# Attenuation vector in heterogeneous, weakly dissipative, anisotropic media

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## Summary

Attenuation vector $\mathbf{A}$ plays an important role in the investigation of waves propagating in dissipative, isotropic or anisotropic media. It is defined as the imaginary part of the complex-valued slowness vector $\mathbf{p} = \mathbf{P} + \mathrm{i}\mathbf{A}$. In this paper, simple expressions for the attenuation vector $\mathbf{A}$ are derived for seismic body waves propagating in heterogeneous, weakly dissipative, anisotropic media. Ray-theory perturbation method is used, in which the dissipative medium is considered to be a small perturbation of perfectly elastic medium. The general perturbation procedure proposed by Klimeš (2002) is very suitable for this purpose, as it does not require to perform ray tracing in dissipative media; the computation of rays in the background perfectly elastic medium, followed by quadratures along them, is quite sufficient. It is shown that the waves propagating in a heterogeneous, weakly dissipative, anisotropic or isotropic medium are, in general, inhomogeneous. This means that their attenuation vector $\mathbf{A}$ is not parallel to the propagation vector $\mathbf{P}$. This holds also for waves generated by a point source, propagating in homogeneous anisotropic dissipative media. Exception is a wave generated by a point source in a homogeneous isotropic dissipative medium, where the generated wave is homogeneous. Thus, the commonly used concept of homogeneous waves, used often in isotropic dissipative media, cannot be used in anisotropic dissipative media. The situation is different for plane waves propagating in homogeneous dissipative isotropic or anisotropic media. In this case, the homogeneity or inhomogeneity of the plane wave depends on initial conditions, which may be chosen freely. It is also shown that the intrinsic attenuation factor $\mathcal{A}^{in}$, defined as twice the projection of $\mathbf{A}$ into the direction of the reference ray, does not depend on the weak inhomogeneity of the wave, and represents a convenient measure of material dissipation. The intrinsic attenuation factor is used to determine the position and direction dependent quality factor $Q$ for heterogeneous anisotropic weakly dissipative media.

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# 1 Introduction

In seismic ray methods, perturbation approaches play a very important role. We assume that a model, in which we wish to study the wave propagation, differs only little from another model, called the *background*, or *reference* model. The solution in the perturbed model can be then sought in the form of a power series in the deviations of the perturbed and reference models. When only the first term of the series is considered, we speak of the first-order perturbation expansion, or briefly the *first-order perturbation*.

Often, the perturbations of the travel time and/or of its derivatives in perfectly elastic media are sought. The relevant first-order perturbation formula for the travel time has been known for a long time. For isotropic reference and perturbed models, see Aki & Richards (1980), where many other references can be found. For anisotropic, perfectly elastic perturbed model, see Červený (1982), Červený & Jech (1982), Hanyga (1982), Farra (1989), Jech & Pšenčík (1989), Chapman & Pratt (1992), etc. For a review and other references, see Červený (2001, sec. 3.9). In these approaches, a reference ray is traced in reference, isotropic or anisotropic, medium, and the first-order travel-time correction is sought by quadratures along the reference ray. Even second-order perturbation of the travel time can be calculated in a similar way. Farra (1999) proposed a procedure for calculating the second-order travel-time perturbation, based on the calculation of first-order perturbation of the reference ray. A different procedure by Snieder & Sambridge (1992) is also based on the perturbation of rays. These procedures can be also adopted to compute the first-order perturbation of the slowness vector. For example, the procedure by Farra (1999) was used to calculate first-order perturbation of the slowness vector by Červený (2002).

A fully general procedure for computation of higher-order perturbations of the travel time and its derivatives (slowness vector, etc.) along the reference ray was proposed by Klimeš (2002). There are two great advantages of the procedure proposed by Klimeš (2002): a) The procedure is fully based on the computation and quadratures along the reference ray in the reference medium. The reference ray itself need not be perturbed. b) The procedure is quite universal and general, and may be recursively used to calculate the perturbations of any order of the travel time and its derivatives of any order. In the procedure, the dynamic ray tracing along the reference ray in the reference medium plays a very important role. For more details, see Section 2 and Klimeš (2002). A brief exposition of the procedure can be also found in Červený, Klimeš & Pšenčík (2007, sec. 3).

The above discussed procedures may be applied even to weakly dissipative, isotropic and anisotropic, media. In dissipative media, however, the rays are complex-valued. The complex-valued ray-tracing procedures for dissipative media usually do not belong to standard computing facilities at seismological institutions. For this reason, it is useful to choose the reference medium perfectly elastic, and consider a real-valued reference ray in perfectly elastic media. Standard, generally available routines can be then used to perform ray tracing and dynamic ray tracing, both for isotropic and anisotropic media. The results for weakly dissipative (perturbed) medium are then obtained by quadratures along the real-valued reference ray. For example, we can describe the reference medium by real-valued elastic moduli, and the perturbed medium by complex-valued viscoelastic moduli, in which the real-valued part is the same as in reference medium. Consequently, the perturbation of the medium is imaginary-valued. This approach immediately yields that the first-order perturbation of travel time is imaginary-valued, and represents the attenuation along the ray. For isotropic weakly dissipative media, see Moczo et al. (1987), and for anisotropic weakly dissipative media, see Gajewski & Pšenčík (1992).

The slowness vector $\mathbf{p}$ in the perturbed dissipative medium is also complex-valued in this case, $\mathbf{p} = \mathbf{P} + \mathrm{i}\mathbf{A}$, where $\mathbf{P}$ is called the propagation vector, and $\mathbf{A}$ the attenuation vector. For $\mathbf{P}$ and $\mathbf{A}$ parallel, we speak of *homogeneous waves*, but for $\mathbf{P}$ and $\mathbf{A}$ nonparallel, we speak of *inhomogeneous waves*. The inhomogeneous waves play a very important role in dissipative media, see Declercq et al. (2005). For this reason, it is of a great interest to know, whether (or when) the imaginary-valued perturbation of elastic moduli generates homogeneous or inhomogeneous waves, propagating in perturbed dissipative medium.

The first attempt to study the slowness vector of waves propagating in weakly dissipative media by perturbation methods was described in Červený & Pšenčík (2002). The authors used the procedure by Farra (1999), modified by Červený (2002), for imaginary-valued perturbation of elastic moduli. They derived a complex-valued perturbed ray in perturbed weakly dissipative medium. From the equation for the complex-valued ray, they easily obtained the complex-valued perturbation of the slowness vector. The method of Farra (1999) is very suitable to find the equation of the complex-valued ray in weakly dissipative medium. In the study of the perturbation of the slowness vector, however, the computation of the perturbed complex-valued ray is a nonnecessary complication. Nevertheless, the results suggest that the waves in weakly dissipative anisotropic media *are in general inhomogeneous*.

In this brief paper, we will study the complex-valued slowness vector $\mathbf{p} = \mathbf{P} + \mathrm{i}\mathbf{A}$ in a weakly dissipative, isotropic or anisotropic, heterogeneous medium, using the method of Klimeš (2002). The advantage of this method is that it does not require to calculate the complex-valued perturbed ray in the dissipative medium. The procedure is quite straightforward. For heterogeneous media, it requires one integration along the reference real-valued ray. For homogeneous media, the integration is not required, and the result is obtained analytically, by closed-form formulae.

Briefly to the content of this paper. In Section 2, the general perturbation procedure proposed by Klimeš (2002) is briefly described. Specific equations for the perturbation of slowness vector along a reference ray are presented. Section 3 introduces the Hamiltonian used in the paper, corresponding to the imaginary-valued perturbation of elastic moduli. Section 4 presents final perturbation equations for the slowness vector in heterogeneous weakly dissipative, isotropic or anisotropic media. Particular attention is devoted to the attenuation vector $\mathbf{A}$. Specific analytical results for homogeneous media are discussed in Section 5.

In the whole paper, we use Cartesian coordinates $x_i$, and time $t$. The lower case Roman indices take values 1,2,3, the uppercase indices 1,2. The Greek indices take any value from 1,2,... . The Einstein summation convention over Roman indices is used. For time-harmonic waves, we consider the exponential time factor $\exp[-\mathrm{i}\omega t]$, where $\omega$ is a fixed, real-valued and positive circular frequency.

## 2 General perturbation procedure

The procedure was proposed by Klimeš (2002), where it is also described in detail. Here we will explain it only briefly, and take over the needed equations from Klimeš (2002), without a derivation.

We will use the same notation and terminology as Klimeš (2002). We assume that the Hamiltonian $\mathcal{H}$ is a function of phase-space coordinates (spatial coordinates $x_i$, slowness vector components $p_i$), and of any number of model parameters $f_\kappa$,

$$\mathcal{H} = \mathcal{H}(x_i, p_j, f_\kappa) \ . \tag{1}$$

The travel time and other quantities in the model are assumed to be functions of spatial coordinates $x_i$ and model parameters $f_\kappa$,

$$\tau = \tau(x_i, f_\kappa) \ . \tag{2}$$

We also introduce the following notations for partial derivatives: The partial derivatives with respect to spatial Cartesian coordinates $x_i$ are denoted by lower-case Roman subscripts following a comma, taking the value of 1, 2 or 3. Similarly, the partial derivatives with respect to model parameters by the lower-case Greek subscripts, following a comma, taking any integer value. The partial derivatives with respect to slowness-vector components are denoted by lower-case Roman superscripts, following a comma, taking the value of 1, 2 or 3. We present two examples:

$$\tau_{,ij\ldots n\alpha\beta\ldots\nu} = \frac{\partial}{\partial f_\alpha}\frac{\partial}{\partial f_\beta}\cdots\frac{\partial}{\partial f_\nu}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\cdots\frac{\partial}{\partial x_n}\tau(x_i, f_\kappa) \ , \tag{3}$$

$$\mathcal{H}^{,ab\ldots f}_{,ij\ldots n\alpha\beta\ldots\nu} = \frac{\partial}{\partial f_\alpha}\frac{\partial}{\partial f_\beta}\cdots\frac{\partial}{\partial f_\nu}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\cdots\frac{\partial}{\partial x_n}\frac{\partial}{\partial p_a}\frac{\partial}{\partial p_b}\cdots\frac{\partial}{\partial p_f}\mathcal{H}(x_i, p_j, f_\kappa) \ . \tag{4}$$

We consider a known "reference ray" in the reference model and assume that the dynamic ray tracing along the reference ray has been performed. The dynamic ray tracing system consists of six ordinary differential equations for $Q_{ia}$ and $P_{ia}$ along the ray, and reads

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\gamma_3}Q_{ia} &= \mathcal{H}^{,i}_{,j}Q_{ja} + \mathcal{H}^{,ij}P_{ja} \ , \\ \frac{\mathrm{d}}{\mathrm{d}\gamma_3}P_{ia} &= -\mathcal{H}_{,ij}Q_{ja} - \mathcal{H}^{,j}_{,i}P_{ja} \ . \end{aligned} \tag{5}$$

At present, the dynamic ray tracing system is included in most of computer packages based on the ray method, and its solutions for proper initial conditions does not cause any difficulties. We define the $3 \times 3$ matrices $Q_{ia}$ and $P_{ia}$, computed by dynamic ray tracing, in the following way: $Q_{ia}$ represent the partial derivative of $x_i$, with respect to the ray parameter $\gamma_a$, and $P_{ia}$ the partial derivative of the slowness vector component $p_i = \tau_{,i}$ with respect to the same ray parameter,

$$Q_{ia} = \partial x_i / \partial \gamma_a \ , \qquad P_{ia} = \partial \tau_{,i} / \partial \gamma_a \ . \tag{6}$$

Here $\gamma_1$ and $\gamma_2$ are the take-off ray parameters, and $\gamma_3$ the independent parameter (variable) along the ray, determined by the Hamiltonian. Here we shall use $\gamma_3$ to correspond to the travel time along the unperturbed ray. The $3 \times 3$ matrix $Q_{ia}$ is usually called the spreading matrix. Note that $Q_{ia}$ and $P_{ia}$ are real-valued and are assumed to be known. It also follows from (6) that

$$P_{ia} = \tau_{,ij} Q_{ja} \ . \tag{7}$$

Klimeš (2002) derived a general equation for $\tau_{,ij...n\alpha\beta...\nu}$, that is for the perturbation of any order $(\alpha\beta...\nu)$ of the travel-time derivative of any order $(ij...n)$. It reads

$$\tau_{,ij...n\alpha\beta...\nu} = T_{ab...f\alpha\beta...\nu} Q^{-1}_{ai} Q^{-1}_{bj} ... Q^{-1}_{fn} \ , \tag{8}$$

where

$$\begin{aligned} T_{ab...f\alpha\beta...\nu}(\gamma_3) &= T_{ab...f\alpha\beta...\nu}(\gamma_3^0) \\ &+ \int_{\gamma_3^0}^{\gamma_3} K_{ij...n\alpha\beta...\nu} Q_{ia} Q_{jb} ... Q_{nf} \mathrm{d}\gamma_3' \ . \end{aligned} \tag{9}$$

The kernels $K_{ij...n\alpha\beta...\nu}$ are given in Klimeš (2002, eq.21). They contain the partial derivatives of $\tau$ of a lower order then the computed $\tau_{,ij...n\alpha\beta...\nu}$. Consequently, eqs. (8) and (9) must be used recursively, starting from lower-order partial derivatives. For higher-order derivatives of $\tau$ the algorithm may lead to cumbersome computations. For lower-order derivatives, however, it yields simple results.

Here we will not give the general expressions for $K_{ij..n\alpha\beta..\nu}$, as we do not need them. We only present the first three expressions:

$$K_{\alpha} = -\mathcal{H}_{,\alpha} \ , \tag{10}$$
$$K_{i\alpha} = -\mathcal{H}_{,i\alpha} - \mathcal{H}^{,r}_{,\alpha} \tau_{,ri} \ , \tag{11}$$
$$K_{\alpha\beta} = -\mathcal{H}_{,\alpha\beta} - \mathcal{H}^{,r}_{,\alpha} \tau_{,r\beta} - \mathcal{H}^{,r}_{,\beta} \tau_{,r\alpha} - \mathcal{H}^{,rs} \tau_{,r\alpha} \tau_{,s\beta} \ . \tag{12}$$

The kernel (10) can be used to determine the first-order perturbation of the travel time, the kernel (11) the first-order perturbation of the slowness vector, and the kernel (12) the second-order perturbation of the travel time. Note that $T_{3\alpha}(\gamma_3)$ can be computed without the integration;

$$T_{3\alpha}(\gamma_3) = -\mathcal{H}_{,\alpha}(\gamma_3) \ . \tag{13}$$

Then the perturbation of the slowness vector, $\tau_{,i\alpha}(\gamma_3)$, can be computed using the formula:

$$\tau_{,i\alpha}(\gamma_3) = T_{a\alpha}(\gamma_3) Q^{-1}_{ai}(\gamma_3) \ , \tag{14}$$

where

$$
\begin{aligned}
T_{K\alpha}(\gamma_3) &= T_{K\alpha}(\gamma_3^0) - \int_{\gamma_3^0}^{\gamma_3} [\mathcal{H}_{,i\alpha}(\gamma_3')Q_{iK}(\gamma_3') + \mathcal{H}^{\,i}_{,\alpha}(\gamma_3')P_{iK}(\gamma_3')]\mathrm{d}\gamma_3' \,, \\
T_{3\alpha}(\gamma_3) &= -\mathcal{H}_{,\alpha}(\gamma_3) \,. \qquad (15)
\end{aligned}
$$

Equation (14), with (15), represents the final expression for the perturbation of the slowness vector. The Einstein summation convention is applied to Roman (but not Greek) indices. Thus, the computation of $\tau_{,i\alpha}(\gamma_3)$ requires three quadratures along the reference ray (i=1,2,3).

We can also express eqs. (14) and (15) for $\tau_{,i\alpha}(\gamma_3)$ in a matrix form. We introduce $3 \times 1$ column matrices $\boldsymbol{\tau}$ and $\mathbf{T}$ as follows:

$$
\boldsymbol{\tau} = \begin{pmatrix} \tau_{,1\alpha} \\ \tau_{,2\alpha} \\ \tau_{,3\alpha} \end{pmatrix} \,, \qquad \mathbf{T} = \begin{pmatrix} T_{1\alpha} \\ T_{2\alpha} \\ T_{3\alpha} \end{pmatrix} \,. \qquad (16)
$$

Then (14) yields

$$
\boldsymbol{\tau}(\gamma_3) = \mathbf{Q}^{-T}(\gamma_3)\mathbf{T}(\gamma_3) \,. \qquad (17)
$$

Here $\boldsymbol{\tau}(\gamma_3)$ is a sought matrix, $\mathbf{T}(\gamma_3)$ is given by (16) with (15), and $\mathbf{Q}(\gamma_3)$ is the spreading matrix, known from dynamic ray tracing.

For completeness, we shall also write the expression for the second-order perturbation of the travel time, although we shall not need it in this paper. It reads

$$
\tau_{,\alpha\beta}(\gamma_3) = T_{\alpha\beta}(\gamma_3) \,, \qquad (18)
$$

with

$$
\begin{aligned}
T_{\alpha\beta}(\gamma_3) &= T_{\alpha\beta}(\gamma_3^0) \\
&- \int_{\gamma_3^0}^{\gamma_3} [\mathcal{H}_{,\alpha\beta} + \mathcal{H}^{;r}_{,\alpha}\tau_{,r\beta} + \mathcal{H}^{;r}_{,\beta}\tau_{,r\alpha} + \mathcal{H}^{;rs}\tau_{,r\alpha}\tau_{,s\beta}]\mathrm{d}\gamma_3' \,. \qquad (19)
\end{aligned}
$$

All functions in the integral are functions of $\gamma_3'$.

# 3 Hamiltonians

In the zero-order approximation of the ray method, the displacement vector of time-harmonic waves $\mathbf{u}(\mathbf{x}, t)$ may be expressed in the following form:

$$
\mathbf{u}(\mathbf{x}, t) = a(\mathbf{x})\mathbf{U}(\mathbf{x}) \exp[-\mathrm{i}\omega(t - \tau(\mathbf{x}))] \,. \qquad (20)
$$

Here $\omega$ is a real-valued, positive, fixed frequency, $\tau(\mathbf{x})$ represents the travel time, $\mathbf{U}(\mathbf{x})$ the normalized polarization vector ($\mathbf{U} \cdot \mathbf{U} = 1$), and $a(\mathbf{x})$ the amplitude factor. The functions $\tau(\mathbf{x})$, $\mathbf{U}(\mathbf{x})$ and $a(\mathbf{x})$ should be determined from the elastodynamic equation, and from boundary conditions. For inhomogeneous anisotropic media, the source-free elastodynamic equation reads:

$$
(c_{ijkl}u_{k,l})_{,j} = \rho \ddot{u}_i \,, \qquad (21)
$$

where $\rho = \rho(\mathbf{x})$ is the density, and $c_{ijkl} = c_{ijkl}(\mathbf{x})$ elastic moduli. See Fedorov (1968), Musgrave (1970), Helbig (1994), and many others. For perfectly elastic media, $c_{ijkl}$ are real-valued and frequency-independent, but for viscoelastic media, they are complex-valued and may depend on $\omega$,

$$c_{ijkl} = c^R_{ijkl} - \mathrm{i}c^I_{ijkl} \ . \tag{22}$$

We do not write explicitely $\omega$ as an argument of $c_{ijkl}$, as we consider $\omega$ fixed. The minus sign in (22) is related to the minus sign in the exponential factor $\exp[-\mathrm{i}\omega t]$, see (20). We consider only such "physical" models, for which $c^R_{ijkl}$ in the Voigt notation are positive definite, and $c^I_{ijkl}$ positive-definite or zero. The case of $c^I_{ijkl} = 0$ corresponds to the perfectly elastic medium.

Inserting (20) into (21) yields

$$-N_i(\mathbf{U}) + \mathrm{i}\omega^{-1}M_i(\mathbf{U}) + \omega^{-2}L_i(\mathbf{U}) = 0 \ , \tag{23}$$

where

$$\begin{aligned} N_i(\mathbf{U}) &= c_{ijkl}p_jp_lU_k - \rho U_i \ , \\ M_i(\mathbf{U}) &= c_{ijkl}p_jU_{k,l} + (c_{ijkl}p_lU_k)_{,j} \ , \\ L_i(\mathbf{U}) &= (c_{ijkl}U_{k,l})_{,j} \ . \end{aligned} \tag{24}$$

Here $p_i$ are the Cartesian components of the slowness vector $\mathbf{p}$, given by relation

$$p_i = \tau_{,i} = \partial\tau/\partial x_i \ . \tag{25}$$

In perfectly elastic media, $\tau(\mathbf{x})$, $\mathbf{p}(\mathbf{x})$ and $\mathbf{U}(\mathbf{x})$ are frequency independent. Consequently, $N_i$, $M_i$ and $L_i$ in (23) are also frequency independent. Since (23) should be satisfied for any frequency $\omega$, $N_i$, $M_i$ and $L_i$ should equal zero. We shall consider here $N_i(\mathbf{U}) = 0$, that is

$$a_{ijkl}p_jp_lU_k = U_i \ , \qquad i = 1, 2, 3 \ . \tag{26}$$

Here

$$a_{ijkl}(\mathbf{x}) = c_{ijkl}(\mathbf{x})/\rho(\mathbf{x}) \ , \tag{27}$$

are the density-normalized elastic moduli. Equation (26) will play a basic role in our treatment.

For viscoelastic media, $c_{ijkl}$ may depend on frequency, and the argumentation leading to $N_i(\mathbf{U}) = 0$ cannot be strictly used. Still, however, for smoothly varying media and weak frequency dependence of $c_{ijkl}$, we may use (26) as a good approximation also for viscoelastic media, particularly if we consider fixed $\omega \gg 0$.

We also introduce the generalized Christoffel matrix $\Gamma_{ik}(\mathbf{x}, \mathbf{p})$ by the relation

$$\Gamma_{ik}(\mathbf{x}, \mathbf{p}) = a_{ijkl}(\mathbf{x})p_jp_l \ . \tag{28}$$

Then the basic equation (26) can be expressed in a simpler form

$$\Gamma_{ik}U_k = U_i \ . \tag{29}$$

We will strictly distinguish between the generalized Christoffel matrix (28) and the Christoffel matrix $\Gamma_{ik} = a_{ijkl}n_j n_l$, where $\mathbf{n}$ is a *known* real-valued unit vector.

We now derive two important equations from (29).

**First**, we multiply (29) by $U_i$ and obtain

$$\Gamma_{ik}U_iU_k = a_{ijkl}p_jp_lU_iU_k = 1 \ . \tag{30}$$

The product $U_iU_k$ is a function of $x_n$ and $p_m$, and can be expressed as follows

$$U_iU_k = D_{ik}/D \ , \tag{31}$$

where

$$D_{ij} = \tfrac{1}{2}\epsilon_{ikl}\epsilon_{jrs}(\Gamma_{lr} - \delta_{lr})(\Gamma_{ls} - \delta_{ls}) \ , \tag{32}$$

and $\epsilon_{ijk}$ is the Levi-Civitta symbol. See Červený (2001). Thus, the function $\Gamma_{ik}U_iU_k$ is a function of $x_n$ and $p_m$ and equals 1 at any point of the model, for any slowness vector $\mathbf{p}$. Consequently, we may take this function to represent the Hamiltonian. Actually, we can also multiply it by any real-valued positive constant. For covenience, we use the multiplicative constant 1/2 and introduce the Hamiltonian as follows:

$$\mathcal{H}(x_n, p_m) = \tfrac{1}{2}\Gamma_{ik}U_iU_k = \tfrac{1}{2}a_{ijkl}p_jp_lU_iU_k \ . \tag{33}$$

Let us emphasize that we have not used the eigenvalue problem of $\Gamma_{ik}$ to derive the Hamiltonian (33).

**Second**, we multiply (29) by $U_i^*$,

$$\Gamma_{ik}U_i^*U_k = U_iU_i^* \ . \tag{34}$$

This yields an alternative version of the Hamiltonian $\mathcal{H}^A$:

$$\mathcal{H}^A(x_n, p_m) = \tfrac{1}{2}a_{ijkl}p_jp_lU_i^*U_k/(U_n^*U_n) \ . \tag{35}$$

Here $U_n^*U_n$ is a real-valued scalar.

We can use the Hamiltonian in any of the two forms, (33) or (35). The results of this paper do not depend on the differences between (33) and (35). For this reason, we shall use the Hamiltonian (33). It should be, however, noted that the choice may play an important role in certain applications, particularly if we are interested in the energy-related quantities. In this case, the choice of the Hamiltonian (35) would be closer to considering time-averaged energy-related quantities.

Now we will construct the Hamiltonian $\mathcal{H}(x_i, p_j, f_\kappa)$, where $f_\kappa$ are model parameters. We introduce one model parameter $f_\alpha$ by modifying the elastic/viscoelastic moduli in the following way

$$a_{ijkl}(x_n, f_\alpha) = a^R_{ijkl}(x_n) - \mathrm{i}f_\alpha a^I_{ijkl}(x_n) \ . \tag{36}$$

For $f_\alpha = 0$, the moduli $a_{ijkl}(x_n, f_\alpha)$ correspond to the perfectly elastic media, with $a_{ijkl} = a^R_{ijkl}$. Similarly, for $f_\alpha = 1$, the moduli $a_{ijkl}(x_n, f_\alpha)$ correspond to the viscoelastic media, with $a_{ijkl} = a^R_{ijkl} - \mathrm{i}a^I_{ijkl}$.

Hamiltonian $\mathcal{H}(x_i, p_j, f_\alpha)$ may be then introduced in several ways. We introduce it simply in such a way that we replace $a_{ijkl}$ in (33) by $a_{ijkl}(x_n, f_\alpha)$, given by (36). This also implies that $U_k$ is a function of $f_\alpha$, $U_k = U_k(f_\alpha)$, see (26). We obtain

$$\mathcal{H}(x_n, p_m, f_\alpha) = \tfrac{1}{2} a_{ijkl}(x_n, f_\alpha) p_j p_l U_i(f_\alpha) U_k(f_\alpha) \ . \tag{37}$$

We, of course, tacitly assume that $a_{ijkl}$, $U_i$ and $U_k$ are also functions of position $x_n$, and $U_i, U_k$ functions of $p_m$.

Equation (37) is a final form of Hamiltonian that we shall use in the following.

# 4 Perturbation from perfectly elastic to viscoelastic media

In this section, we shall specify the general perturbation equations for travel time and slowness vector, derived in Section 2, to perturbations from perfectly elastic to viscoelastic media. For this purpose, we shall use the Hamiltonian (37). We consider a heterogeneous anisotropic medium.

We now derive expressions for partial derivatives of the Hamiltonian (37) $\mathcal{H}_{,\alpha}$, $\mathcal{H}^{;i}_{,\alpha}$ and $\mathcal{H}_{,i\alpha}$, that we need in equations (14)–(15) for $\tau_{,i\alpha}$. We first derive equations for $\mathcal{H}^{;i}$ and $\mathcal{H}_{,i}$. For $\mathcal{H}^{;i}$, we obtain

$$\begin{aligned} \mathcal{H}^{;i} &= \partial\mathcal{H}/\partial p_i = a_{ijkl} p_l U_j U_k + \Gamma_{nk} U_k \partial U_n/\partial p_i \\ &= a_{ijkl} p_l U_j U_k + U_n \partial U_n/\partial p_i = a_{ijkl} p_l U_j U_k \ . \end{aligned}$$

Here we have taken into account that $\Gamma_{nk} U_k = U_n$, see (26), and that $U_n \partial U_n/\partial p_i = \frac{1}{2}\partial(U_n U_n)/\partial p_i = 0$, as $\mathbf{U}$ is a unit vector. Similarly we obtain the partial derivative $\mathcal{H}_{,i}$. Together:

$$\begin{aligned} \mathcal{H}^{;i} &= \partial\mathcal{H}/\partial p_i = a_{ijkl} p_l U_j U_k \ , \\ \mathcal{H}_{,i} &= \partial\mathcal{H}/\partial x_i = \tfrac{1}{2} a_{njkl,i} p_j p_l U_n U_k \ . \end{aligned} \tag{38}$$

Here $\mathcal{H}^{;i}$ is the energy-velocity vector (called also the ray-velocity vector), tangent to the ray. From (37) and (38), we can easily compute the "perturbation" partial derivatives $\mathcal{H}_{,\alpha}$, $\mathcal{H}^{;i}_{,\alpha}$ and $\mathcal{H}_{,i\alpha}$:

$$\begin{aligned} \mathcal{H}_{,\alpha} &= -\tfrac{1}{2} \mathrm{i} a^I_{ijkl} p_j p_l U_i U_k \ , \\ \mathcal{H}^{;i}_{,\alpha} &= -\mathrm{i} a^I_{ijkl} p_l U_j U_k \ , \\ \mathcal{H}_{,i\alpha} &= -\tfrac{1}{2} \mathrm{i} a^I_{njkl,i} p_j p_l U_n U_k \ . \end{aligned} \tag{39}$$

These are the final equations for partial derivatives needed in (14)–(15). The polarization vectors $U_i$ in (39) are still functions of $f_{,\alpha}$. In (12)–(14), we take $U_i$ for $f_{,\alpha} = 0$, i.e. for the reference medium. Then all the three partial derivatives in (39), $\mathcal{H}_{,\alpha}$, $\mathcal{H}^{;i}_{,\alpha}$ and $\mathcal{H}_{,i\alpha}$ are imaginary-valued. In the following, we consider $U_i$ for $f_{,\alpha} = 0$, that is for the reference medium.

Now we express equations (17) and (15) for the Hamiltonian (37), considered in this section. The quantities $\mathcal{H}_{,\alpha}$, $\mathcal{H}^{;i}_{,\alpha}$ and $\mathcal{H}_{,i\alpha}$ are then given by (39). We take into account that the Cartesian components of the attenuation vector $\mathbf{A}$ are given by the relation $A_i = \mathrm{Im}\tau_{,i\alpha}$, and obtain from (17):

$$\mathbf{A}(\gamma_3) = \mathbf{Q}^{-T}(\gamma_3)\mathrm{Im}\mathbf{T}(\gamma_3) \ , \tag{40}$$

where $\mathbf{T}(\gamma_3)$ is given by (16) with (15). Consequently

$$\begin{aligned} \mathrm{Im}T_{K\alpha}(\gamma_3) &= \mathrm{Im}T_{K\alpha}(\gamma_3^0) \\ &+ \int_{\gamma_3^0}^{\gamma_3} \left[\tfrac{1}{2}\mathcal{B}_i(\gamma_3')Q_{iK}(\gamma_3') + \mathcal{W}_i(\gamma_3')P_{iK}(\gamma_3')\right] \mathrm{d}\gamma_3' \ , \\ \mathrm{Im}T_{3\alpha}(\gamma_3) &= \tfrac{1}{2}\mathcal{A}^{in}(\gamma_3) \ . \end{aligned} \tag{41}$$

Here $\mathcal{A}^{in}$ is the so-called intrinsic attenuation factor, given by the relation,

$$\mathcal{A}^{in} = a^I_{ijkl}p_jp_lU_iU_k \ . \tag{42}$$

Vector $\boldsymbol{\mathcal{B}}$ (with Cartesian components $\mathcal{B}_i$) is given by the relation:

$$\mathcal{B}_i = \tfrac{1}{2}a^I_{njkl,i}p_jp_lU_nU_k \ . \tag{43}$$

It depends on the spatial derivatives of $a^I_{njkl}$, i.e. on the spatial variations of dissipation parameters $a^I_{njkl}$. For homogeneous media, $\boldsymbol{\mathcal{B}} = \mathbf{0}$. Finally, vector $\boldsymbol{\mathcal{W}}$ (with Cartesian components $\mathcal{W}_i$) is given by the relation:

$$\mathcal{W}_i = a^I_{ijkl}p_lU_jU_k \ . \tag{44}$$

The vector $\mathcal{W}_i$ is related to the perturbation of the energy-velocity vector $\mathcal{H}^{;i}$, see (38). The scalar $\mathcal{A}^{in}$ and vectors $\mathcal{B}_i$ and $\mathcal{W}_i$ are real-valued.

It may be useful to express (40) in terms of covariant basis vectors connected with the reference ray. We introduce the $3 \times 3$ matrices of contravariant basis vectors $\mathbf{Q}$ and of covariant basis vectors $\mathbf{S}$ in the following way:

$$\mathbf{Q} = [\mathbf{Q}_1, \mathbf{Q}_2, \boldsymbol{\mathcal{U}}] \ , \qquad \mathbf{S} = [\mathbf{f}_1, \mathbf{f}_2, \mathbf{p}] \ . \tag{45}$$

Here $\boldsymbol{\mathcal{U}}$ is the energy-velocity vector, tangent to the reference ray, and $\mathbf{p}$ the slowness vector, perpendicular to the wavefront. Both these vectors are calculated during ray tracing of the reference ray. The $3 \times 3$ matrix $\mathbf{Q}$ equals the spreading matrix, computed by dynamic ray tracing. $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are tangent to the wavefront. The covariant basis vectors $\mathbf{f}_1$ and $\mathbf{f}_2$ can be determine from $\mathbf{Q}_1, \mathbf{Q}_2, \boldsymbol{\mathcal{U}}$ as follows:

$$\mathbf{f}_1 = \frac{\mathbf{Q}_2 \times \boldsymbol{\mathcal{U}}}{\boldsymbol{\mathcal{U}}^T(\mathbf{Q}_1 \times \mathbf{Q}_2)} \ , \qquad \mathbf{f}_2 = \frac{\boldsymbol{\mathcal{U}} \times \mathbf{Q}_1}{\boldsymbol{\mathcal{U}}^T(\mathbf{Q}_1 \times \mathbf{Q}_2)} \ . \tag{46}$$

The vectors $\mathbf{f}_1$ and $\mathbf{f}_2$ are perpendicular to the reference ray. The $3 \times 3$ matrices $\mathbf{Q}$ and $\mathbf{S}$ are related as follows:

$$\mathbf{S}(\gamma_3) = \mathbf{Q}^{-T}(\gamma_3) \ . \tag{47}$$

Note that all basis vectors are real-valued.

Using (47), Equation (40) can be expressed as follows:

$$\mathbf{A}(\gamma_3) = \mathbf{S}(\gamma_3)\mathrm{Im}\mathbf{T}(\gamma_3) \ . \tag{48}$$

An explicit form of (48) is:

$$\mathbf{A}(\gamma_3) = \mathbf{f}_1(\gamma_3)\mathrm{Im}T_{1\alpha}(\gamma_3) + \mathbf{f}_2(\gamma_3)\mathrm{Im}T_{2\alpha}(\gamma_3) + \tfrac{1}{2}\mathcal{A}^{in}(\gamma_3)\mathbf{p}(\gamma_3) \ . \tag{49}$$

Equation (49), with $\mathrm{Im}T_{K\alpha}(\gamma_3)$ given by (41) and $\mathcal{A}^{in}(\gamma_3)$ by (42), represents the final result of this paper. It expresses the attenuation vector $\mathbf{A}$ at any point of the reference ray, specified by $\gamma_3$, in terms of covariant basis vectors $\mathbf{f}_1(\gamma_3)$, $\mathbf{f}_2(\gamma_3)$ and $\mathbf{p}(\gamma_3)$, and of $\mathrm{Im}T_{1\alpha}(\gamma_3)$, $\mathrm{Im}T_{2\alpha}(\gamma_3)$ and $\mathrm{Im}T_{3\alpha}(\gamma_3) = \frac{1}{2}\mathcal{A}^{in}(\gamma_3)$ given by (41)–(44). All quantities in (49) can be simply computed along the reference ray.

In certain applications, it may be useful to consider an alternative expression, in which the attenuation vector $\mathbf{A}$ is expressed in terms of the slowness vector $\mathbf{p}$ and of vectors $\mathbf{Q}_1$ and $\mathbf{Q}_2$,perpendicular to $\mathbf{p}$. The general expression for $\mathbf{A}$ is then rather complicated, but simplifies considerably in special cases. Let us consider a unit vector $\mathbf{n}(\gamma_3) = \mathbf{p}(\gamma_3)/|\mathbf{p}(\gamma_3)|$. Then we obtain

$$\begin{aligned} \mathbf{f}_1 &= -\mathbf{n}[\boldsymbol{\mathcal{U}}^T(\mathbf{Q}_2 \times \mathbf{n})]/\Delta - (\mathbf{n} \times \mathbf{Q}_2)(\boldsymbol{\mathcal{U}}^T\mathbf{n})/\Delta \ , \\ \mathbf{f}_2 &= -\mathbf{n}[\boldsymbol{\mathcal{U}}^T(\mathbf{n} \times \mathbf{Q}_1)]/\Delta + (\mathbf{n} \times \mathbf{Q}_1)(\boldsymbol{\mathcal{U}}^T\mathbf{n})/\Delta \ , \end{aligned} \tag{50}$$

where

$$\Delta = \boldsymbol{\mathcal{U}}^T(\mathbf{Q}_1 \times \mathbf{Q}_2) \ . \tag{51}$$

All quantities in (50) and (51) are taken at $\gamma_3$. Inserting (50) into (49) yields the final expression:

$$\begin{aligned} \mathbf{A}(\gamma_3) = \ &+ \ [(\mathbf{n} \times \mathbf{Q}_1)\mathrm{Im}T_{2\alpha} - (\mathbf{n} \times \mathbf{Q}_2)\mathrm{Im}T_{1\alpha}](\boldsymbol{\mathcal{U}}^T\mathbf{n})/\Delta \\ &+ \ \mathbf{p}[\tfrac{1}{2}\mathcal{A}^{in} - \mathcal{C}\boldsymbol{\mathcal{U}}^T(\mathbf{n} \times \mathbf{Q}_1)\mathrm{Im}T_{2\alpha}/\Delta - \mathcal{C}\boldsymbol{\mathcal{U}}^T(\mathbf{Q}_2 \times \mathbf{n})\mathrm{Im}T_{1\alpha}/\Delta] \ . \end{aligned} \tag{52}$$

All quantities in (52) are again taken at $\gamma_3$.

There are three "perturbation quantities", which play an important role in the determination of the attenuation vector $\mathbf{A}$ along the reference ray in an inhomogeneous weakly dissipative anisotropic media, namely $\mathcal{A}^{in}$, $\mathcal{B}_i$ and $\mathcal{W}_i$. The most important of them is the scalar, real-valued, intrinsic attenuation factor $\mathcal{A}^{in}$. It plays a basic role in the computation of quality factor in the dissipative medium. Multiplying (49) by $\boldsymbol{\mathcal{U}}(\gamma_3)$ and taking into account that $\mathbf{f}_I^T\boldsymbol{\mathcal{U}} = 0$ and $\mathbf{p}^T\boldsymbol{\mathcal{U}} = 1$ at any point of the reference ray, we obtain

$$\boldsymbol{\mathcal{U}}^T(\gamma_3)\mathbf{A}(\gamma_3) = \tfrac{1}{2}\mathcal{A}^{in}(\gamma_3) \ . \tag{53}$$

Thus, the projection of the attenuation vector $\mathbf{A}$ into the ray velocity vector $\boldsymbol{\mathcal{U}}$ equals $\frac{1}{2}\mathcal{A}^{in}$. The intrinsic attenuation factor depends only on the intrinsic attenuation parameters $a^I_{ijkl}$ of the medium, and does not depend on the inhomogeneity of the wave under consideration.

Červený & Pšenčík (2007a,b) showed that the quantity $(\mathcal{A}^{in})^{-1}$ can be used as a generalization of the quality factor $Q$ for anisotropic media. Their derivation was based on the definition of $Q$ given by Buchen (1971). They also showed that the expression for $Q$, derived by Gajewski & Pšenčík (1992) using different concepts than here, fully coincides with the definition $Q^{-1} = \mathcal{A}^{in}$.

One important note. The computation of the intrinsic attenuation factor $\mathcal{A}^{in}$ (or the quality factor $Q$) does not require any integration along the reference ray. Moreover, it does not require to perform dynamic ray tracing along the reference ray. The knowledge of the slowness vector $\mathbf{p}$, of the polarization vector $\mathbf{U}$ and of the intrinsic attenuation parameters $a^I_{ijkl}$ at a point of the reference ray is fully sufficient to determine $\mathcal{A}^{in}$ (or $Q$) at that point.

Let us now investigate the homogeneity/inhomogeneity of the wave propagating in heterogeneous, weakly dissipative, anisotropic medium. Equation (49) shows that the wave propagating in a weakly dissipative, anisotropic, heterogeneous medium is, in general, inhomogeneous. Remind the reader that the wave is called inhomogeneous if the attenuation vector $\mathbf{A}$ has nonvanishing component(s) in the direction perpendicular to the slowness vector. We can investigate these components considering $\mathbf{Q}^T_K\mathbf{A}$, as $\mathbf{Q}_K$ are perpendicular to the propagation vector. Multiplying (49) by $\mathbf{Q}_K(\gamma_3)$, we obtain

$$\mathbf{Q}^T_K(\gamma_3)\mathbf{A}(\gamma_3) = \mathrm{Im}T_{K\alpha}(\gamma_3) \ . \tag{54}$$

Here $\mathrm{Im}T_{K\alpha}(\gamma_3)$ are given by (41). In general, $\mathrm{Im}T_{K\alpha}(\gamma_3)$, $K = 1, 2$, are different from zero. This shows that the wave propagating in weakly dissipative anisotropic heterogeneous media are, in general, inhomogeneous. In heterogeneous media, the exceptions are very rare. For homogeneous media, see the next section.

Contrary to the expression (53) for $\mathcal{A}^{in}$, the determination of $\mathrm{Im}T_{K\alpha}(\gamma_3)$ requires the dynamic ray tracing along the reference ray, and the integration of $\mathcal{B}_i$ and $\mathcal{W}_i$ along the ray, see (41). The perturbation vector $\mathcal{W}_i$ is related to the perturbation of the energy-velocity vector, and $\mathcal{B}_i$ gives the spatial derivatives of $\mathcal{A}^{in}$. Actually, in the expression for $\mathcal{B}_i$, the vectors $\mathbf{p}$ and $\mathbf{U}$ are fixed, and the spatial derivatives are related only to $a^I_{njkl}$.

The first-order perturbation equations (48) or (49), supplemented by (41) and (46), can be used to compute the attenuation vector $\mathbf{A}$ in weakly dissipative anisotropic heterogeneous media along any reference ray, along which also dynamic ray tracing has been performed. Equations remain valid for homogeneous media, for isotropic media, for plane waves, for waves generated by a point source and for waves generated at a curved initial surface. They are valid for inhomogeneous waves, and also for homogeneous waves, if they exist.

# 5 Homogeneous medium

It may be useful to derive also certain analytical solutions of (49) for some simple cases. Particularly the analytical solutions for *homogeneous* media may be valuable. There are, at least, two reasons for it. First: Such analytical solutions contribute considerably to better understanding of various effects of wave propagation in weakly dissipative media. Second: The analytical solutions for homogeneous media may be used to compare our results with results known from plane wave propagation in homogeneous viscoelastic anisotropic media, see Červený & Pšenčík (2005,2006,2007a).

For homogeneous anisotropic media, the dynamic ray tracing system (5) simplifies considerably, as $\mathcal{H}^{,i}_{,j} = 0$ and $\mathcal{H}_{,ij} = 0$. The dynamic ray tracing system then reads

$$\mathrm{d}Q_{ia}/\mathrm{d}\gamma_3 = \mathcal{H}^{,ij} P_{ja} \ , \quad \mathrm{d}P_{ia}/\mathrm{d}\gamma_3 = 0 \ , \quad i = 1,2,3 \ . \tag{55}$$

The solution of this system is straightforward.

$$\begin{aligned} Q_{ia}(\gamma_3) &= Q_{ia}(\gamma_3^0) + \mathcal{H}^{,ij}(\gamma_3^0) P_{ja}(\gamma_3^0)(\gamma_3 - \gamma_3^0) \ , \\ P_{ia}(\gamma_3) &= P_{ia}(\gamma_3^0) \ . \end{aligned} \tag{56}$$

For $a = 3$, the solution simplifies,

$$Q_{i3}(\gamma_3) = \mathcal{U}_i(\gamma_3) \ , \qquad P_{i3}(\gamma_3) = 0 \ . \tag{57}$$

All the quantities in (40) are taken in the reference medium. Moreover, for a homogeneous medium, we also obtain,

$$\mathcal{B}_i = 0 \ . \tag{58}$$

Consequently, the attenuation vector $\mathbf{A}(\gamma_3)$ at any point of the reference ray in a homogeneous medium is again given by (40), where $\mathbf{Q}$ is given by (56) and (57), and $\mathrm{Im}\mathbf{T}(\gamma_3)$ by relations

$$\begin{aligned} \mathrm{Im}T_{K\alpha}(\gamma_3) &= \mathrm{Im}T_{K\alpha}(\gamma_3^0) + \mathcal{W}_i(\gamma_3^0) P_{iK}(\gamma_3^0)(\gamma_3 - \gamma_3^0) \ , \\ \mathrm{Im}T_{3\alpha}(\gamma_3) &= \tfrac{1}{2}\mathcal{A}^{in}(\gamma_3^0) \ . \end{aligned} \tag{59}$$

We shall now consider two special cases of initial conditions. In Section 5.1, we treat the plane wave initial conditions, and in Section 5.2, we consider the point source.

## 5.1 Plane waves

Consider a plane wave propagating in homogeneous weakly dissipative anisotropic medium, in an arbitrary direction. Denote the unit normal perpendicular to the wavefront by $\mathbf{n}$. Then

$$\begin{array}{ll} P_{iK}(\gamma_3) = P_{iK}(\gamma_3^0) = 0 & Q_{iK}(\gamma_3) = Q_{iK}(\gamma_3^0) \ , \\ \mathrm{Im}T_{K\alpha}(\gamma_3) = \mathrm{Im}T_{K\alpha}(\gamma_3^0) \ , & \mathrm{Im}T_{3\alpha}(\gamma_3) = \tfrac{1}{2}A^{in}(\gamma_3^0) \ . \end{array} \tag{60}$$

The vectors $\mathbf{Q}_1(\gamma_3^0)$ and $\mathbf{Q}_2(\gamma_3^0)$ are situated in the wavefront, but otherwise they can be chosen freely. We choose $\mathbf{Q}_1$, $\mathbf{Q}_2$, $\mathbf{n}$ to form an orthogonal right-handed triplet of unit vectors.

We now insert (60) into (52). Taking into account that $\mathbf{n} \times \mathbf{Q}_1 = \mathbf{Q}_2$, $\mathbf{n} \times \mathbf{Q}_2 = -\mathbf{Q}_1$, and $\Delta = \boldsymbol{\mathcal{U}}^T\mathbf{n} = \mathcal{C}$, we obtain

$$\begin{aligned} \mathbf{A}(\gamma_3) = \ &+ \ (\mathbf{Q}_2 \mathrm{Im}T_{2\alpha} + \mathbf{Q}_1 \mathrm{Im}T_{1\alpha}) \\ &+ \ \mathbf{p}[\tfrac{1}{2}\mathcal{A}^{in} - (\boldsymbol{\mathcal{U}}^T\mathbf{Q}_2)\mathrm{Im}T_{2\alpha} - (\boldsymbol{\mathcal{U}}^T\mathbf{Q}_1)\mathrm{Im}T_{1\alpha}] \ . \end{aligned} \tag{61}$$

This is the final expression for the plane wave propagating in a homogeneous weakly dissipative anisotropic medium. It clearly demonstrates that the plane wave may be homogeneous or inhomogeneous, depending on the choice of initial conditions $\mathrm{Im}T_{1\alpha}$ and $\mathrm{Im}T_{2\alpha}$. If we choose $\mathrm{Im}T_{1\alpha} = \mathrm{Im}T_{2\alpha} = 0$, the plane wave is homogeneous. If at least one of the quantities $\mathrm{Im}T_{1\alpha}$ or $\mathrm{Im}T_{2\alpha}$ is nonvanishing, the plane wave is inhomogeneous.

The plane waves propagating in homogeneous weakly dissipative anisotropic media were also studied by Hayes & Rivlin (1974), Deschamps & Assouline (2000), Zhu & Tsvankin (2006, 2007), Červený & Pšenčík (2007a,b), etc. Červený & Pšenčík (2007a) derived the attenuation vector directly from exact solutions. We will now show that both results are the same. We choose the initial conditions as follows

$$\mathrm{Im}T_{1\alpha} = D \ , \quad \mathrm{Im}T_{2\alpha} = 0 \ . \tag{62}$$

and call $D$ the inhomogeneity parameter. Using then the notation $\mathbf{m} = \mathbf{Q}_1$, we obtain from (61):

$$\mathbf{A} = D\mathbf{m} + (\tfrac{1}{2}\mathcal{A}^{in} - (\boldsymbol{\mathcal{U}}^T\mathbf{m})D)\mathbf{p} \ . \tag{63}$$

This coincides fully with equation derived by Červený and Pšenčík (2007a, eq.(31)).

## 5.2 Waves generated by a point source

Consider a wave generated by a point source situated at $S$, propagating to a receiver $R$ in a homogeneous weakly dissipative anisotropic medium. Then the reference ray is a straight line connecting points $S$ and $R$, parallel to the energy velocity vector $\boldsymbol{\mathcal{U}}$. We do not treat here the problem of determination of the relevant slowness vector $\mathbf{p}$ from $\boldsymbol{\mathcal{U}}$, and assume that the slowness vector $\mathbf{p}$ corresponding to given $\boldsymbol{\mathcal{U}}$ is known. There may be several such $\mathbf{p}$, and we consider any of them. (The results for different $\mathbf{p}$ corresponding to the same $\boldsymbol{\mathcal{U}}$ will be, of course, different.) The initial conditions for dynamic ray tracing are then $Q_{iK}(\gamma_3^0) = 0$, $P_{iK}(\gamma_3^0) \neq 0$. The vectors $\mathbf{P}_1$ and $\mathbf{P}_2$ are situated in the slowness surface at $S$ and are perpendicular to the reference ray, $P_{iK}\mathcal{U}_i = 0$. Actually, they can be chosen arbitrarily, but must be perpendicular to the reference ray $\boldsymbol{\mathcal{U}}$. Then we obtain, see (56),

$$\begin{aligned} Q_{iK}(\gamma_3) &= \mathcal{H}^{,ij}(\gamma_3^0)P_{jK}(\gamma_3^0)(\gamma_3 - \gamma_3^0) \ , \\ Q_{i3}(\gamma_3) &= \mathcal{U}_i(\gamma_3^0) \ . \end{aligned} \tag{64}$$

The initial value of $\mathrm{Im}T_{K\alpha}(\gamma_3^0)$ at a point source vanishes (see Klimeš, 2002). Then (59) yields

$$\begin{aligned} \mathrm{Im}T_{K\alpha}(\gamma_3) &= \mathcal{W}_i(\gamma_3^0)P_{iK}(\gamma_3^0)(\gamma_3 - \gamma_3^0) \ , \\ \mathrm{Im}T_{3\alpha}(\gamma_3) &= \tfrac{1}{2}\mathcal{A}^{in}(\gamma_3^0) \ . \end{aligned} \tag{65}$$

When the vector $\boldsymbol{\mathcal{W}}$ is parallel to $\boldsymbol{\mathcal{U}}$, $\mathrm{Im}T_{K\alpha}(\gamma_3)$ vanishes. For a more detailed treatment of this case see later in this section.

To calculate $\mathbf{A}(\gamma_3)$, it would be, of course, possible to use directly (40), with $\mathbf{Q}(\gamma_3)$ given by (64) and $\mathrm{Im}T_{k\alpha}(\gamma_3)$ by (65). We shall, however, use an alternative approach, which will offer us a clearer physical insight. We express $\mathbf{A}(\gamma_3)$ in terms of covariant basis vectors $\mathbf{P}_1$, $\mathbf{P}_2$ and $\mathbf{p}$ connected with the reference ray, where $\mathbf{P}_1$ and $\mathbf{P}_2$ are perpendicular to the ray, and $\mathbf{p}$ is the slowness vector,

$$A_i = a_1 P_{i1} + a_2 P_{i2} + \tfrac{1}{2}\mathcal{A}^{in} p_i \ . \tag{66}$$

Here $a_1$ and $a_2$ should be determined. We determine them from known expressions for $\mathrm{Im}T_{K\alpha}(\gamma_3)$ given by (65), using the relation (54). In a componental form, we can express (54) as follows:

$$\mathrm{Im}T_{K\alpha} = Q_{iK}(a_1 P_{i1} + a_2 P_{i2}) \ . \tag{67}$$

Inserting (64) and (65) into (67) yields

$$\begin{aligned} \mathcal{W}_i P_{i1} &= a_1 D_{11} + a_2 D_{22} \ , \\ \mathcal{W}_i P_{i2} &= a_1 D_{12} + a_2 D_{22} \ , \end{aligned} \tag{68}$$

where

$$D_{MK} = P_{iM}\mathcal{H}^{,ij} P_{jK} \ . \tag{69}$$

The $2\times 2$ matrix $\mathbf{D}$ is related to the curvature matrix of the slowness surface. The solution of equations (68) for $a_1$ and $a_2$ is as follows:

$$\begin{aligned} a_1 &= \mathcal{W}_i(P_{i1}D_{22} - P_{i2}D_{21})/\det\mathbf{D} \ , \\ a_2 &= \mathcal{W}_i(P_{i2}D_{11} - P_{i1}D_{12})/\det\mathbf{D} \ . \end{aligned} \tag{70}$$

We now denote

$$\mathcal{W}^{(1)} = \mathcal{W}_i P_{i1} \qquad \mathcal{W}^{(2)} = \mathcal{W}_i P_{i2} \ . \tag{71}$$

The real-valued scalars $\mathcal{W}^{(1)}$ and $\mathcal{W}^{(2)}$ represent the projections of vector $\boldsymbol{\mathcal{W}}$ into basis vectors $\mathbf{P}_1$ and $\mathbf{P}_2$, respectively. Then we obtain,

$$\begin{aligned} a_1 &= (\mathcal{W}^{(1)}D_{22} - \mathcal{W}^{(2)}D_{21})/\det\mathbf{D} \ , \\ a_2 &= (\mathcal{W}^{(2)}D_{11} - \mathcal{W}^{(1)}D_{12})/\det\mathbf{D} \ . \end{aligned} \tag{72}$$

The final expression for the attenuation vector $\mathbf{A}$ is as follows, see (66),

$$\mathbf{A} = \frac{\mathcal{W}^{(1)}D_{22} - \mathcal{W}^{(2)}D_{21}}{\det\mathbf{D}}\mathbf{P}_1 + \frac{\mathcal{W}^{(2)}D_{11} - \mathcal{W}^{(1)}D_{12}}{\det\mathbf{D}}\mathbf{P}_2 + \tfrac{1}{2}\mathcal{A}^{in}\mathbf{p} \ . \tag{73}$$

Equation (73) is simplified if a special choice of covariant basis vector $\mathbf{P}_1$ and $\mathbf{P}_2$ is considered. Assume that $\mathbf{P}_1$ is taken in the plane specified by vectors $\boldsymbol{\mathcal{U}}$ and $\boldsymbol{\mathcal{W}}$. Then $\mathcal{W}^{(2)} = 0$, and (73) yields

$$\mathbf{A} = \frac{\mathcal{W}^{(1)}}{\det\mathbf{D}}(D_{22}\mathbf{P}_1 - D_{12}\mathbf{P}_2) + \tfrac{1}{2}\mathcal{A}^{in}\mathbf{p} \ . \tag{74}$$

This is the final expression for attenuation vector $\mathbf{A}$ in terms of covariant basis vectors $\mathbf{P}_1$, $\mathbf{P}_2$ and $\mathbf{p}$.

Similarly as in (52), we can also express the attenuation vector $\mathbf{A}$ in terms of the slowness vector $\mathbf{p}$ and of two vectors perpendicular to $\mathbf{p}$. This gives us a simple possibility to discuss the inhomogeneity of the wave under consideration. We again denote $\mathbf{n}$ the real-valued unit vector along $\mathbf{p}$, and introduce the vectors $\mathbf{h}_1$ and $\mathbf{h}_2$, using relations:

$$\mathbf{h}_1 = \mathbf{n} \times (\mathbf{P}_1 \times \mathbf{n}) \ , \qquad \mathbf{h}_2 = \mathbf{n} \times (\mathbf{P}_2 \times \mathbf{n}) \ . \tag{75}$$

Here $\mathbf{h}_1$ and $\mathbf{h}_2$ are perpendicular to the slowness vector $\mathbf{p}$. Then we use the obvious decomposition

$$\mathbf{P}_1 = \mathbf{n}(\mathbf{P}_1^T\mathbf{n}) + \mathbf{h}_1 \ , \qquad \mathbf{P}_2 = \mathbf{n}(\mathbf{P}_2^T\mathbf{n}) + \mathbf{h}_2 \ . \tag{76}$$

Inserting (76) into (74) yields

$$\begin{aligned} \mathbf{A} &= \frac{\mathcal{W}^{(1)}}{\det \mathbf{D}}(D_{22}\mathbf{h}_1 - D_{12}\mathbf{h}_2) \\ &+ \mathbf{p}[\tfrac{1}{2}\mathcal{A}^{in} + \frac{\mathcal{C}\mathcal{W}^{(1)}}{\det \mathbf{D}}(D_{22}(\mathbf{P}_1^T\mathbf{n}) - D_{12}(\mathbf{P}_2^T\mathbf{n}))] \ . \end{aligned} \tag{77}$$

As we can see, the inhomogeneity of the wave is influenced by $\mathcal{W}^{(1)}$ and by the curvature of the slowness surface. The wave is always homogeneous if $\mathcal{W}^{(1)} = 0$. As $\mathcal{W}^{(1)} = \boldsymbol{\mathcal{W}}^T\mathbf{P}_1$, the wave is homogeneous if $\boldsymbol{\mathcal{W}}$ is parallel to $\boldsymbol{\mathcal{U}}$. It can be easily shown that $\boldsymbol{\mathcal{W}}$ is parallel to $\boldsymbol{\mathcal{U}}$ for isotropic weakly dissipative media, but rarely for anisotropic weakly dissipative media. Consequently, the final result is: waves generated by a point source, propagating in homogeneous isotropic weakly dissipative media, are always homogeneous. Contrary to it, the waves generated by a point source, propagating in homogeneous anisotropic weakly dissipative media are, in general, inhomogeneous.


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### References


Aki, K. & Richards, P., 1980. *Quantitative seismology. Theory and methods.* Freeman, San Francisco.

Buchen, P.V., 1971. Plane waves in linear viscoelastic media. *Geophys. J.R. astr. Soc.*, **23**, 531–542.

Červený, V., 1982. Direct and inverse kinematic problems for inhomogeneous anisotropic media - Linearization approach. *Contrib. Geophys. Inst. Slov. Acad. Sci.*, **13**, 127–133.

Červený, V., 2001. *Seismic Ray Theory.* Cambridge Univ. Press, Cambridge.

Červený, V., 2002. Perturbation methods for two-point ray tracing in anisotropic inhomogeneous media, in *Seismic waves in complex 3-D structures*, Report 12, pp. 89–97. Charles University, Faculty of Mathematics and Physics, Dept. of Geophysics, Praha. (Available online at "http://sw3d.mff.cuni.cz".)

Červený, V. & Jech, J., 1982. Linearized solutions of kinematic problems of seismic body waves in inhomogeneous slightly anisotropic media. *J. Geophys.*, **51**, 96–104.

Červený, V., Klimeš, L. & Pšenčík, I., 2007. Seismic ray method: Recent developments. *Advances in Geophysics*, **48**, 1–126.

Červený, V., & Pšenčík, I., 2002. Complex two-point rays in anisotropic inhomogeneous weakly dissipative media, in *Seismic waves in complex 3-D structures*, Report 12, pp. 241–245. Charles University, Faculty of Mathematics and Physics, Dept. of Geophysics, Praha. (Available online at "http://sw3d.mff.cuni.cz".)

Červený, V. & Pšenčík, I., 2005. Plane waves in viscoelastic anisotropic media. I. Theory. *Geophys. J. Int.*, **161**, 197–212.

Červený, V. & Pšenčík, I., 2006. Energy flux in viscoelastic anisotropic media. *Geophys. J. Int.*, **166**, 1299–1317.

Červený, V., & Pšenčík, I., 2007a. Weakly inhomogeneous plane waves in anisotropic weakly dissipative media, in *Seismic waves in complex 3-D structures*, Report 17, pp. 00–00. Charles University, Faculty of Mathematics and Physics, Dept. of Geophysics, Praha. (Available online at "http://sw3d.mff.cuni.cz".)

Červený, V., & Pšenčík, I., 2007b. Quality factor $Q$ in dissipative anisotropic media, in *Seismic waves in complex 3-D structures*, Report 17, pp. 00–00. Charles University, Faculty of Mathematics and Physics, Dept. of Geophysics, Praha. (Available online at "http://sw3d.mff.cuni.cz".)

Chapman, C.H. & Pratt, R.G., 1992. Traveltime tomography in anisotropic media: I. Theory. *Geophys. J. Int.*, **109**, 1–19.

Declercq, N.F., Briers, R., Degrieck, J. & O. Leroy, 2005. The history and properties of ultrasonic inhomogeneous waves. *IEEE Transactions on Ultrasonics, Ferroelectrics, and Frequency Control*, **52**, 776–791.

Deschamps, M. & Assouline, P., 2000. Attenuation along the Poynting vector direction of inhomogeneous plane waves in absorbing and anisotropic solids. *Acustica*, **86**, 295–302.

Farra, V., 1989. Ray perturbation theory for heterogeneous hexagonal anisotropic medium. *Geophys. J. Int.*, **99**, 723–737.

Farra, V., 1999. Computation of second-order traveltime perturbation by Hamiltonian ray theory. *Geophys. J. Int.*, **136**, 205–217.

Fedorov, F.I., 1968. *Theory of Elastic Waves in Crystals.* Plenum, New York.

Gajewski, D. & Pšenčík, I., 1992. Vector wavefield for weakly attenuating anisotropic media by the ray method. *Geophysics*, **57**, 27–38.

Hayes, M.A. & R.S. Rivlin, 1974. Plane wave in linear viscoelastic materials. *Quart. Appl. Math.*, **32**, 113–121.

Hanyga, A., 1982. The kinematic inverse problem for weakly laterally inhomogeneous anisotropic media. *Tectonophysics*, **90**, 253–262.

Helbig, K., 1994. *Foundations of Anisotropy for Exploration Seismics.* Pergamon, Oxford.

Jech, J. & Pšenčík, I., 1989. First-order perturbation method for anisotropic media. *Geophys. J. Int.*, **99**, 369–376.

Klimeš, L., 2002. Second-order and higher-order perturbations of travel time in isotropic and anisotropic media. *Stud. Geophys. Geod.*, **46**, 213–248.

Moczo, P., Bard, P.-Y. & Pšenčík, I., 1987. Seismic response of two-dimensional absorbing structures by the ray method. *J. Geophys.*, **62**, 38–49.

Musgrave, M.I.P., 1970. *Crystal acoustics.* Holden Day, San Francisco.

Snieder, R. & Sambridge, M. 1992. Ray perturbation theory for travel times and ray paths in 3-D heterogeneous media. *Geophys. J. Int.*, **109**, 294–322.

Zhu, Y.P. & Tsvankin, I., 2006. Plane-wave propagation in attenuative transversely isotropic media. *Geophysics*, **71**, T17–T30.

Zhu, Y.P. & Tsvankin, I., 2007. Plane-wave attenuation anisotropy in orthorhombic media. *Geophysics*, **72**, D9–D19.