structural Gabor functions. We derive the equations for the linearized stochastic wavefield inversion based on the above mentioned relation between the structural Gabor functions and the scattered sensitivity Gaussian packets. Whereas the standard seismic migration algorithms replace wavefield inversion by simple back projection, our algorithm allows for the true inversion of the complete set of seismograms recorded for all shots.

2. Structural Gabor functions and sensitivity Gaussian packets

2.1. Material parameters and complex–valued structural Gabor functions

Let functions $m_i(\mathbf{x})$ represent material parameters describing the elastic medium, e.g., density, density–reduced elastic moduli in the anisotropic medium, P–wave and S–wave propagation velocities in the isotropic medium, or their combinations. In this paper, the lower–case subscripts index individual material parameters of the medium, and $\mathbf{x}$ represents the three spatial coordinates. For examples of selecting the sets of material parameters refer to Appendix A.

In stochastic inversion, we assume that functions $m_i(\mathbf{x})$ are realizations of random quantities. We define the *velocity model* described by the smooth material parameters

$$m_i^0(\mathbf{x}) \quad . \tag{1}$$

This model is assumed to represent the statistical mean model constrained by the data available for its construction and by the smoothness requirements of the ray methods.

To calculate the scattered wavefield, we decompose the deviations

$$\Delta m_i(\mathbf{x}) = m_i(\mathbf{x}) - m_i^0(\mathbf{x}) \tag{2}$$

of the material parameters from the velocity model into *structural Gabor functions*

$$G_\alpha(\mathbf{x}) = \exp[\mathrm{i}\mathbf{k}_\alpha{}^{\mathrm{T}}(\mathbf{x} - \mathbf{x}^\alpha) - \tfrac{1}{2}(\mathbf{x} - \mathbf{x}_\alpha)^{\mathrm{T}}\mathbf{K}_\alpha(\mathbf{x} - \mathbf{x}_\alpha)] \tag{3}$$

indexed by lower-case Greek subscripts. Gabor function G_α is centred at spatial position $\mathbf{x}_\alpha$ and has structural wavenumber vector $\mathbf{k}_\alpha$. The envelope and wavefront curvature of Gabor function G_α is described by symmetric complex–valued 3×3 matrix $\mathbf{K}_\alpha$ with the positive–definite real part.

We denote by $U_{i\alpha}$ the first–order Born approximation of the complex–valued wavefield scattered by the perturbation of material parameter $m_i(\mathbf{x})$ in the form of Gabor function $G_\alpha(\mathbf{x})$. This approximate scattered wavefield $U_{i\alpha}$ is composed of a few Gaussian packets (Klimeš, 2007). For different sources, we obtain different scattered wavefields $U_{i\alpha}$ composed of different Gaussian packets. The wavefield is then approximately composed of the wavefield calculated in the velocity model $m_i^0(\mathbf{x})$ and of the scattered wavefields $U_{i\alpha}$. Note that the complex–valued wavefields $U_{i\alpha}$ represent the analytical signals, and that the corresponding real–valued wavefields are their real parts (Červený, 2001, eq. A.1.2; Klimeš, 2007, eq. 14).

We assume that the structural perturbations $\Delta m_i(\mathbf{x})$ are real–valued, and express them in form

$$\Delta m_i(\mathbf{x}) = \sum_\alpha [c_{i\alpha} G_\alpha(\mathbf{x}) + c_{i\alpha*} G_{\alpha*}(\mathbf{x})] \quad . \tag{4}$$

Here index $_{\alpha*}$ corresponds to the Gabor function $G_{\alpha*}(\mathbf{x}) = [G_\alpha(\mathbf{x})]^*$ complex–conjugate to $G_\alpha(\mathbf{x})$, with $\mathbf{k}_{\alpha*} = -\mathbf{k}_\alpha$ and $\mathbf{K}_{\alpha*} = (\mathbf{K}_\alpha)^*$. For example, we may index the structural Gabor functions so that the couple of indices $_\alpha$ and $_{\alpha*}$ corresponds to the couple of consecutive odd and even indices. Coefficient $c_{i\alpha*} = (c_{i\alpha})^*$ is complex–conjugate to coefficient $c_{i\alpha}$. For each material parameter $m_i(\mathbf{x})$ and each couple $G_\alpha(\mathbf{x})$ and $G_{\alpha*}(\mathbf{x})$ of complex–conjugate structural Gabor functions we have the couple $c_{i\alpha}$ and $c_{i\alpha*}$ of complex conjugate coefficients, i.e., two independent real–valued coefficients $\mathrm{Re}(c_{i\alpha})$ and $\mathrm{Im}(c_{i\alpha})$. We thus rewrite equation (4) as

$$\Delta m_i(\mathbf{x}) = \sum_\alpha \{2\mathrm{Re}(c_{i\alpha})\mathrm{Re}[G_\alpha(\mathbf{x})] - 2\mathrm{Im}(c_{i\alpha})\mathrm{Im}[G_\alpha(\mathbf{x})]\} \quad . \tag{5}$$

The real–valued wavefield scattered by the Gabor function $c_{i\alpha} G_\alpha(\mathbf{x})$ in (4) is

$$\mathrm{Re}[c_{i\alpha} U_{i\alpha}] = \mathrm{Re}(c_{i\alpha})\mathrm{Re}[U_{i\alpha}] - \mathrm{Im}(c_{i\alpha})\mathrm{Im}[U_{i\alpha}] \quad , \tag{6}$$

and the real–valued wavefield scattered by the Gabor function $c_{i\alpha*} G_{\alpha*}(\mathbf{x})$ in (4) is

$$\mathrm{Re}[(c_{i\alpha})^* U_{i\alpha*}] = \mathrm{Re}(c_{i\alpha})\mathrm{Re}[U_{i\alpha*}] + \mathrm{Im}(c_{i\alpha})\mathrm{Im}[U_{i\alpha*}] \quad . \tag{7}$$

Let us emphasize that the complex–valued wavefield $U_{i\alpha*}$ is not complex–conjugate to $U_{i\alpha}$ and propagates in different directions.

For fixed α, Gaussian packets composing wavefields $U_{i\alpha}$ corresponding to individual material parameters differ by their amplitudes only. In numerical calculations, we thus work with Gaussian packets common to all material parameters. Each Gaussian packet is supplemented with the set of amplitudes corresponding to individual coefficients $c_{i\alpha}$ of material parameters. For the dependence of the amplitudes on the selection of material parameters refer to Appendix A.

2.2. Material parameters and real–valued structural Gabor functions

Hereinafter, we assume that coefficient $m_{i\alpha}$ is real–valued and represents either $\mathrm{Re}(c_{i\alpha})$ or $-\mathrm{Im}(c_{i\alpha})$. We define,

$$m_{i\alpha} = \mathrm{Re}(c_{i\alpha}) \quad , \tag{8}$$

$$g_\alpha(\mathbf{x}) = 2\mathrm{Re}[G_\alpha(\mathbf{x})] \quad , \tag{9}$$

$$u_{i\alpha} = \mathrm{Re}[U_{i\alpha}] + \mathrm{Re}[U_{i\alpha*}] \quad , \tag{10}$$

and

$$m_{i\alpha*} = -\mathrm{Im}(c_{i\alpha}) \quad , \tag{11}$$

$$g_{\alpha*}(\mathbf{x}) = 2\mathrm{Im}[G_\alpha(\mathbf{x})] \quad , \tag{12}$$

$$u_{i\alpha*} = \mathrm{Im}[U_{i\alpha}] - \mathrm{Im}[U_{i\alpha*}] \quad . \tag{13}$$

Then the real–valued perturbation (5) can be expressed in form

$$\Delta m_i(\mathbf{x}) = \sum_\alpha m_{i\alpha}\, g_\alpha(\mathbf{x}) \quad . \tag{14}$$

The real–valued wavefield scattered by real–valued perturbation (14) reads

$$\sum_{i,\alpha} m_{i\alpha} u_{i\alpha} \quad , \tag{15}$$

see equations (6) and (7).

2.3. Decomposition of a given structural perturbation

In this section, we assume that function $\Delta m_i(\mathbf{x})$ representing a structural perturbation, is known. We wish to decompose given function $\Delta m_i(\mathbf{x})$ in form (14). We are thus searching for linear operator projecting function $\Delta m_i(\mathbf{x})$ to coefficients $m_{i\alpha}$. We express this operator in form

$$m_{i\alpha} = \sum_{\beta} F_{\alpha\beta} \int \mathrm{d}\mathbf{x}\, g_\beta(\mathbf{x})\, \Delta m_i(\mathbf{x}) \quad , \tag{16}$$

where we have to determine $F_{\alpha\beta}$. Equation (14) with (16) substituted for $m_{i\alpha}$ imply matrix equation

$$B_{\alpha\delta} = \sum_{\beta} \sum_{\gamma} B_{\alpha\beta} F_{\beta\gamma} B_{\gamma\delta} \tag{17}$$

for $F_{\alpha\beta}$. Here

$$B_{\alpha\beta} = \int \mathrm{d}\mathbf{x}\, g_\alpha(\mathbf{x}) g_\beta(\mathbf{x}) \quad . \tag{18}$$

For the algorithm of calculating matrix $B_{\alpha\beta}$, refer to Klimeš (2008a). Since $B_{\alpha\beta}$ is a singular positive–semidefinite matrix, $F_{\alpha\beta}$ is not determined by equation (17) uniquely and we may impose on $F_{\alpha\beta}$ an additional condition. We select the solution $F_{\alpha\beta}$ of equation (17) which has the minimum squared norm

$$\sum_{\alpha} \sum_{\beta} F_{\alpha\beta} F_{\alpha\beta} \tag{19}$$

(Daubechies, 1992). This means that

$$F_{\alpha\beta} = B^{-1}_{\alpha\beta} \tag{20}$$

are the components of the generalized inverse matrix to singular positive–semidefinite matrix (18). Consequently, coefficients $m_{i\alpha}$ of the decomposition (14) of function $\Delta m_i(\mathbf{x})$ represent the solution of equation

$$\sum_{\beta} B_{\alpha\beta} m_{i\beta} = \int \mathrm{d}\mathbf{x}\, g_\alpha(\mathbf{x})\, \Delta m_i(\mathbf{x}) \tag{21}$$

which has the minimum squared norm

$$\sum_{\alpha} m_{i\alpha} m_{i\alpha} \quad . \tag{22}$$

3. Linearized inversion

3.1. Recorded seismograms

The data for inversion are represented by all measured seismogram components at all times, at all receivers and for all sources. We index here this huge amount of data by an upper–case roman subscript, and denote the measured seismograms by s_I. Subscript $_I$ thus represents all combinations of sources, receivers, times and components.

We shall invert difference

$$\Delta s_I = s_I - s_I^0 \tag{23}$$

between the recorded seismograms s_I and seismograms s_I^0 calculated in the velocity model $m_i^0(\mathbf{x})$.

The synthetic seismograms required for inversion are calculated for the configurations corresponding to the measured seismograms. We denote by $u_{I(i\alpha)}$ the synthetic seismograms corresponding to approximate wavefield $u_{i\alpha}$ scattered by the perturbation of material parameter $m_i(\mathbf{x})$ in the form of Gabor function $g_\alpha(\mathbf{x})$. The seismograms corresponding to synthetic wavefield (15) then read

$$\sum_{i,\alpha} m_{i\alpha} u_{I(i\alpha)} \quad . \tag{24}$$

3.2. Least–square inversion

In the naive least–square inversion, scattered wavefield (24) is fitted to differences (23) by minimizing objective function

$$\sum_I \left(\sum_{i,\alpha} m_{i\alpha} u_{I(i\alpha)} - \Delta s_I \right)^2 \quad . \tag{25}$$

The objective function is minimized by coefficients $m_{i\alpha}$ satisfying equation

$$\sum_{j,\beta} U_{(i,\alpha)(j\beta)} m_{j\beta} = \sum_I u_{I(i\alpha)} \Delta s_I \quad , \tag{26}$$

where

$$U_{(i,\alpha)(j\beta)} = \sum_I u_{I(i\alpha)} u_{I(j\beta)} \quad . \tag{27}$$

Since $U_{(i,\alpha)(j\beta)}$ is a singular positive–semidefinite matrix, $m_{i\alpha}$ is not determined by equation (26) uniquely and we may impose on $m_{i\alpha}$ an additional condition. We select the solution $m_{i\alpha}$ of equation (26) which has the minimum squared norm (22).

3.3. Stochastic inversion

Let $\widetilde{g}_\alpha(\mathbf{x})$ be any set of functions satisfying equation

$$\sum_\alpha g_\alpha(\mathbf{x}_1) \widetilde{g}_\alpha(\mathbf{x}_2) = \delta(\mathbf{x}_1 - \mathbf{x}_2) \quad . \tag{28}$$

We express functions $\widetilde{g}_\alpha(\mathbf{x})$ as linear combinations of functions $g_\alpha(\mathbf{x})$,

$$\widetilde{g}_\alpha(\mathbf{x}) = \sum_\beta F_{\alpha\beta}\, g_\beta(\mathbf{x}) \quad . \tag{29}$$

Inserting (29) into (28), we find out that matrix $F_{\alpha\beta}$ must satisfy equation (17). One example of set $\widetilde{g}_\alpha(\mathbf{x})$ is the dual frame (Daubechies, 1992, sec. 3.2) to $g_\alpha(\mathbf{x})$, obtained for $F_{\alpha\beta}$ selected according to equation (20).

We decompose deviation $\Delta m_i(\mathbf{x})$ by convolution with (28) into structural Gabor functions $g_\alpha(\mathbf{x})$ which generate seismograms $u_{I(i\alpha)}$. Difference (23) between the recorded seismograms s_I and seismograms s_I^0 calculated in the velocity model $m_i^0(\mathbf{x})$ then reads

$$\Delta s_I = \delta s_I + \sum_{i,\alpha} u_{I(i\alpha)} \int \mathrm{d}\mathbf{x}\, \widetilde{g}_\alpha(\mathbf{x}) \Delta m_i(\mathbf{x}) \quad , \tag{30}$$

where δs_I is the random noise.

We assume a linear inversion of difference Δs_I. The *a posteriori model* obtained by linearized inversion is

$$\widetilde{m}_i(\mathbf{x}) = m_i^0(\mathbf{x}) + \sum_I R_{iI}(\mathbf{x}) \Delta s_I \quad . \tag{31}$$

Inserting (30) into (31), we arrive at

$$\widetilde{m}_i(\mathbf{x}) = m_i^0(\mathbf{x}) + \sum_I R_{iI}(\mathbf{x}) \Big[\delta s_I + \sum_{j,\beta} u_{I(j\beta)} \int \mathrm{d}\mathbf{x}'\, \widetilde{g}_\beta(\mathbf{x}') \Delta m_j(\mathbf{x}') \quad . \tag{32}$$

We define the a priori *model covariance function*

$$C_{ij}(\mathbf{x}_1, \mathbf{x}_2) = \langle \Delta m_i(\mathbf{x}_1)\ \Delta m_j(\mathbf{x}_2) \rangle \quad , \tag{33}$$

and the *data covariance function*

$$T_{IJ} = \langle \delta s_I\ \delta s_J \rangle \quad , \tag{34}$$

where brackets $\langle\,\rangle$ represent the mean value. We assume no correlation between $\Delta m_i(\mathbf{x})$ and δs_I.

The random error

$$\delta m_i(\mathbf{x}) = \widetilde{m}_i(\mathbf{x}) - m_i(\mathbf{x}) \tag{35}$$

of the a posteriori model $\widetilde{m}_i(\mathbf{x})$ is

$$\delta m_i(\mathbf{x}) = -\Delta m_i(\mathbf{x}) + \sum_I R_{iI}(\mathbf{x}) \Big[\delta s_I + \sum_{j,\beta} u_{I(j\beta)} \int \mathrm{d}\mathbf{x}'\, \widetilde{g}_\beta(\mathbf{x}') \Delta m_j(\mathbf{x}') \Big] \quad . \tag{36}$$

The *a posteriori covariance function* is

$$\widetilde{C}_{ij}(\mathbf{x}_1, \mathbf{x}_2) = \langle \delta m_i(\mathbf{x}_1)\ \delta m_j(\mathbf{x}_2) \rangle \quad . \tag{37}$$

We insert (36) into (37) and consider definitions (33) and (34) of the a priori covariance functions. The a posteriori covariance function then reads

$$\begin{aligned} \widetilde{C}_{ij}(\mathbf{x}_1, \mathbf{x}_2) = C_{ij}(\mathbf{x}_1, \mathbf{x}_2) &- \sum_I R_{iI}(\mathbf{x}_1) \Psi_{jI}(\mathbf{x}_2) - \sum_I \Psi_{iI}(\mathbf{x}_1) R_{jI}(\mathbf{x}_2) \\ &+ \sum_I \sum_J R_{iI}(\mathbf{x}_1) R_{jJ}(\mathbf{x}_2) S_{IJ} \quad , \end{aligned} \tag{38}$$

where

$$\Psi_{iI}(\mathbf{x}_1) = \sum_{j,\alpha} \int \mathrm{d}\mathbf{x}_2\, C_{ij}(\mathbf{x}_1, \mathbf{x}_2) u_{I(j\alpha)} \widetilde{g}_\alpha(\mathbf{x}_2) \tag{39}$$

are the *seismogram back–projective functions*, and

$$S_{IJ} = T_{IJ} + \Theta_{IJ} \tag{40}$$

is the *complete seismogram covariance function.* Here

$$\Theta_{IJ} = \sum_{i,\alpha}\sum_{j,\beta} u_{I(i\alpha)} u_{J(j\beta)} \Theta_{(i\alpha)(j\beta)} \quad , \tag{41}$$

with

$$\Theta_{(i\alpha)(j\beta)} = \int \mathrm{d}\mathbf{x}_1 \int \mathrm{d}\mathbf{x}_2 \, C_{ij}(\mathbf{x}_1, \mathbf{x}_2)\, \widetilde{g}_\alpha(\mathbf{x}_1)\, \widetilde{g}_\beta(\mathbf{x}_2) \quad , \tag{42}$$

is the *a priori geometrical seismogram covariance function.* We may insert (29) into (42), and express matrix (42) in form

$$\Theta_{(i\alpha)(j\beta)} = \sum_{\gamma}\sum_{\delta} F_{\alpha\gamma} F_{\beta\delta} C_{(i\gamma)(j\delta)} \quad , \tag{43}$$

where

$$C_{(i\alpha)(j\beta)} = \int \mathrm{d}\mathbf{x}_1 \int \mathrm{d}\mathbf{x}_2 \, C_{ij}(\mathbf{x}_1, \mathbf{x}_2)\, g_\alpha(\mathbf{x}_1)\, g_\beta(\mathbf{x}_2) \quad . \tag{44}$$

The objective function is the integral variance

$$\sum_i \int \mathrm{d}\mathbf{x}\, \widetilde{C}_{ii}(\mathbf{x},\mathbf{x}) = \sum_i \Big[\int \mathrm{d}\mathbf{x}\, C_{ii}(\mathbf{x},\mathbf{x}) - \int \mathrm{d}\mathbf{x} \sum_I R_{iI}(\mathbf{x}) \Psi_{iI}(\mathbf{x})$$
$$- \int \mathrm{d}\mathbf{x} \sum_I R_{iI}(\mathbf{x}) \Psi_{iI}(\mathbf{x}) + \int \mathrm{d}\mathbf{x} \sum_I \sum_J R_{iI}(\mathbf{x}) R_{iJ}(\mathbf{x}) S_{IJ} \Big] \ . \tag{45}$$

Minimizing the objective function with respect to $R_{iI}(\mathbf{x})$, we obtain equation

$$\sum_J R_{iJ}(\mathbf{x}) S_{JI} = \Psi_{iI}(\mathbf{x}) \quad . \tag{46}$$

Matrix S_{IJ} is regular and thus invertible,

$$\boxed{R_{iI}(\mathbf{x}) = \sum_J \Psi_{iJ}(\mathbf{x}) S^{-1}_{JI} \quad .} \tag{47}$$

This equation is a counterpart to the well–known equation for the stochastic travel–time inversion (Klimeš, 2002, eq. 58).

Unfortunately, the dimension of S_{IJ} corresponds to the data and is too large for inversion. We now assume that we can invert matrix T_{IJ} analytically and express the equation for $R_{iI}(\mathbf{x})$ in a different form. We multiply equation (46) with (39), (40) and (41) by matrix T^{-1}_{IJ} and arbitrary vector Δs_J from the right, and obtain implication

$$\sum_I \sum_J u_{I(i\alpha)}\, T^{-1}_{IJ}\, \Delta s_J = 0 \quad \Longrightarrow \quad \sum_J R_{iJ}(\mathbf{x})\, \Delta s_J = 0 \quad . \tag{48}$$

It is thus possible to compose the kernel $R_{iJ}(\mathbf{x})$ of operator (31) of finite number of functions $R_{i(j\beta)}(\mathbf{x})$, each function multiplied by data–projecting vector $\sum_I u_{I(j\beta)}\, T^{-1}_{IJ}$,

$$R_{iJ}(\mathbf{x}) = \sum_{j,\beta} R_{i(j\beta)}(\mathbf{x}) \sum_I u_{I(j\beta)}\, T^{-1}_{IJ} \quad . \tag{49}$$

We express functions $R_{i(j\beta)}(\mathbf{x})$ as linear combinations of functions $g_\alpha(\mathbf{x})$,

$$R_{i(j\beta)}(\mathbf{x}) = \sum_\alpha g_\alpha(\mathbf{x}) R_{(i\alpha)(j\beta)} \quad . \tag{50}$$

Then

$$R_{iJ}(\mathbf{x}) = \sum_\alpha \sum_{j,\beta} g_\alpha(\mathbf{x}) R_{(i\alpha)(j\beta)} \sum_I u_{I(j\beta)}\, T^{-1}_{IJ} \quad . \tag{51}$$

Inserting (51) into (46) with (39), (40) and (41), and integrating with $g_\alpha(\mathbf{x})$, we find out that

$$\sum_\gamma \sum_{j,\delta} \sum_{l,\beta} B_{\alpha\gamma} R_{(i\gamma)(j\delta)} \left[\sum_{k,\epsilon} W_{(j\delta)(k\epsilon)} \Theta_{(k\epsilon)(l\beta)} + \delta_{jl}\delta_{\delta\beta} \right] u_{I(l\beta)} = \sum_{j,\beta} \sum_\gamma C_{(i\alpha)(j\beta)} F_{\gamma\beta} u_{I(j\gamma)} \,, \tag{52}$$

where

$$W_{(i\alpha)(j\beta)} = \sum_I \sum_J u_{I(i\alpha)} u_{J(j\beta)}\, T^{-1}_{IJ} \quad . \tag{53}$$

We hope that we can invert matrix T_{IJ} and calculate matrix $W_{(i\alpha)(j\beta)}$ analytically.

We define matrix $\mathbf{\Theta}$ with elements (42), matrix $\mathbf{W}$ with elements (53), matrix $\mathbf{R}$ with elements $R_{(i\alpha)(j\beta)}$, matrix $\mathbf{F}$ with elements

$$F_{(i\alpha)(j\beta)} = \delta_{ij} F_{\alpha\beta} \quad , \tag{54}$$

and matrix $\mathbf{B}$ with elements

$$B_{(i\alpha)(j\beta)} = \delta_{ij} B_{\alpha\beta} \quad . \tag{55}$$

Matrices $\mathbf{F}$ and $\mathbf{B}$ are thus composed of smaller matrices $F_{\alpha\beta}$ and $B_{\alpha\beta}$, respectively. The algorithm of calculating matrix $B_{\alpha\beta}$ has been proposed by Klimeš (2008a), and is coded in program `sgfmat.for` of package `FORMS` (Bucha & Bulant, 2008). Generalized inverse matrix $\mathbf{B}^{-1}$ then has elements

$$B^{-1}_{(i\alpha)(j\beta)} = \delta_{ij} B^{-1}_{\alpha\beta} \quad , \tag{56}$$

where $B^{-1}_{\alpha\beta}$ are the components of the generalized inverse matrix to singular positive–semidefinite matrix (18). Equation (52) yields matrix equation

$$\mathbf{BR}\,(\mathbf{W\Theta W} + \mathbf{W}) = \mathbf{CF}^{\mathrm{T}}\mathbf{W} \quad , \tag{57}$$

where

$$\mathbf{\Theta} = \mathbf{FCF}^{\mathrm{T}} \quad , \tag{58}$$

see (43). Since $\mathbf{B}$ and $\mathbf{W\Theta W} + \mathbf{W}$ are singular positive–semidefinite matrices, $\mathbf{R}$ is not determined by equation (57) uniquely and we may impose on $\mathbf{R}$ an additional condition. Analogously to (19), we minimize objective function

$$\mathrm{tr}(\mathbf{R}^{\mathrm{T}}\mathbf{R}) \tag{59}$$

under condition (57),

$$\mathbf{R} = \mathbf{B\Lambda}(\mathbf{W\Theta W} + \mathbf{W}) \quad , \tag{60}$$

where $\mathbf{\Lambda}$ is the matrix of Lagrange multipliers. We rewrite equation (57) as

$$\mathbf{BRW}\,(\mathbf{\Theta W} + \mathbf{1}) = \mathbf{CF}^{\mathrm{T}}\mathbf{W} \quad . \tag{61}$$

Since the characteristic values of matrix $\mathbf{\Theta W}$ are not negative, matrix $\mathbf{\Theta W} + \mathbf{1}$ is regular, and

$$\mathbf{BRW} = \mathbf{CF}^{\mathrm{T}}\mathbf{W}\,(\mathbf{\Theta W} + \mathbf{1})^{-1} \quad . \tag{62}$$

We see that the right–hand side of equation (62) must annul the vectors from the null space of matrix $\mathbf{B}$ from the left–hand side, and the vectors from the null space of matrix $\mathbf{W}$ from the right–hand side. We rewrite equation (60) as

$$\mathbf{R} = \mathbf{B\Lambda}\,(\mathbf{W\Theta} + \mathbf{1})\,\mathbf{W} \quad , \tag{63}$$

and see that matrix $\mathbf{R}$ annuls the vectors from the null space of matrix $\mathbf{B}$ from the left–hand side, and the vectors from the null space of matrix $\mathbf{W}$ from the right–hand side. Then

$$\mathbf{R} = \mathbf{B}^{-1}\mathbf{CF}^{\mathrm{T}}\mathbf{W}\,(\mathbf{\Theta W} + \mathbf{1})^{-1}\mathbf{W}^{-1} \quad , \tag{64}$$

where $\mathbf{B}^{-1}$ is the generalized inverse matrix to singular positive–semidefinite matrix $\mathbf{B}$, and $\mathbf{W}^{-1}$ is the generalized inverse matrix to singular positive–semidefinite matrix $\mathbf{W}$.

Matrix $F_{\alpha\beta}$ satisfies equation (17). If we insert (58) into (57), we see that matrix $F_{\alpha\beta}$ appears in (57) in the product $\mathbf{WFC}$ with elements $\sum_j\sum_\beta\sum_\gamma W_{(i\alpha)(j\beta)}F_{\beta\gamma}C_{(j\gamma)(k\delta)}$ only. Vectors n_α from the null space of matrix $B_{\alpha\beta}$ correspond to identically zero sum (14). These vectors thus satisfy identity $\sum_\gamma n_\gamma C_{(j\gamma)(k\delta)} = 0$. If the sum (24) corresponding to zero sum (14) is always zero, vectors n_α from the null space of matrix $B_{\alpha\beta}$ satisfy identity $\sum_\beta W_{(i\alpha)(j\beta)} n_\beta = 0$, and there is no difference between choice

$$F_{\alpha\beta} = B^{-1}_{\alpha\beta} \tag{65}$$

and another matrix $F_{\alpha\beta}$ satisfying (17). If the sum (24) corresponding to zero sum (14) is not always zero, it is reasonable to reduce the error of forward operator (30) by avoiding zero combinations of functions $g_\beta(\mathbf{x})$ by option (65). We thus choose

$$\mathbf{F} = \mathbf{B}^{-1} \quad . \tag{66}$$

Inserting (58) with (66) into (64), we obtain

$$\mathbf{R} = \mathbf{B}^{-1}\mathbf{CB}^{-1}\mathbf{W}\,(\mathbf{B}^{-1}\mathbf{CB}^{-1}\mathbf{W} + \mathbf{1})^{-1}\mathbf{W}^{-1} \quad , \tag{67}$$

which can also be expressed as

$$\boxed{\mathbf{R} = [\mathbf{1} - (\mathbf{B}^{-1}\mathbf{CB}^{-1}\mathbf{W} + \mathbf{1})^{-1}]\,\mathbf{W}^{-1} \quad . \qquad (68)}$$

Consequently, inserting (67) into (51), the vector of coefficients $m_{i\alpha}$ of the decomposition (14) of the solution of the inverse problem reads

$$\mathbf{m} = \mathbf{B}^{-1}\mathbf{CB}^{-1}\mathbf{W}\,(\mathbf{B}^{-1}\mathbf{CB}^{-1}\mathbf{W} + \mathbf{1})^{-1}\mathbf{W}^{-1}\mathbf{s} \quad , \tag{69}$$

where the components of vector $\mathbf{s}$ are

$$s_{(i\alpha)} = \sum_I\sum_J u_{I(i\alpha)}\,T^{-1}_{IJ}\,\Delta s_J \quad . \tag{70}$$

We hope that we can invert matrix T_{IJ} analytically and calculate vector $s_{(i\alpha)}$ semianalytically.

Inserting (68) into (51), the vector of coefficients $m_{i\alpha}$ of the decomposition (14) of the solution of the inverse problem alternatively reads

$$\mathbf{m} = [\mathbf{1} - (\mathbf{B}^{-1}\mathbf{CB}^{-1}\mathbf{W} + \mathbf{1})^{-1}]\,\mathbf{W}^{-1}\mathbf{s} \quad . \tag{71}$$

We denote by $\mathbf{b}$ the solution of singular equation

$$\mathbf{W}\mathbf{b} = \mathbf{s} \tag{72}$$

which has the minimum squared norm

$$\mathrm{tr}(\mathbf{b}^{\mathrm{T}}\mathbf{b}) \quad , \tag{73}$$

and by $\mathbf{a}$ the solution of regular equation

$$(\mathbf{B}^{-1}\mathbf{C}\mathbf{B}^{-1}\mathbf{W} + \mathbf{1})\,\mathbf{a} = \mathbf{b} \quad . \tag{74}$$

The vector of coefficients $m_{i\alpha}$ of the decomposition (14) of the solution of the inverse problem may then be expressed using equation (69) as

$$\mathbf{m} = (\mathbf{B}^{-1}\mathbf{C}\mathbf{B}^{-1}\mathbf{W})\,\mathbf{a} \quad , \tag{75}$$

or using equation (71) as

$$\mathbf{m} = \mathbf{b} - \mathbf{a} \quad . \tag{76}$$

Unfortunately, calculation of inverse matrix $B^{-1}_{\alpha\beta}$, which constitutes matrix $\mathbf{B}^{-1}$ through (56) and is thus necessary for determination of matrix $\mathbf{B}^{-1}\mathbf{C}\mathbf{B}^{-1}\mathbf{W}$, may represent a considerable numerical problem.

The null space of matrix (18), which generates the null space of matrix $\mathbf{B}$ through (55), corresponds to vanishing linear combinations of real–valued structural Gabor functions $g_\alpha(\mathbf{x})$. The null space of matrix $\mathbf{B}$ is thus a subspace of the null space of matrix $\mathbf{C}$. The null space of matrix $\mathbf{C}$ may or may not be greater than the null space of matrix $\mathbf{B}$, depending on the positivity of the model covariance function $C_{ij}(\mathbf{x}_1, \mathbf{x}_2)$, see (44). The null space of matrix $\mathbf{B}$ is also approximately (because scattered Gaussian packets are approximate) a subspace of the null space of matrix $\mathbf{W}$. The null space of matrix $\mathbf{W}$ will often be greater than the null space of matrix $\mathbf{B}$ because many scattered Gaussian packets will not be recorded.

3.4. Low–noise limit

For decreasing noise, the positive–definite data covariance matrix T_{IJ} decreases and its inverse T^{-1}_{IJ} increases. As a consequence, the positive eigenvalues of matrix $\mathbf{W}$ defined by (53) increase.

It follows from the definition (72) and (73) of vector $\mathbf{b}$ that vector $\mathbf{b}$ does not contain vectors from the null space of matrix $\mathbf{W}$. It then follows from (74) that vector $\mathbf{a}$ does not contain vectors from the null space of matrix $\mathbf{W}$, and from (76) that vector $\mathbf{m}$ does not contain vectors from the null space of matrix $\mathbf{W}$.

As vector $\mathbf{a}$ does not contain vectors from the null space of matrix $\mathbf{W}$, equation (74) implies that vector $\mathbf{a}$ converge to zero for increasing positive eigenvalues of matrix $\mathbf{W}$. Vector $\mathbf{b}$ thus represents the low–noise limit of the solution of our inverse problem, and

$$\mathbf{R} = \mathbf{W}^{-1} \tag{77}$$

represents the low–noise limit of equation (68).

4. Overview of wavefield inversion using the sensitivity Gaussian packets

The algorithm of the linearized inversion of the complete set of seismograms recorded for all shots should be composed of the following steps:

(a) We specify the finite set of structural Gabor functions (3) to which the wanted deviations (2) of the material parameters from the given velocity model will be decomposed. The coefficients of this decomposition (4) are unknown. In each spatial position, we consider structural Gabor functions differing in the structural wavenumber and in the orientation of the structural wavenumber vector. If possible, the shape of the Gabor functions should be optimized, see Section 4.1. If possible, the Gabor functions should form a frame, see Section 4.2.

(b) For each source and each Gabor function $G_\alpha(\mathbf{x})$, we calculate the Gaussian packets forming scattered wavefield $U_{i\alpha}$ according to the paper by Klimeš (2007). Each Gaussian packet is supplemented with the set of amplitudes corresponding to unknown coefficients $c_{i\alpha}$ of individual material parameters according to Appendix A. It is sufficient to select Gaussian packets arriving to the corresponding receivers only.

(c) We determine the coefficients $c_{i\alpha}$ of decomposition (4) of the perturbations (2) of the material parameters by approximate linearized inversion according to Section 3 of this paper.

For a particular source, each structural Gabor function generates at most a few Gaussian packets propagating in determined directions. If the Gaussian packets do not fall into the aperture covered by the receivers and into the legible frequency band, the corresponding information about the Gabor function is lost. This situation improves with more differently positioned sources. If we have many sources, the Gaussian packets propagating from a Gabor function may be lost during the measurement corresponding to one source, but recorded during the measurement corresponding to another, differently positioned source. However, the problem is not only to record the Gaussian packets from a Gabor function, but to record them in as many different measurement configurations as to resolve several material parameters.

4.1. Optimization of the Gabor representation

We outline here a desirable optimization procedure of obtaining the finite set of structural Gabor functions (3).

At each point $\mathbf{x}_\alpha$, we consider all source–receiver pairs and all corresponding combinations of elementary waves. For each combination of elementary waves from the source and from the receiver, we obtain a set of structural wavenumbers $\mathbf{k}_\alpha$. For each structural wavenumber $\mathbf{k}_\alpha$, we also obtain the linear dependence of the complex–valued matrix describing the initial shape of the scattered Gaussian packet on symmetric complex–valued matrix $\mathbf{K}_\alpha$ (Klimeš, 2007, eqs. 237–239). For each point $\mathbf{x}_\alpha$ and each structural wavenumber $\mathbf{k}_\alpha$, we should consider all these linear dependences and minimize the widths of the scattered Gaussian packets by optimizing matrix $\mathbf{K}_\alpha$.

The obtained dependence of matrix $\mathbf{K}_\alpha$ on the structural phase–space coordinates $\mathbf{x}_\alpha$ and $\mathbf{k}_\alpha$ then determines the phase–space metric tensor. We then need to cover the target phase–space region by points $(\mathbf{x}_\alpha, \mathbf{k}_\alpha)$ distributed evenly with respect to this phase–space metric tensor. These points represent the phase–space central points of the structural Gabor functions.

Refer to Klimeš (2008d) for an attempt to specify the structural Gabor functions to some extent optimally in the very simple case of a homogeneous medium, and to program `sgfhom.for` of package `FORMS` (Bucha & Bulant, 2008) for the corresponding FORTRAN 77 code.

4.2. Frames composed of Gabor functions

In the standard *Gabor transform*, matrix $\mathbf{K}_\alpha$, which controls the shape of Gabor functions, is real–valued and independent of both position $\mathbf{x}_\alpha$ and wavenumber vector $\mathbf{k}_\alpha$. In this case, at least the 1–D Gabor functions form a frame (Daubechies, 1992, sec. 3.4.4 B; Klimeš, 2008b), which we shall call the *Gabor transform frame*. The estimation of the frame bounds of the n–D Gabor transform with complex–valued matrix $\mathbf{K}_\alpha$ has been conjectured by Klimeš (2008c) but has not been proved.

In the standard 1–D *Gabor wavelet transform*, matrix $\mathbf{K}_\alpha$ is real–valued, is independent of position, but is proportional to the square of the wavenumber $|\mathbf{k}_\alpha|$. In this case, the Gabor functions form a frame (Daubechies, 1992, sec. 3.3.5 C), which we shall call the *Gabor wavelet frame*.

In our case of structural Gabor functions, matrix $\mathbf{K}_\alpha$ is complex–valued and is proportional to the wavenumber $|\mathbf{k}_\alpha|$. For the wavefield inversion, we thus desire a frame formed from these Gabor functions whose linear dependence of matrix $\mathbf{K}_\alpha$ on the wavenumber is inbetween the Gabor transform frame and the Gabor wavelet frame. We might refer to this frame as the *Gabor wave packet frame* because this wavenumber or frequency dependence of the envelopes of Gabor functions is typical for wave phenomena. Moreover, because of optimization of the envelope of structural Gabor functions, we wish matrix $\mathbf{K}_\alpha$ were smoothly varying with position $\mathbf{x}_\alpha$ and direction of wavenumber vector $\mathbf{k}_\alpha$.

We thus need to generalize the estimation of the frame bounds and of discretization error from the Gabor transform frame and the Gabor wavelet frame to the Gabor wave packet frame, then from real–valued matrix $\mathbf{K}_\alpha$ to complex–valued matrix $\mathbf{K}_\alpha$, then from 1–D to 2–D and 3–D, and finally from matrix $\mathbf{K}_\alpha$ proportional to the wavenumber $|\mathbf{k}_\alpha|$ only to matrix $\mathbf{K}_\alpha$ also at least slightly varying with position $\mathbf{x}_\alpha$ and direction of wavenumber vector $\mathbf{k}_\alpha$.

Appendix A: Material parameters and reflectivity function

In this appendix, we index the three spatial coordinates by the lower–case superscripts in order to distinguish them from material parameters of the medium indexed by the lower–case subscripts. In this appendix, we use the Einstein summation over the repetitive lower–case superscripts.

For fixed α, Gaussian packets composing wavefields $U_{i\alpha}$ corresponding to individual material parameters differ by their amplitudes only. In numerical calculations, we thus work with Gaussian packets common to all material parameters, each Gaussian packet being supplemented with the set of amplitudes corresponding to coefficients $c_{i\alpha}$ of individual material parameters.

The initial amplitude of a Gaussian packet from the wavefield scattered by structural Gabor function G_α is proportional to reflectivity function r_α (Klimeš, 2007, eq. 241). We approximate the dependence of reflectivity function r_α on coefficients $c_{i\alpha}$ of individual material parameters by linear Taylor expansion

$$r_\alpha = \sum_i c_{i\alpha}\, r_{i\alpha} \quad . \tag{A1}$$

We then obtain the amplitude corresponding to coefficient $c_{i\alpha}$ by substituting derivative $r_{i\alpha}$ for reflectivity function r_α.

The reflectivity function depends on coefficients ϱ_α and c_α^{ijkl} corresponding to density and elastic moduli through approximate equation

$$r_\alpha = (2\varrho)^{-1}(\varrho_\alpha\, E^i e^i - c_\alpha^{ijkl}\, E^i P^j e^k p^l) \tag{A2}$$

(Klimeš, 2007, eq. 68). Here P_i and E_i are the slowness vector and the polarization vector of the incident wave, and p_i and e_i are the slowness vector and the polarization vector of the scattered Gaussian packet. Density ϱ corresponds to the velocity model. For a selected set of material parameters $m_i(\mathbf{x})$, we insert linear Taylor expansion of coefficients ϱ_α and c_α^{ijkl} with respect to coefficients $c_{i\alpha}$ into (A2). We now present 3 examples of selecting the set of material parameters and determining the corresponding derivatives $r_{i\alpha}$.

General anisotropic medium can be parametrized by 22 material parameters $m_i(\mathbf{x})$, where $m_1(\mathbf{x}) = \varrho(\mathbf{x})$ and $m_2(\mathbf{x})$ to $m_{22}(\mathbf{x})$ represent 21 density normalized elastic moduli $a^{ijkl}(\mathbf{x})$. The elastic moduli then depend on the selected material parameters through

$$c^{ijkl} = \varrho\, a^{ijkl} \quad . \tag{A3}$$

The linear Taylor expansion of coefficients c_α^{ijkl} with respect to coefficients $c_{i\alpha}$ reads

$$c_\alpha^{ijkl} \approx \varrho_\alpha\, a^{ijkl} + a_\alpha^{ijkl}\, \varrho \quad , \tag{A4}$$

where a_α^{ijkl} represent $c_{2\alpha}$ to $c_{22\alpha}$. Inserting (A4) into (A2), we arrive at approximate Taylor expansion

$$r_\alpha = \varrho_\alpha\, (2\varrho)^{-1}(E^i e^i - a^{ijkl} E^i P^j e^k p^l) - a_\alpha^{ijkl}\, 2^{-1} E^i P^j e^k p^l \tag{A5}$$

of the reflectivity function with respect to coefficients $c_{i\alpha}$. We see that

$$r_{1\alpha} = (2\varrho)^{-1}(E^i e^i - a^{ijkl} E^i P^j e^k p^l) \quad , \tag{A6}$$

and that

$$r_\alpha^{ijkl} = -\frac{1}{2} \sum_{(i,j,k,l)} E^i P^j e^k p^l \quad , \tag{A7}$$

where r_α^{ijkl} represent $r_{2\alpha}$ to $r_{22\alpha}$. Here $\sum_{(i,j,k,l)}$ denotes the summation over all permutations of indices i, j, k and l.

Isotropic medium can be parametrized by 3 material parameters $m_1(\mathbf{x}) = \varrho(\mathbf{x})$, $m_2(\mathbf{x}) = v_{\rm P}(\mathbf{x})$ and $m_3(\mathbf{x}) = v_{\rm S}(\mathbf{x})$, where $v_{\rm P}$ represents the P–wave propagation velocity and $v_{\rm S}$ represents the S–wave propagation velocity. The elastic moduli then depend on the selected material parameters through

$$c^{ijkl} = \varrho\,[(v_{\rm P}^2 - 2v_{\rm S}^2)\,\delta^{ij}\delta^{kl} + v_{\rm S}^2\,\delta^{ik}\delta^{jl} + v_{\rm S}^2\,\delta^{il}\delta^{jk}] \quad . \tag{A8}$$

The linear Taylor expansion of coefficients c_α^{ijkl} with respect to coefficients ϱ_α, $v_{\rm P\,\alpha}$ and $v_{\rm S\,\alpha}$ reads

$$\begin{aligned} c_\alpha^{ijkl} \approx\ & \varrho_\alpha\,[(v_{\rm P}^2 - 2v_{\rm S}^2)\,\delta^{ij}\delta^{kl} + v_{\rm S}^2\,\delta^{ik}\delta^{jl} + v_{\rm S}^2\,\delta^{il}\delta^{jk}] \\ & + v_{\rm P\,\alpha}\,2\varrho v_{\rm P}\,\delta^{ij}\delta^{kl} - v_{\rm S\,\alpha}\,2\varrho v_{\rm S}\,(2\delta^{ij}\delta^{kl} - \delta^{ik}\delta^{jl} - \delta^{il}\delta^{jk}) \quad . \end{aligned} \tag{A9}$$

Inserting (A9) into (A2), we arrive at approximate Taylor expansion

$$\begin{aligned} r_\alpha =\ & \varrho_\alpha\,(2\varrho)^{-1}[E^ie^i - (v_{\rm P}^2 - 2v_{\rm S}^2)E^iP^ie^kp^k - v_{\rm S}^2(E^ie^iP^kp^k + E^ip^iP^ke^k)] \\ & - v_{\rm P\,\alpha}\,v_{\rm P}E^iP^ie^kp^k + v_{\rm S\,\alpha}\,v_{\rm S}(2E^iP^ie^kp^k - E^ie^iP^kp^k - E^ip^iP^ke^k) \end{aligned} \tag{A10}$$

of the reflectivity function with respect to coefficients $c_{i\alpha}$. We see that

$$r_{1\alpha} = (2\varrho)^{-1}[E^ie^i - (v_{\rm P}^2 - 2v_{\rm S}^2)E^iP^ie^kp^k - v_{\rm S}^2(E^ie^iP^kp^k + E^ip^iP^ke^k)] \quad , \tag{A11}$$

$$r_{2\alpha} = -v_{\rm P}E^iP^ie^kp^k \quad , \tag{A12}$$

$$r_{3\alpha} = v_{\rm S}(2E^iP^ie^kp^k - E^ie^iP^kp^k - E^ip^iP^ke^k) \quad . \tag{A13}$$

For scattering P→P, equations (A10)–(A14) simplify to

$$r_\alpha = -\varrho_\alpha\,(2\varrho)^{-1}(1 - E^ie^i)[1 - 2v_{\rm S}^2v_{\rm P}^{-2}(1 + E^ie^i)] - v_{\rm P\,\alpha}\,v_{\rm P}^{-1} + v_{\rm S\,\alpha}\,2v_{\rm S}v_{\rm P}^{-2}[1 - (E^ie^i)^2]\,, \tag{A14}$$

$$r_{1\alpha} = -(2\varrho)^{-1}(1 - E^ie^i)[1 - 2v_{\rm S}^2v_{\rm P}^{-2}(1 + E^ie^i)] \quad , \tag{A15}$$

$$r_{2\alpha} = -v_{\rm P}^{-1} \quad , \tag{A16}$$

$$r_{3\alpha} = 2v_{\rm S}v_{\rm P}^{-2}[1 - (E^ie^i)^2] \quad . \tag{A17}$$

For scattering P→S, S→P or S→S, equations (A10)–(A14) simplify to

$$r_\alpha = \varrho_\alpha\,(2\varrho)^{-1}[E^ie^i - v_{\rm S}^2(E^ie^iP^kp^k + E^ip^iP^ke^k)] - v_{\rm S\,\alpha}\,v_{\rm S}\,(E^ie^iP^kp^k + E^ip^iP^ke^k)\,, \tag{A18}$$

$$r_{1\alpha} = (2\varrho)^{-1}[E^ie^i - v_{\rm S}^2(E^ie^iP^kp^k + E^ip^iP^ke^k)] \quad , \tag{A19}$$

$$r_{2\alpha} = 0 \quad , \tag{A20}$$

$$r_{3\alpha} = -v_{\rm S}\,(E^ie^iP^kp^k + E^ip^iP^ke^k) \quad . \tag{A21}$$

Liquid isotropic medium can be parametrized by 2 material parameters $m_1(\mathbf{x}) = \varrho(\mathbf{x})$ and $m_2(\mathbf{x}) = v_{\rm P}(\mathbf{x})$, where $v_{\rm P}$ represents the P–wave propagation velocity. The elastic moduli then depend on the selected material parameters through

$$c^{ijkl} = \varrho\,v_{\rm P}^2\,\delta^{ij}\delta^{kl} \quad . \tag{A22}$$

The linear Taylor expansion of coefficients c_α^{ijkl} with respect to coefficients ϱ_α and $v_{\rm P\,\alpha}$ reads

$$c_\alpha^{ijkl} \approx \varrho_\alpha\,v_{\rm P}^2\delta^{ij}\delta^{kl} + v_{\rm P\,\alpha}\,2\varrho v_{\rm P}\,\delta^{ij}\delta^{kl} \quad . \tag{A23}$$

Inserting (A23) into (A2), we arrive at approximate Taylor expansion

$$r_\alpha = -\varrho_\alpha\,(2\varrho)^{-1}(1 - E^ie^i) - v_{\rm P\,\alpha}\,v_{\rm P}^{-1} \tag{A24}$$

of the reflectivity function with respect to coefficients $c_{i\alpha}$. We see that

$$r_{1\alpha} = -(2\varrho)^{-1}(1 - E^ie^i) \quad , \tag{A25}$$

$$r_{2\alpha} = -v_{\rm P}^{-1} \quad . \tag{A26}$$

Acknowledgements

The research has been supported by the Grant Agency of the Czech Republic under contract 205/07/0032, by the Ministry of Education of the Czech Republic within research project MSM0021620860, and by the members of the consortium "Seismic Waves in Complex 3–D Structures" (see "http://sw3d.cz").

References

Bucha, V. & Bulant, P. (eds.) (2008): SW3D–CD–12 (DVD ROM). In: *Seismic Waves in Complex 3-D Structures, Report 18*, pp. 245–245, Dep. Geophys., Charles Univ., Prague.

Červený, V. (2001): *Seismic Ray Theory.* Cambridge Univ. Press, Cambridge.

Daubechies, I. (1992): *Ten Lectures on Wavelets.* SIAM, Philadelphia.

Klimeš, L. (2002): Application of the medium covariance functions to travel–time tomography. *Pure appl. Geophys.*, **159**, 1791–1810, online at "http://sw3d.cz".

Klimeš, L. (2007): Sensitivity of seismic waves to the structure. In: *Seismic Waves in Complex 3-D Structures, Report 17*, pp. 27–61, Dep. Geophys., Charles Univ., Prague, online at "http://sw3d.cz".

Klimeš, L. (2008a): Scalar products of the structural Gabor functions. In: *Seismic Waves in Complex 3-D Structures, Report 18*, pp. 87–93, Dep. Geophys., Charles Univ., Prague.

Klimeš, L. (2008b): Frame bounds and discretization error of the 1–D Gabor transform. In: *Seismic Waves in Complex 3-D Structures, Report 18*, pp. 95–108, Dep. Geophys., Charles Univ., Prague.

Klimeš, L. (2008c): A conjecture on the frame bounds of the multidimensional Gabor transform with complex–valued envelopes. In: *Seismic Waves in Complex 3-D Structures, Report 18*, pp. 109–114, Dep. Geophys., Charles Univ., Prague.

Klimeš, L. (2008d): Optimization of the structural Gabor functions in a homogeneous velocity model for a zero-offset surface seismic reflection survey. In: *Seismic Waves in Complex 3-D Structures, Report 18*, pp. 115–127, Dep. Geophys., Charles Univ., Prague.

Tarantola, A. (1987): *Inverse Problem Theory.* Elsevier, Amsterdam.