Ray-theoretical expansion of inhomogeneous plane waves in homogeneous isotropic elastic media

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SUMMARY

Inhomogeneous plane waves are defined as waves with an exponential decrease of amplitude along a plane wavefront and are usually treated in ray theory as waves with complex-valued slowness and polarization vectors and complex rays. The inhomogeneous waves can also be treated in real ray theory, provided higher-order ray approximations are considered. In real ray theory, the ray expansion of the inhomogeneous waves consists of an infinite number of terms and may converge to an exact solution. The inhomogeneous P wave is linearly polarized in the zero-order ray approximation and propagates with the velocity of the homogeneous wave. Higher-order ray approximations modify and correct the kinematic as well as dynamic attributes of the zero-order ray solution. They produce a correct elliptical polarization of the P wave and cause the propagation velocity to be dependent on the wave inhomogeneity.

1. INTRODUCTION

Inhomogeneous plane waves are defined as waves with an exponential decrease of amplitude along the plane wavefront and play a key role in theory of waves propagating in attenuative media. The inhomogeneous waves are usually treated in ray theory as waves

with complex-valued slowness and polarization vectors (Richards 1984; Caviglia & Morro 1992; Carcione 2001; Shuvalov 2001; Červený & Pšenčík 2005a,b). Introducing complex-valued quantities, however, causes difficulties in ray theory. Rays become trajectories in complex space and ray tracing becomes significantly complicated (Thomson 1997; Chapman et al. 1999; Kravtsov, Forbes & Asatryan 1999; Hanyga & Seredyňska 2000; Kravtsov 2005). Moreover, even very basic problems such as a complex ray tracing in media with plane interfaces poses essential troubles. Also the physical interpretation of the ray-theoretical quantities is more involved. For these reasons, it is desirable to develop theory of propagation of inhomogeneous waves in the framework of real ray theory. The real ray theory is well established and if functional on inhomogeneous waves it can remarkably spread its field of applicability.

The most straightforward way how to generalize the real ray theory on propagation of inhomogeneous waves is to apply the perturbation approach or to introduce real space-time rays. Such theories have been proposed (Suchy 1972; Rönnmark 1984; Kravtsov & Orlov 1990; Caviglia & Morro 1992; Gajewski & Pšenčík 1992; Thomson 1997; Klimeš 2002; Kravtsov 2005), and found applications in wave propagation in weakly attenuative media. Here I suggest an alternative real ray approach based on considering higher-order ray approximations. Formulas for calculating the higher-order ray approximations and for constructing the complete ray expansion are well known (Babich 1961; Červený 1972; Červený, Molotkov & Pšenčík 1977; Červený 2001), but used rather exceptionally (Daley & Hron 1987; Kiselev & Roslov 1991; Hron & Zheng 1993; Popov & Camerlynck 1995; Goldin & Ashkarin 1999; Vavryčuk & Yomogida 1995, 1996; Vavryčuk 1997, 1999, 2001). Here, I show that the ray expansion of inhomogeneous waves is meaningful and the complete ray series is well defined. The ray series has an infinite number of terms and converges to the exact solution under some conditions. I discuss properties of rays of inhomogeneous waves and their relation to the energy flux. All calculations are performed under the assumption of plane P waves propagating in a homogeneous isotropic elastic medium, which is a simple problem suitable for understanding the basic idea. The validity of the results is, however, more general, because more complicated and more realistic problems can be treated in an analogous way.

2. HIGHER-ORDER APPROXIMATIONS OF RAY THEORY

In ray theory, we seek a solution of the elastodynamic equation in a form of the ray series (Červený 2001):

$$u_i(\mathrm{x},t) = \exp\left[-i\omega\left(t-\tau\right)\right]\cdot\sum_{K=0}^{\infty}\left(\frac{i}{\omega}\right)^K U_i^{(K)}(\mathrm{x}) \quad , \tag{1}$$

K denotes the order of the ray approximation, $\tau(\mathrm{x})$ is the travel time, and $\mathrm{U}^{(K)}(\mathrm{x})$ is the ray amplitude, which can be decomposed into the additional and principal components $\mathrm{U}^{(K)\perp}$ and $\mathrm{U}^{(K)\|}$,

$$\mathrm{U}^{(K)} = \mathrm{U}^{(K)\perp} + \mathrm{U}^{(K)\|} \ , \tag{2}$$

specified for the P wave as

$$\mathrm{U}^{(K)\|} = U^{(K)\|}\mathrm{g}^{P} \ , \tag{3}$$

$$\mathrm{U}^{(K)\perp} = U^{(K)\perp SV}\mathrm{g}^{SV} + U^{(K)\perp SH}\mathrm{g}^{SH} \quad . \tag{4}$$

Vectors g^{P}, g^{SV} and g^{SH} are the polarization vectors. They form an orthogonal system of unit vectors: g^{P} is parallel to the slowness vector $\mathrm{p} = \partial\tau/\partial\mathrm{x}$, g^{SV} and g^{SH} are perpendicular to p. The slowness vector is expressed as $\mathrm{p} = \mathrm{n}/\alpha$, where n defines its direction, and $\alpha = \sqrt{(\lambda+2\mu)/\rho}$ is the P-wave propagation velocity.

Each term of the ray series is determined in two steps. In the first, the additional component $\mathrm{U}^{(K)\perp}$ is calculated as (Červený *et al.* 1977, Eqs 2.16, 2.19)

$$U_i^{(K)\perp} = -\frac{\alpha^2}{\lambda+\mu}\left(g_i^{SV}g_k^{SV} + g_i^{SH}g_k^{SH}\right)\left[M_k\left(\mathrm{U}^{PK-1)}\right) - L_k\left(\mathrm{U}^{(K-2)}\right)\right], \tag{5}$$

and the second, the principal component $\mathrm{U}^{(K)\|}$ is calculated by solving the transport equation. (Červený *et al*. 1977, Eqs 2.24, 2.26). Specifying it for a homogeneous isotropic medium, we get

$$\frac{dU^{(K)\|}}{d\tau}+\frac{U^{(K)\|}}{2J}\frac{dJ}{d\tau}=\frac{1}{2\rho}\left[L_n(\mathrm{U}^{(K-1)})-M_n(\mathrm{U}^{(K)\perp})\right]g_n^P \ , \tag{6}$$

where J is the Jacobian between the Cartesian and ray coordinates (Červený 2001, eq. 3.10.20), and the differential vector operators N, M and L are defined as follows:

$$N_i(\mathrm{U})=-\rho U_i+(\lambda+\mu)p_i p_j U_j+\mu p_j p_j U_i \ ,$$

$$M_i(\mathrm{U})=(\lambda+\mu)\left[p_j\frac{\partial U_j}{\partial x_i}+p_i\frac{\partial U_j}{\partial x_j}+U_j\frac{\partial p_i}{\partial x_j}\right]+\mu\left[2p_j\frac{\partial U_i}{\partial x_j}+U_i\frac{\partial p_j}{\partial x_j}\right] , \tag{7}$$

$$L_i(\mathrm{U})=(\lambda+\mu)\frac{\partial^2 U_j}{\partial x_i\partial x_j}+\mu\frac{\partial^2 U_i}{\partial x_j\partial x_j} \ .$$

3. RAY EXPANSION

Let us consider a time-harmonic inhomogeneous plane P wave propagating along the x-axis. The zero-order ray approximation is described by the following displacement vector:

$$\mathrm{u}^{(0)}(\mathrm{x},t)=\mathrm{U}^{(0)}(\mathrm{x})\exp[-i\omega(t-px)] \ , \tag{8}$$

where ω is positive and $p=1/\alpha$. Since the plane wave is inhomogeneous, the zero-order ray amplitude $\mathrm{U}^{(0)}(\mathrm{x})$ is expressed as

$$\mathrm{U}^{(0)}(\mathrm{x})=\hat{U}^{(0)}(x)\exp[-Dz]\,\mathrm{g}^P \ , \tag{9}$$

where $\hat{U}^{(0)}(x)$ is the zero-order scalar amplitude, from which exponential factor $\exp[-Dz]$ is removed, g^P is the P-wave polarization vector, and D is the wave inhomogeneity. The P, SV and SH polarization vectors are defined as:

$$\mathrm{g}^P = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathrm{g}^{SV} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathrm{g}^{SH} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}. \tag{10}$$

Imposing the initial condition

$$\mathrm{u}(x = 0, z, t = 0) = \exp[-Dz]\,\mathrm{g}^P, \tag{11}$$

we get

$$\hat{U}^{(0)}(x = 0) = 1. \tag{12}$$

Taking into account that rays of plane waves are parallel to the x-axis, the ray Jacobian equals 1 and transport equation (6) yields:

$$U^{(K)\|} = \frac{1}{2\rho}\int_{x_0}^{x}\left[L_n(\mathrm{U}^{(K-1)}) - M_n(\mathrm{U}^{(K)\perp})\right]g_n^P d\tau + C_0^{(K)}, \tag{13}$$

where $C_0^{(K)}$ are the integration constants, which must be determined from initial conditions. To simplify the problem, we impose condition (11) which guarantees that all integration constants for $K > 0$ are zero.

Applying equations (5) and (13), we can calculate all higher-order ray approximations from the zero-order approximation (8-9) and subsequently construct a complete ray expansion. The ray expansion has an infinite number of terms consisting of principal as well as additional components. The additional components of the first five ray amplitudes read (exponential $\exp[-Dz]$ term is omitted):

$$U^{(0)\perp SV} = 0,$$

$$U^{(1)\perp SV} = \sqrt{2C},$$

$$U^{(2)\perp SV} = \sqrt{2C}C\tau, \tag{14}$$

$$U^{(3)\perp SV} = \sqrt{2C}\left(\frac{1}{2}C^2\tau^2 + C\right),$$

$$U^{(4)\perp SV} = \sqrt{2C}\left(\frac{1}{6}C^3\tau^3 + \frac{3}{2}C^2\tau\right),$$

The additional components polarized in the direction of the SH wave, $U^{(K)\perp SH}$, are zero. The principal components of the first five ray amplitudes are expressed as follows (exponential term $\exp[-Dz]$ is omitted):

$$U^{(0)\parallel} = 1,$$

$$U^{(1)\parallel} = C\tau,$$

$$U^{(2)\parallel} = \frac{1}{2}C^2\tau^2, \tag{15}$$

$$U^{(3)\parallel} = \frac{1}{6}C^3\tau^3 + \frac{1}{2}C^2\tau,$$

$$U^{(4)\parallel} = \frac{1}{24}C^4\tau^4 + \frac{1}{2}C^3\tau^2,$$

where

$$C = \frac{1}{2}\alpha^2 D^2 \ . \tag{16}$$

The rules for calculating ray approximations of arbitrary order K are in Appendix A. Summing all terms of the P-wave ray expansion, we can express the additional and principal components as

$$\mathrm{u}^{\parallel}(\mathrm{x},t) = \mathrm{g}^{P} \exp(-Dz) \exp[-i\omega(t - p_C x)] \ , \tag{17}$$

$$\mathrm{u}^{\perp}(\mathrm{x},t) = \frac{iD}{\omega p_C} \mathrm{g}^{SV} \exp(-Dz) \exp[-i\omega(t - p_C x)] \ , \tag{18}$$

and write the complete ray expansion in the following compact form:

$$\mathrm{u}(\mathrm{x},t) = \mathrm{g}_C^{P} \exp(-Dz) \exp[-i\omega(t - p_C x)] \ , \tag{19}$$

where

$$\mathrm{g}_C^{P} = \left(1, 0, \frac{iD}{\omega p_C}\right)^T , \quad p_C = \frac{1}{\alpha}\sqrt{1 + \frac{\alpha^2 D^2}{\omega^2}} , \ D \le \frac{\omega}{\alpha} \quad . \tag{20}$$

Quantities g_C^{P} and p_C are called the "corrected" polarization vector and the "corrected" slowness.

4. EXACT SOLUTION

Let us calculate the exact solution for the inhomogeneous plane P wave directly from the elastodynamic equation. The wave can be expressed as:

$$\mathrm{u}(\mathrm{x},t)=\mathrm{g}^{P}\exp\left[-i\omega(t-\mathrm{s}\cdot\mathrm{x})\right], \tag{21}$$

where polarization vector g^{P} and slowness vector s are, in general, complex-valued. Since the wave propagates along the *x*-axis, the slowness vector can be expressed as

$$\mathrm{s}=\left(p_E,0,i\frac{D}{\omega}\right)^{T}. \tag{22}$$

where p_E is the real-valued slowness, and D is the real-valued wave inhomogeneity. Inserting (21) into the elastodynamic equation we obtain the following complex-valued eikonal equation,

$$\alpha^2 s_j s_j=1. \tag{23}$$

Consequently, the real-valued slowness p_E and the unit polarization vector g_E^P come out as

$$p_E=\frac{1}{\alpha}\sqrt{1+\frac{\alpha^2D^2}{\omega^2}},\quad \mathrm{g}_E^P=\frac{1}{\sqrt{p_E^2+\dfrac{D^2}{\omega^2}}}\left(p_E,0,i\frac{D}{\omega}\right)^{T}. \tag{24}$$

Renormalizing g_E^P according to the initial condition (11) we obtain

$$\mathrm{g}_E^P=\left(1,0,\frac{iD}{\omega p_E}\right)^{T}. \tag{25}$$

Note that complex-valued polarization vectors mean physically that the wave is elliptically polarized.

5. EXACT VERSUS RAY SOLUTIONS

Comparing (20) with (24-25), we find that the ray series yields an exact result. Hence, the complete ray series is capable to correct the approximate zero-order term: the approximate slowness $p = 1/\alpha$ is corrected to the exact slowness $p_C = p\sqrt{1+\alpha^2 D^2/\omega^2}$, and the approximate linear polarization (10) is corrected to the exact elliptical polarization (25). It implies that higher-order ray approximations can modify and correct kinematic as well as dynamic attributes of the zero-order ray approximation.

Obviously, this correction works only under the condition that the ray series converges. This constrains plausible values of wave inhomogeneity *D*, which must be less than or equal to ω/α. For $D > \omega/\alpha$, the ray series diverges. For $D \le \omega/\alpha$, the ray series converges, and the lower the value of *D*, the faster the convergence. Consequently, the lower the value of *D*, the lower the number of the higher-order ray approximations being sufficient to meet to required accuracy. For $D = 0$, the inhomogeneous wave becomes homogeneous, all higher-order ray approximations vanish, and the zero-order ray approximation describes the exact solution.

The convergence of the ray series depends also on the travel time. The larger the travel time, the slower the convergence. If we require the higher-order ray approximations to be negligible, then $\frac{C\tau}{\omega} = \frac{1}{2}\frac{\alpha^2 D^2 \tau}{\omega}$ must be small.

6. RAYS AND ENERGY FLUX

Similarly as for any other wave having constant or varying amplitude along the wavefront, the ray of an inhomogeneous wave is normal to the wavefront and coincides with the energy flux direction of the zero-order ray approximation. Obviously, this direction can be different from the direction of the exact energy flux, which is time dependent for an elliptically polarized inhomogeneous P wave. The difference between the ray and the direction of the exact energy flux is produced by higher-order ray approximations. They cause that energy can partly flow through walls of the ray tube.

The discrepancy between ray and energy flux directions is well known for wave fields generated by point sources, provided that observation points are close to a source. In this case, the anomalies in the energy flow are associated with the existence of near-field waves, which are also described by the higher-order ray approximations (Vavryčuk & Yomogida 1995, 1996; Vavryčuk 2001).

7. DISCUSSION

The calculations indicate that inhomogeneous waves can be treated as any other wave with a varying amplitude along a wavefront provided the inhomogeneity of the wave is less than or equal to ω/c. If we further require the higher-order approximations to be negligible, the product $\frac{C\tau}{\omega}$ must be small. For attenuative media, this condition should not be very restrictive. For low values of C, which are generated in weakly dissipative media, rays can be traced to far distances and large travel times. For high values of C, generated in strongly dissipative media, energy is strongly dissipated and it is sufficient to trace rays to short distances and small travel times.

If the above convergence criteria are satisfied, the rays of inhomogeneous waves can be calculated in elastic media as trajectories in the real space using standard ray-tracing procedures. Also ray amplitudes can be calculated in smooth media using standard transport equation, and in media with interfaces using reflection/transmission coefficients derived for homogeneous waves. The calculation of rays and amplitudes of the inhomogeneous waves in attenuative media will be more involved but keeps within the real ray theory. Practical aspects of the proposed approach such as its efficiency and accuracy when applied to realistic wave propagation problems should be investigated in later numerical studies.

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REFERENCES

Babich, V.M., 1961. Ray method for the computation of the intensity of wave fronts in elastic inhomogeneous anisotropic medium, in: *Problems of the Dynamic Theory of Propagation of Seismic Waves*, 5, Leningrad University Press, 36-46, republished in: *Geophys. J. Int.*, 118, 379-383, 1994.

Carcione, J.M., 2001. *Wave Fields in Real Media: Wave Propagation in Anisotropic, Anelastic and Porous Media*, Pergamon, Amsterdam.

Caviglia, G. & Morro, A., 1992. *Inhomogeneous Waves in Solids and Fluids*, World Scientific, London.

Červený, V., 1972. Seismic rays and ray intensities in inhomogeneous anisotropic media, *Geophys. J. Roy. Astr. Soc.*, 29, 1-13.

Červený, V., 2001. *Seismic Ray Theory*, Cambridge University Press, Cambridge, U.K.

Červený, V., Molotkov, I. & Pšenčík, I., 1977. *Ray Method in Seismology*, Charles University Press, Prague.

Červený, V. & Pšenčík, I., 2005a. Plane waves in viscoelastic anisotropic media – I. Theory, *Geophys. J. Int.*, 161, 197-212.

Červený, V. & Pšenčík, I., 2005b. Plane waves in viscoelastic anisotropic media – II. Numerical examples. *Geophys. J. Int.*, 161, 213-229.

Chapman, S.J., Lawry, J.M.H., Ockendon, J.R. & Tew, R.H., 1999. On the theory of complex rays. *SIAM Review*, 41, 417-509.

Daley, P.F. & Hron, F., 1987. Reflection of an incident spherical P wave on a free surface (near-vertical incidence), *Bull. Seism. Soc. Am.*, 77, 1057-1070.

Gajewski, D. & Pšenčík, I., 1992. Vector wavefields for weakly attenuating anisotropic media by the ray method, *Geophysics*, 57, 27-38.

Goldin, S.V. & Ashkarin, L.I., 1991. Ray analysis of seismic waves in model situations, in *Methods of Computation and Interpretation of Seismic Wavefields*, 95-125, eds Krylov, S.V. & Goldin, S.V., Nauka, Novosibirsk (in Russian).

Hanyga, A. & Seredyńska, M., 2000. Ray tracing in elastic and viscoelastic media, *Pure appl. Geophys*., 157, 679-717.

Hron, F. & Zheng, B.S., 1993. On the longitudinal component of the particle motion carried by the shear PS wave reflected from the free surface at normal incidence, *Bull. Seism. Soc. Am*., 83, 1610-1616.

Kiselev, A.P. & Roslov, Yu.V., 1991. Use of additional components for numerical modeling of polarization anomalies of elastic body waves, *Sov. Geol. Geophys*., 32, 105-114.

Klimeš, L., 2002. Second-order and higher-order perturbations of travel time in isotropic and anisotropic media, *Stud. Geophys. Geod.*, 46, 213-248.

Kravtsov, Yu.A., 2005. *Geometrical Optics in Engineering Physics*, Alpha Science, Harrow, U.K.

Kravtsov, Yu.A., Forbes, G.W. & Asatryan, A.A., 1999. Theory and applications of complex rays, in *Progress in Optics*, ed. E. Wolf (Amsterdam, Elsevier 1999), 39, 3-62.

Kravtsov, Yu.A. & Orlov, Yu.I., 1990. *Geometrical Optics of Inhomogeneous Media*, Springer-Verlag, Berlin.

Popov, M.M. & Camerlynck, C., 1995. Second term of the ray series and validity of ray theory, *J. Geophys. Res.*, 101, 817-826.

Richards, P.G., 1984. On wavefronts and interfaces in anelastic media, *Bull. Seism. Soc. Am.,* 74, 2157-2165.

Rönnmark, K., 1984. Ray tracing in dissipative media, *Annales Geophysicae*, 2, 57-60.

Shuvalov, A.L., 2001. On the theory of plane inhomogeneous waves in anisotropic elastic media, *Wave Motion*, 34, 401-429.

Suchy, K., 1972. Ray tracing in an anisotropic absorbing medium, *J. Plasma Physics*, 8, 53-65.

Thomson, C.J., 1997. Complex rays and wave packets for decaying signals in inhomogeneous, anisotropic and anelastic media, *Stud. Geophys. Geod.*, 41, 345-381.

Vavryčuk, V., 1997. Elastodynamic and elastostatic Green tensors for homogeneous weak transversely isotropic media, *Geophys. J. Int*., 130, 786-800.

Vavryčuk, V., 1999. Applicability of higher-order ray theory for S wave propagation in inhomogeneous weakly anisotropic elastic media, *J. Geophys. Res.*, 104, No. B12, 28.829-28.840.

Vavryčuk, V., 2001. Exact elastodynamic Green functions for simple types of anisotropy derived from higher-order ray theory, *Stud. Geophys. Geod.*, 45, 67-84.

Vavryčuk, V. & Yomogida, K., 1995. Multipolar elastic fields in homogeneous isotropic media by higher-order ray approximations, *Geophys. J. Int.*, 121, 925-932.

Vavryčuk, V. & Yomogida, K., 1996. SH-wave Green tensor for homogeneous transversely isotropic media by higher-order approximations in asymptotic ray theory, *Wave Motion*, 23, 83-93.

APPENDIX A: RECURSIVE EQUATIONS FOR RAY AMPLITUDES OF THE P WAVE

The additional and principal components of the ray amplitudes of the *K*th-order can be obtained using the following recursive formulas:

1. For $K \le 2$, the additional components read:

$$U^{(0)\perp SV} = 0, \quad U^{(1)\perp SV} = \sqrt{2C}, \quad U^{(2)\perp SV} = \sqrt{2C}C\tau, \tag{A1}$$

and the principal components read:

$$U^{(0)\parallel} = 1, \quad U^{(1)\parallel} = C\tau, \quad U^{(2)\parallel} = \frac{1}{2}C^2\tau^2, \tag{A2}$$

where

$$C = \frac{1}{2}\alpha^2 D^2 \quad . \tag{A3}$$

2. For $K > 2$, the amplitudes $U^{(K)\perp}$ and $U^{(K)\parallel}$ can be calculated from the amplitudes $U^{(K-1)\perp}$ and $U^{(K-2)\parallel}$ in the following way. Assuming that

$$U^{(K-1)\perp} = \sum_{n=0}^{K-1} A_n \tau^n, \quad U^{(K-1)\parallel} = \sum_{n=1}^{K-1} B_n \tau^n, \tag{A4}$$

where are A_n and B_n are constants and τ is the travel time, then

$$U^{(K)\perp} = A_0 C\tau + A_1\left(\frac{1}{2}C\tau^2 + 1\right) + \sum_{n=2}^{K-1} A_n\left(C\frac{\tau^{n+1}}{n+1} + \frac{1}{2}n\tau^{n-1}\right), \tag{A5}$$

$$U^{(K)\parallel} = \frac{1}{2}B_1 C\tau^2 + \sum_{n=2}^{K-1} B_n\left(C\frac{\tau^{n+1}}{n+1} + \frac{1}{2}n\tau^{n-1}\right). \tag{A6}$$