# Weakly inhomogeneous plane waves in anisotropic, weakly dissipative media

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## Summary

Weakly inhomogeneous time-harmonic plane waves propagating in homogeneous anisotropic, weakly dissipative media are studied using the perturbation method. Only dissipation mechanisms, which can be described in the framework of a linear viscoelasticity, are considered. As a reference (non-perturbed) case, plane waves with a real-valued slowness vector propagating in perfectly elastic anisotropic media are used. Simple approximate expressions for the complex-valued slowness and polarization vectors of weakly inhomogeneous plane waves propagating in anisotropic, weakly dissipative media are derived. A special attention is devoted to the imaginary part of the slowness vector, known as the attenuation vector, which is responsible for the amplitude attenuation of a plane wave. The derived approximate expression for the attenuation vector depends on the material (intrinsic) dissipation parameters as well as on the inhomogeneity of the plane wave. Its projection into the direction of the energy-velocity vector yields, however, the intrinsic attenuation factor, which does not depend on the inhomogeneity of the wave, and which thus represents a very suitable measure of the material dissipation. The derived expression for the intrinsic attenuation factor is valid for media of unrestricted anisotropy and weak dissipation and for homogeneous as well as weakly inhomogeneous plane waves. The intrinsic attenuation factor is inversely proportional to the scalar quantity, which, in isotropic viscoelastic media, corresponds to the well-known quality factor $Q$. Its generalization to anisotropic weakly viscoelastic media is directionally dependent. Numerical examples are presented, in which the accuracy of the approximate formulae based on the perturbation method is studied. The results indicate that the presented perturbation results are sufficiently accurate for the use in practical applications. Strong directivity of the intrinsic attenuation factor shows its great potential for the solution of inverse problems.

**Keywords**: anisotropy, attenuation, inhomogeneous plane waves, perturbation methods, viscoelasticity.

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# 1 Introduction

Wave propagation in viscoelastic anisotropic media is an important subject in contemporary seismology, seismic exploration, acoustics, ultrasonic nondestructive testing of materials, etc. In seismology and seismic exploration, examples are wave propagation in shallow consolidated sediments, reservoir rocks, zones of partial melting, mining areas, etc. The properties of seismic waves in such media have been mostly investigated by direct numerical solution of equation of motion. For a detailed description of such methods, with many results of computations and other references, see Carcione (2001, 2007).

An important insight into the wave processes in viscoelastic anisotropic media can be obtained by the use of plane waves. In propagation of plane waves in unbounded homogeneous viscoelastic media, a special role is played by inhomogeneous plane waves. The basic problem in their study consists in the specification of their slowness vectors. Several possibilities, namely the directional, componential and mixed specifications of the slowness vector, are described by Červený (2004), Červený & Pšenčík (2005a,b). The most suitable of them is the mixed specification, which is quite universal and can be applied, without limitations, to homogeneous as well as inhomogeneous plane waves, propagating in perfectly elastic or viscoelastic, isotropic or anisotropic media. With the mixed specification, the computation of an inhomogeneous plane wave propagating in a specified direction leads, in general, to a solution of an algebraic equation of the sixth degree with complex-valued coefficients. In special cases, the algebraic equation of the sixth degree can be factorized into two equations, one of the fourth and the other of the second degree.

Why we are interested in inhomogeneous plane waves? In an excellent review article on ultrasonic inhomogeneous waves, Declercq et al. (2005) give an answer when they write: "In general, whenever damping plays a role, inhomogeneous waves are part of the game". Waves propagating in a viscoelastic anisotropic medium from a point source to a given receiver are always inhomogeneous, even if the medium is homogeneous. In a smooth medium, the inhomogeneity of the wave is usually weak. It may, however, increase considerably if there are strong heterogeneities and structural interfaces (post-critical transmission) in the medium. Inhomogeneity of waves propagating from a point source to a receiver is given by the configuration and properties of the medium. One of the reasons why we concentrate on inhomogeneous plane waves propagating in homogeneous viscoelastic, isotropic or anisotropic media is that their inhomogeneity can be controlled. Let us mention that inhomogeneous plane waves are not only a theoretical concept. Laboratory sources of inhomogeneous plane waves are now available and used in ultrasonic nondestructive testing (Deschamps & Hosten, 1989; Declercq et al., 2005).

Although the medium dissipation and inhomogeneity of waves in smooth media are usually weak in seismology and seismic exploration, they play an important role, as they are responsible for the exponential decay of amplitudes along their travel paths. The dissipative effects are cumulative, i.e., for long travel paths the complete dissipation effects may be rather large, and thus well measurable. Consequently, the dissipative effects can be used in inversion schemes to offer additional independent parameters characterizing the medium under consideration.

For study of weakly inhomogeneous waves propagating in weakly dissipative media it is natural to use the perturbation methods. The perturbation methods lead to simple but sufficiently accurate expressions, often of great practical importance. The use of perturbation methods has a long tradition in the study of wave propagation in perfectly elastic anisotropic media, in which the reference medium is isotropic, and anisotropy is considered as a perturbation. Often, the perturbation methods have been used to find travel times in weakly anisotropic media, see, for example, Červený (1982), Červený & Jech (1982), Hanyga (1982), Jech & Pšenčík (1989), Chapman & Pratt (1992), Farra (1999), Červený (2001), Klimeš (2002), Pšenčík & Farra (2005). The perturbation methods were, however, also used to compute the whole wavefield, see, e.g., Farra (1989), Nowack & Pšenčík (1991), Pšenčík & Farra (2007), etc.

The reference (unperturbed) case considered in this paper is a homogeneous plane wave (with a real-valued slowness vector) propagating in a perfectly elastic anisotropic medium. Weak inhomogeneity of the wave and weak viscoelasticity of the medium are considered as perturbations. A similar perturbation problem has been studied by Hayes & Rivlin (1974), who, however, used a different approach. They assumed that the imaginary parts of the complex-valued viscoelastic moduli are small compared to their real parts, and that the imaginary part of the slowness vector $\mathbf{p}$ (called the attenuation vector $\mathbf{A}$) is small compared to its real part (called the propagation vector $\mathbf{P}$). The inhomogeneity of the wave under consideration, however, was not explicitly specified. It could be easily specified explicitly with the mixed specification of the slowness vector, proposed by Červený (2004), Červený & Pšenčík (2005a). The mixed specification simplifies considerably the treatment and yields simple explicit results. Another perturbation problem, similar to the one discussed in this paper, has been studied by Gajewski & Pšenčík (1992) using the ray theory. Although Gajewski & Pšenčík (1992) did not consider inhomogeneous waves, they arrived to an important approximate expression for the quality factor in anisotropic dissipative media, equal to the expression derived and discussed in this paper.

The main goal of this paper is to derive the first-order perturbation formulae for the slowness and polarization vectors of weakly inhomogeneous plane waves propagating in weakly viscoelastic anisotropic media. The advantage of considering plane waves consists in the fact that the exact solutions are available for them so that the approximate, first-order perturbation results, can be easily compared with the exact ones.

The paper is organized in the following way. In Section 2, the exact equations for time-harmonic homogeneous and inhomogeneous plane waves propagating in viscoelastic anisotropic media are reviewed. The equations are valid for media of unrestricted viscoelasticity and anisotropy, and for unrestricted inhomogeneity of plane waves. In Section 3, weakly inhomogeneous plane waves propagating in weakly dissipative anisotropic media are treated. In Section 3.1, the reference case is specified. A special attention is devoted to the first-order perturbation of the slowness vector and to the attenuation vector, see Section 3.2. The intrinsic attenuation factor is also introduced there. Section 3.3 is devoted to the first-order perturbation of the polarization vector. Section 4 contains results of numerical tests, in which the accuracy of the first-order perturbation formulae is studied by comparison with exact results. Presented examples indicate that accuracy of derived first-order expressions could be sufficient for practical purposes. The calculated example of intrinsic attenuation factor shows a strong directivity of this quantity.

In the whole paper, symbols $x_i$ denote Cartesian coordinates and $t$ denotes time. The lower-case indices $i, j, k, l, ...$ take the values of 1,2,3. The Einstein summation convention over repeated indices is used. For time-harmonic wave fields, we consider the exponential time factor $\exp[-\mathrm{i}\omega t]$, where $\omega$ is a fixed, real-valued, positive circular frequency. Parallelly with the tensor notation $a_{ijkl}$ of the complex-valued density-normalized viscoelastic moduli, we also use the Voigt matrix notation $A_{\alpha\beta}$, where $\alpha, \beta, ...$ take the values of 1,2,..,6.

## 2 Time-harmonic plane waves in homogeneous viscoelastic anisotropic media

Time-harmonic homogeneous and inhomogeneous plane waves propagating in a homogeneous unbounded viscoelastic anisotropic medium can be represented by the expression

$$u_i(x_j, t) = aU_i \exp[-\mathrm{i}\omega(t - p_n x_n)] \ . \tag{1}$$

Here $u_i, U_i$ and $p_i$ are complex-valued Cartesian components of the displacement vector $\mathbf{u}$, the normalized polarization vector $\mathbf{U}$, ($U_i U_i = 1$), and the slowness vector $\mathbf{p}$, respectively. The symbol $a$ denotes a complex-valued scalar amplitude factor. The polarization vector $\mathbf{U}$, the slowness vector $\mathbf{p}$ and the amplitude factor $a$ are independent of $x_j$ and $t$. We emphasize the use of sign "$-$" in the exponential factor $\exp[-\mathrm{i}\omega(...)]$. If we considered $\exp[+\mathrm{i}\omega(...)]$, all resulting complex-valued quantities would be complex-conjugate to those obtained from eq. (1).

The displacement vector (1) must satisfy the equation of motion and the generalized Hooke's law for homogeneous viscoelastic anisotropic solids:

$$\tau_{ij,j} = \rho \dot{v}_i \tag{2}$$

and

$$\tau_{ij} = c_{ijkl} e_{kl} \ , \tag{3}$$

where $\tau_{ij}$ denotes the stress tensor, $e_{ij}$ denotes the strain tensor, $e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$, $v_i$ is the particle velocity, $v_i = \dot{u}_i$, and $\rho$ is the density. The symbol $c_{ijkl}(\omega)$ denotes the tensor of complex-valued frequency-dependent viscoelastic moduli. As we consider $\omega$ fixed, we do not explicitly indicate frequency dependence of $c_{ijkl}$ in the following. For more detailed discussion of the equation of motion and of the generalized Hooke's law, see, e.g., Červený & Pšenčík (2005a). Inserting eqs (1) and (3) into eq. (2), we find out that eq. (1) represents a plane wave only if $U_1, U_2$ and $U_3$ satisfy the system of linear equations

$$(\Gamma_{ik} - \delta_{ik})U_k = 0 \ . \tag{4}$$

Here $\Gamma_{ik}$ is the generalized Christoffel matrix (containing components $p_j$ of the slowness vector $\mathbf{p}$ instead of components $n_j$ of a unit directional vector $\mathbf{n}$ appearing in the standard Christoffel matrix), given by the relation

$$\Gamma_{ik}(p_n) = a_{ijkl} p_j p_l \ . \tag{5}$$

In eq. (5), $a_{ijkl}$ are complex-valued density-normalized viscoelastic moduli, $a_{ijkl} = c_{ijkl}/\rho$. The condition of solvability of the system of equations (4) reads

$$\det(\Gamma_{ik} - \delta_{ik}) = 0 \ . \tag{6}$$

This is the constraint relation for the slowness vector $\mathbf{p}$.

It has been common to specify the complex-valued slowness vector $\mathbf{p}$ as follows:

$$\mathbf{p} = \mathbf{P} + \mathrm{i}\mathbf{A} \ . \tag{7}$$

Here $\mathbf{P}$ and $\mathbf{A}$ are real-valued vectors, called the propagation and attenuation vectors, respectively. The angle made by the vectors $\mathbf{P}$ and $\mathbf{A}$ is called the attenuation angle $\gamma$, $\cos\gamma = \mathbf{P}\cdot\mathbf{A}/|\mathbf{P}||\mathbf{A}|$. For $\gamma \neq 0$, the plane wave is called inhomogeneous, for $\gamma = 0$ homogeneous. Thus, for homogeneous waves, the propagation and attenuation vectors are parallel. Let us note that we call a wave homogeneous even if $\mathbf{A} = 0$, i.e., when the slowness vector is real valued.

For general viscoelastic anisotropic media, the direct determination of vectors $\mathbf{P}$ and $\mathbf{A}$ satisfying (6) is not straightforward. In order to avoid possible complications, we use the so-called mixed specification of the slowness vector $\mathbf{p}$:

$$\mathbf{p} = \sigma\mathbf{n} + \mathrm{i}D\mathbf{m} \ , \qquad \mathbf{m}\cdot\mathbf{n} = 0 \ , \tag{8}$$

see Červený & Pšenčík (2005a). The real-valued, mutually perpendicular unit vectors $\mathbf{n}$ and $\mathbf{m}$ and the real-valued scalar $D$ can be chosen arbitrarily. We call $D$ the inhomogeneity parameter. It is measured in s m$^{-1}$. For $D = 0$, the plane wave is homogeneous, and for $D$ nonzero, it is inhomogeneous. Note that $\mathbf{n}$ is perpendicular to the wavefront of the plane wave under consideration, and $\mathbf{m}$ is tangent to the wavefront. The plane specified by $\mathbf{n}$ and $\mathbf{m}$ is called the propagation-attenuation plane. This plane also contains the vectors $\mathbf{P}$ and $\mathbf{A}$.

The complex-valued quantity $\sigma$ in eq. (8) can be determined from the constraint relation (6). It is also measured in s m$^{-1}$. Inserting eq. (8) into eq. (5) and then eq. (5) into eq. (6), we obtain an algebraic equation of the sixth degree for $\sigma$, with complex-valued coefficients:

$$\det[a_{ijkl}(\sigma n_j + \mathrm{i}Dm_j)(\sigma n_l + \mathrm{i}Dm_l) - \delta_{ik}] = 0 \ . \tag{9}$$

This equation has six complex-valued roots, corresponding to P, S1 and S2 plane waves, propagating in directions of $\mathbf{n}$ and $-\mathbf{n}$, respectively.

Once eq. (9), with $a_{ijkl}$, $\mathbf{n}$, $\mathbf{m}$ and $D$ given, is solved, and its six roots $\sigma$ determined, vectors $\mathbf{P}$ and $\mathbf{A}$ for all six elementary plane waves can be expressed by simple relations:

$$\mathbf{P} = \mathbf{n}\mathrm{Re}\sigma \ , \qquad \mathbf{A} = \mathbf{n}\mathrm{Im}\sigma + D\mathbf{m} \ . \tag{10}$$

The expressions for the phase velocity $\mathcal{C}$ in the direction of $\mathbf{n}$ and for the attenuation angle $\gamma$ of any of the considered waves have the following form:

$$\mathcal{C} = 1/|\mathbf{P}| = 1/|\mathrm{Re}\sigma| \ , \qquad \cos\gamma = \epsilon(\mathrm{Im}\sigma)/[(\mathrm{Im}\sigma)^2 + D^2]^{1/2} \ , \tag{11}$$

where $\epsilon = \mathrm{Re}\sigma/|\mathrm{Re}\sigma| = \pm 1$.

To determine the polarization vector $\mathbf{U}$, we insert the relevant value of $\sigma$ into eq. (8), and then eq. (8) into eq. (4), solve eq. (4) for $U_k$ and normalize the result so that $U_i U_i = 1$. Note that the normalization does not contain multiplication by complex conjugate $U_i^*$.

For more detailed explanations of equations presented above, and for numerical examples, see Červený & Pšenčík (2005a,b; 2006a,b).

# 3 Time-harmonic weakly inhomogeneous plane waves in homogeneous weakly dissipative anisotropic media

The exact computation of inhomogeneous plane waves propagating in viscoelastic anisotropic media is not difficult. It only requires to find complex-valued roots of the algebraic equation of the sixth degree (9). Standard computer routines are available for such a procedure.

In certain cases, however, the numerical solution of the algebraic equation can be substituted by evaluation of simple analytical solution. This is the case of wave propagation in special directions, for which the algebraic equation of the sixth degree factorizes into two equations, one of the fourth and the other of the second degree. For example, for SH waves propagating in the plane of symmetry of a monoclinic (orthorhombic, hexagonal) viscoelastic medium, it is sufficient to solve a quadratic algebraic equation. Similarly, it is sufficient to solve quadratic equations for P or S waves propagating in isotropic viscoelastic media. In all these cases, exact solutions can be expressed in an analytical form. See Červený & Pšenčík (2005a,b; 2006a).

In other cases, the algebraic equation of the sixth degree for $\sigma$ can be solved approximately, using the perturbation method. This applies, for example, to weakly inhomogeneous plane waves (with a small inhomogeneity parameter $D$) propagating in weakly dissipative media (with small $|\mathrm{Im} a_{ijkl}|$). We derive approximate perturbation formulae for such cases in this section.

## 3.1 Reference (unperturbed) case

As a reference case, we consider a homogeneous plane wave (reference wave) propagating in a perfectly elastic anisotropic medium (reference medium) in the direction of the vector $\mathbf{n}$. The homogeneous plane waves with a real-valued slowness vector, propagating in perfectly elastic anisotropic media have well known properties (Fedorov, 1968; Musgrave, 1970; Helbig, 1994; Červený, 2001), and are easy to calculate. In what follows, we assume that all three waves propagating in the anisotropic medium are well separated.

We denote the quantities corresponding to the reference case by the upper indices 0. As the reference medium is perfectly elastic, $a^0_{ijkl}$ (or $A^0_{\alpha\beta}$) are real-valued. We assume that the matrix $A^0_{\alpha\beta}$ is positive definite. Since the plane wave propagating in the unperturbed medium is homogeneous, $D^0 = 0$. The generalized Christoffel matrix $\Gamma^0_{ik}$ in the unperturbed medium is then given by the relation

$$\Gamma^0_{ik} = a^0_{ijkl} p^0_j p^0_l \ . \tag{12}$$

The real-valued slowness vector $\mathbf{p}^0 = \mathbf{P}^0 + \mathrm{i}\mathbf{A}^0$ of the homogeneous plane wave propagating in the unperturbed medium, and the quantities related to it, are given by expressions,

$$\mathbf{p}^0 = \sigma^0 \mathbf{n} \ , \quad \sigma^0 = 1/\mathcal{C}^0 \ , \quad \mathbf{P}^0 = \mathbf{p}^0 \ , \quad \mathbf{A}^0 = \mathbf{0} \ . \tag{13}$$

Here $\mathcal{C}^0$ is the real-valued phase velocity in the reference medium, in the direction of the unit vector $\mathbf{n}$. The quantity $\sigma_0$ is real valued. In the following, we consider only positive values of $\sigma^0$; the negative $\sigma^0$ would correspond to the plane wave propagating against $\mathbf{n}$. Note that the quantity $(\mathcal{C}^0)^2 = 1/\sigma_0^2$ is an eigenvalue of the standard Christoffel matrix $\bar{\Gamma}^0_{ik}$, $\bar{\Gamma}^0_{ik} = a^0_{ijkl} n_j n_l$. The polarization vector $\mathbf{U}^0$ of the considered homogeneous wave is related to the Christoffel matrix $\Gamma^0_{ik}$ by the system of equations (4):

$$(\Gamma^0_{ik} - \delta_{ik}) U^0_k = 0 \ . \tag{14}$$

Here, as in eq. (4), we consider $\mathbf{U}^0$ to be normalized, $U^0_i U^0_i = 1$. The attenuation vector $\mathbf{A}^0$ is zero, as both $\mathrm{Im}\sigma^0$ and $D^0$ are zero in the reference medium, see eq. (10).

In the following derivations, we also need the energy-velocity vector $\boldsymbol{\mathcal{U}}^0$ of a homogeneous plane wave propagating in the perfectly elastic anisotropic reference medium. It is given by the relation:

$$\mathcal{U}^0_i = a^0_{ijkl} p^0_l U^0_j U^0_k \ , \tag{15}$$

see Červený (2001, eq. 2.2.65). Another useful relation follows from (15) and (4):

$$\mathcal{U}^0_i n_i = \mathcal{C}^0 \ . \tag{16}$$

## 3.2 Perturbation of the slowness vector

The perturbed medium is assumed to be anisotropic and weakly viscoelastic,

$$a_{ijkl} = a^0_{ijkl} + \Delta a_{ijkl} \ , \quad \Delta a_{ijkl} = -\mathrm{i} a^I_{ijkl} \tag{17}$$

or alternatively

$$A_{\alpha\beta} = A^0_{\alpha\beta} + \Delta A_{\alpha\beta} \ , \quad \Delta A_{\alpha\beta} = -\mathrm{i} A^I_{\alpha\beta} \ . \tag{18}$$

We assume that the real-valued matrix $A^I_{\alpha\beta}$ is positive definite or zero. If we considered the "+" sign in the exponential factor of eq. (1), the signs of $\Delta a_{ijkl}$ or of $\Delta A_{\alpha\beta}$ in eqs (17) or (18) should be taken opposite. The assumption that $A^I_{\alpha\beta}$ is positive definite or zero remains valid in both cases.

We now expand eq. (4) in the perturbed medium. We use the notations:

$$\Gamma_{ik} = \Gamma^0_{ik} + \Delta\Gamma_{ik} \ , \quad U_k = U^0_k + \Delta U_k \ , \tag{19}$$

where $\Delta\Gamma_{ik}$ and $\Delta U_k$ are the perturbations of $\Gamma^0_{ik}$ and $U^0_k$, respectively. Then eq. (4) can be rewritten as

$$(\Gamma^0_{ik} + \Delta\Gamma_{ik} - \delta_{ik})(U^0_k + \Delta U_k) = 0 \ . \tag{20}$$

Keeping only the first-order perturbations, we obtain

$$\Delta\Gamma_{ik} U^0_k + (\Gamma^0_{ik} - \delta_{ik})\Delta U_k = 0 \ . \tag{21}$$

When deriving eq. (21), we have taken into account eq. (14). If we multiply eq. (21) by $U^0_i$, the last term in eq. (21) vanishes due to eq. (14). This yields a simple equation

$$\Delta\Gamma_{ik} U^0_i U^0_k = 0 \ . \tag{22}$$

This equation can be used to determine the perturbation $\Delta\sigma$ of $\sigma^0$,

$$\sigma = \sigma^0 + \Delta\sigma \ . \tag{23}$$

Using eqs (8) and (13), we obtain, for $\mathbf{n}$ fixed,

$$\mathbf{p} = \mathbf{p}^0 + \Delta\mathbf{p} \ , \qquad \text{where} \qquad \Delta\mathbf{p} = \mathbf{n}\Delta\sigma + \mathrm{i}D\mathbf{m} \ . \tag{24}$$

Here $\mathbf{m}$ is again an arbitrarily chosen unit vector perpendicular to $\mathbf{n}$, specifying with $\mathbf{n}$ the propagation-attenuation plane. The perturbation $\Delta\Gamma_{ik}$ of the generalized Christoffel matrix (12) is given by the relation

$$\Delta\Gamma_{ik} = \Delta a_{ijkl} p^0_j p^0_l + 2a^0_{ijkl}(\Delta\sigma n_j + \mathrm{i}D m_j) p^0_l \ , \tag{25}$$

where we used (17), (23) and (24). Inserting eq.(25) into eq.(22) yields

$$\Delta a_{ijkl} p^0_j p^0_l U^0_i U^0_k + 2a^0_{ijkl}(\Delta\sigma n_j + \mathrm{i}D m_j) p^0_l U^0_i U^0_k = 0 \ . \tag{26}$$

Using eqs (15), (16) and $\Delta a_{ijkl} = -\mathrm{i}a^I_{ijkl}$, see eq.(17), in eq.(26), we obtain

$$-\mathrm{i}a^I_{ijkl} p^0_j p^0_l U^0_i U^0_k + 2\mathcal{C}^0\Delta\sigma + 2\mathrm{i}D\mathcal{U}^0_j m_j = 0 \ . \tag{27}$$

From eq. (27), we can easily obtain the expression for $\Delta\sigma$ we have been looking for:

$$\Delta\sigma = \mathrm{i}(\mathcal{C}^0)^{-1}[\frac{1}{2}a^I_{ijkl} p^0_j p^0_l U^0_i U^0_k - D\mathcal{U}^0_j m_j] \ . \tag{28}$$

Let us note that $\Delta\sigma$ is a purely imaginary quantity. Thus, to determine, approximately, the complex-valued quantity $\sigma$ of a weakly inhomogeneous plane wave (D small), propagating in a weakly dissipative anisotropic medium ($|a^I_{ijkl}|$ small), it is not necessary to solve the algebraic equation (9). It is sufficient to use eq. (23), where $\sigma^0$ is given by eq. (13) and $\Delta\sigma$ by eq. (28). As soon as the approximate value of $\sigma$ is known, eq. (8) can be used to determine, approximately, the slowness vector $\mathbf{p}$.

The propagation vector $\mathbf{P}$ and the attenuation vector $\mathbf{A}$ in the perturbed case can be determined from eq. (10). We obtain

$$\begin{aligned} \mathbf{P} &= \mathbf{P}^0 = \mathbf{n}|\mathrm{Re}\sigma^0| = \mathbf{n}/\mathcal{C}^0 \ , && (29) \\ \mathbf{A} &= \frac{\mathbf{n}}{\mathcal{C}^0}\left[\frac{a^I_{ijkl} p^0_j p^0_l U^0_i U^0_k}{2} - D\mathcal{U}^0_j m_j\right] + D\mathbf{m} \ . && (30) \end{aligned}$$

The perturbed propagation vector $\mathbf{P}$ is the same as the unperturbed propagation vector $\mathbf{P}^0$, if only first-order perturbations are considered, see eq. (29). The perturbed attenuation vector $\mathbf{A}$, however, differs significantly from the unperturbed attenuation vector $\mathbf{A}^0 = \mathbf{0}$, see eq. (30). It has nonzero components to $\mathbf{n}$ and $\mathbf{m}$, and depends on $a^I_{ijkl}$ and $D$. Note that for the determination of the perturbation $\Delta\sigma$ and $\mathbf{A}$ using eqs (28) and (30), it is not necessary to know the perturbation $\Delta\mathbf{U}$.

The attenuation vector $\mathbf{A}$, given by eq. (30), can be used for the approximate evaluation of dissipation of a weakly inhomogeneous plane wave (with $D$ small), propagating in a weakly dissipative medium, specified by complex-valued viscoelastic moduli $a_{ijkl} = a^R_{ijkl} - \mathrm{i}a^I_{ijkl}$ (with $|a^I_{ijkl}|$ small). Equation (30) for $\mathbf{A}$ can be rewritten in the following form:

$$\mathbf{A} = \mathbf{n}\frac{\mathcal{A}^{in}}{2\mathcal{C}^0} + D[\mathbf{m} - \mathbf{n}\frac{(\boldsymbol{\mathcal{U}}^0 \cdot \mathbf{m})}{\mathcal{C}^0}] \; , \tag{31}$$

where $\mathcal{A}^{in}$ is given by the relation:

$$\mathcal{A}^{in} = a^I_{ijkl} p^0_j p^0_l U^0_i U^0_k \tag{32}$$

or alternatively

$$\mathcal{A}^{in} = (\mathcal{C}^0)^{-2} a^I_{ijkl} n_j n_l U^0_i U^0_k \; . \tag{33}$$

The quantity $\mathcal{A}^{in}$ is real valued and always nonnegative, as $A^I_{\alpha\beta}$ is positive definite or zero. We call the quantity $\mathcal{A}^{in}$, given by eq. (32) or (33), the *intrinsic attenuation factor* because it does not depend on the inhomogeneity parameter $D$; it depends on material (intrinsic) dissipation properties $a^I_{ijkl}$ only, no matter if the considered wave is homogeneous or inhomogeneous. For viscoelastic *isotropic* media, the intrinsic attenuation factor $\mathcal{A}^{in}$ equals $Q^{-1}$, where $Q$ is the well-known quality factor, see, e.g., Aki & Richards (1980). Eqs (32), (33) thus offer a simple and straightforward generalization of the quality factor $Q$ for anisotropic attenuating media. See more details later.

Equation (31) shows that the attenuation vector $\mathbf{A}$ depends both on intrinsic dissipation parameters $a^I_{ijkl}$, hidden in the intrinsic attenuation factor $\mathcal{A}^{in}$, and on the inhomogeneity parameter $D$ of the plane wave under consideration. By taking the scalar product of the attenuation vector $\mathbf{A}$, given by eq. (31) and of the energy-velocity vector $\boldsymbol{\mathcal{U}}^0$, given by (15), we obtain a surprisingly simple result for $\mathcal{A}^{in}$:

$$\mathcal{A}^{in} = 2\mathbf{A} \cdot \boldsymbol{\mathcal{U}}^0 \; . \tag{34}$$

Thus, the intrinsic attenuation factor $\mathcal{A}^{in}$, which is not influenced by the inhomogeneity of the wave under consideration, can be determined by computing the amplitude decay along the direction of the reference energy-velocity vector $\boldsymbol{\mathcal{U}}^0$. In the terminology of the ray method, this means computation of the amplitude decay *along a reference ray*. Although eq. (34) is only approximate, it is very general. It is valid for P, S1 and S2 homogeneous or inhomogeneous plane waves propagating in homogeneous, isotropic or anisotropic, viscoelastic media. In order to obtain the intrinsic attenuation factor $\mathcal{A}^{in}$ in a weakly dissipative medium, it is not necessary to consider homogeneous plane waves. The homogeneity or weak inhomogeneity of the plane wave in such a medium does not play any role in the determination of the intrinsic attenuation factor $\mathcal{A}^{in}$ since the term containing the inhomogeneity parameter $D$ is automatically eliminated by projecting the attenuation vector $\mathbf{A}$ on the direction of $\boldsymbol{\mathcal{U}}^0$, see eq. (34). Although $\mathbf{A}$ varies with varying $D$, the scalar product $\mathbf{A}\cdot\boldsymbol{\mathcal{U}}^0$ remains the same and independent of $D$ for weakly dissipative media.

It is interesting to note that the relations for the intrinsic attenuation factor (32) or (33) are valid even for arbitrary high-frequency elementary waves with curved wave fronts, including waves generated by point sources, propagating in *heterogeneous*, isotropic or anisotropic, weakly dissipative media. This has been shown by Gajewski & Pšenčík (1992, eq.7), see also Červený (2001, eq. 5.5.28). In their ray-theory treatment of high-frequency waves propagating in weakly dissipative anisotropic media, Gajewski & Pšenčík (1992) found the equation for the quality factor $Q = 1/\mathcal{A}^{in}$, with $\mathcal{A}^{in}$ given by eq. (32). Eq. (34) shows that their result remains valid even for arbitrary weakly inhomogeneous waves although inhomogeneous waves were not explicitly considered in their study.

Let us mention that a detailed discussion of the intrinsic attenuation (not depending on $D$) of inhomogeneous plane waves propagating in orthorhombic, weakly dissipative media, based on numerical studies can be also found in Deschamps & Assouline (2000). They showed that the quantity, called here the intrinsic attenuation factor, of a weakly inhomogeneous plane wave propagating in a weakly dissipative medium does not depend on the inhomogeneity of the wave under consideration if attenuation is measured along the direction of the Poynting vector (i.e., along the energy-velocity vector).

## 3.3 Perturbation of polarization vectors

If we wish to know the displacement vector $\mathbf{u}$ in the perturbed medium, it is not sufficient to determine the perturbation $\Delta\mathbf{p}$ of the slowness vector, see eqs (24) and (28), but also the perturbation $\Delta\mathbf{U}$ of the polarization vector. In a general anisotropic medium, there are three plane waves, propagating in the given direction $\mathbf{n}$, namely P, S1 and S2 waves. Let us emphasize again that we assume here that the three waves propagate separately. We denote their polarization vectors by $\mathbf{U}^{(1)}$, $\mathbf{U}^{(2)}$and $\mathbf{U}^{(3)}$, and number them arbitrarily. We can expand the polarization vectors as

$$\mathbf{U}^{(m)} = \mathbf{U}^{(m)0} + \Delta\mathbf{U}^{(m)} \ , \tag{35}$$

where $\mathbf{U}^{(m)0}$ are the unperturbed polarization vectors and $\Delta\mathbf{U}^{(m)}$ are their perturbations. The polarization vectors $\mathbf{U}^{(m)0}$ are real-valued, unit and mutually perpendicular so that they satisfy the relation:

$$\mathbf{U}^{(m)0}\mathbf{U}^{(n)0} = \delta_{mn} \ . \tag{36}$$

We require that the perturbed, complex-valued polarization vectors $\mathbf{U}^{(m)}$ are also unit:

$$\mathbf{U}^{(m)} \cdot \mathbf{U}^{(m)} = 1 \tag{37}$$

(no summation over $m$), but they do not need to be mutually perpendicular. Inserting eq. (35) into eq. (37), and keeping only the first-order terms, we obtain

$$\mathbf{U}^{(m)}\mathbf{U}^{(m)} = \mathbf{U}^{(m)0}\mathbf{U}^{(m)0} + 2\mathbf{U}^{(m)0}\Delta\mathbf{U}^{(m)} \ . \tag{38}$$

The requirement (37) then yields

$$\mathbf{U}^{(m)0}\Delta\mathbf{U}^{(m)} = 0\ . \tag{39}$$

It means that the first-order perturbation $\Delta\mathbf{U}^{(m)}$ of $\mathbf{U}^{(m)0}$ must be perpendicular to $\mathbf{U}^{(m)0}$.

Let us now select one of the plane waves and let us specify it by $m = 1$. Due to eq. (39), the perturbation $\Delta\mathbf{U}^{(1)}$ can be expressed in the following form:

$$\Delta\mathbf{U}^{(1)} = \alpha\mathbf{U}^{(2)0} + \beta\mathbf{U}^{(3)0}\ , \tag{40}$$

where $\alpha$ and $\beta$ are first-order perturbation quantities. Equation (21) for the selected wave reads,

$$\Delta\Gamma_{ik}^{(1)}U_k^{(1)0} + (\Gamma_{ik}^{(1)0} - \delta_{ik})\Delta U_k^{(1)} = 0\ . \tag{41}$$

Matrices $\Delta\Gamma_{ik}^{(1)}$ and $\Gamma_{ik}^{(1)0}$ correspond again to the selected wave. Matrix $\Delta\Gamma_{ik}^{(1)}$is given by (25) and $\Gamma_{ik}^{(1)0}$ by (12). Inserting eq. (40) into eq. (41) yields

$$\Delta\Gamma_{ik}^{(1)}U_k^{(1)0} + (\Gamma_{ik}^{(1)0} - \delta_{ik})(\alpha U_k^{(2)0} + \beta U_k^{(3)0}) = 0\ . \tag{42}$$

We first determine $\alpha$; the determination of $\beta$ is analogous. Multiplying eq. (42) by $U_i^{(2)0}$ yields,

$$\begin{aligned}\Delta\Gamma_{ik}^{(1)}U_k^{(1)0}U_i^{(2)0} &+ \alpha(\Gamma_{ik}^{(1)0}U_k^{(2)0}U_i^{(2)0} - U_i^{(2)0}U_i^{(2)0})\\ &+\beta(\Gamma_{ik}^{(1)0}U_k^{(3)0}U_i^{(2)0} - U_i^{(3)0}U_i^{(2)0}) = 0\ .\end{aligned} \tag{43}$$

Let us discuss individual terms in eq. (43). Due to eq. (36), we have $U_i^{(2)0}U_i^{(2)0} = 1$ and $U_i^{(3)0}U_i^{(2)0} = 0$. The quantity $\Gamma_{ik}^{(1)0}$ can be expressed in terms of $\Gamma_{ik}^{(2)0}$,

$$\Gamma_{ik}^{(1)0} = (\mathcal{C}^{(2)0}/\mathcal{C}^{(1)0})^2\Gamma_{ik}^{(2)0}\ , \tag{44}$$

as the two quantities differ only by the components of the slowness vector $\mathbf{p}^{(m)0} = \mathbf{n}/\mathcal{C}^{(m)0}$. Then

$$\begin{aligned}\Gamma_{ik}^{(1)0}U_k^{(2)0}U_i^{(2)0} &= (\mathcal{C}^{(2)0}/\mathcal{C}^{(1)0})^2\Gamma_{ik}^{(2)0}U_k^{(2)0}U_i^{(2)0} = (\mathcal{C}^{(2)0}/\mathcal{C}^{(1)0})^2\ ,\\ \Gamma_{ik}^{(1)0}U_k^{(3)0}U_i^{(2)0} &= 0\ .\end{aligned} \tag{45}$$

In eq. (45) we have used eq. (14). Inserting the preceding equations into eq. (43) yields

$$\Delta\Gamma_{ik}^{(1)}U_k^{(1)0}U_i^{(2)0} + \alpha[(\mathcal{C}^{(2)0}/\mathcal{C}^{(1)0})^2 - 1] = 0\ . \tag{46}$$

Similar equation for $\beta$ can be obtained quite analogically, by multiplying eq. (42) by $U_i^{(3)0}$. From both equations, we obtain

$$\alpha = \frac{\Delta\Gamma_{ik}^{(1)}U_k^{(1)0}U_i^{(2)0}}{1 - (\mathcal{C}^{(2)0}/\mathcal{C}^{(1)0})^2}\ ,\quad \beta = \frac{\Delta\Gamma_{ik}^{(1)}U_k^{(1)0}U_i^{(3)0}}{1 - (\mathcal{C}^{(3)0}/\mathcal{C}^{(1)0})^2}\ . \tag{47}$$

Inserting eqs (47) into eq. (40), we obtain the final expression for $\Delta U_k^{(1)}$:

$$\Delta U_k^{(1)} = \frac{\Delta\Gamma_{ij}^{(1)}U_i^{(2)0}U_j^{(1)0}}{1 - (\mathcal{C}^{(2)0}/\mathcal{C}^{(1)0})^2}U_k^{(2)0} + \frac{\Delta\Gamma_{ij}^{(1)}U_i^{(3)0}U_j^{(1)0}}{1 - (\mathcal{C}^{(3)0}/\mathcal{C}^{(1)0})^2}U_k^{(3)0}\ . \tag{48}$$

Note that the methods how to calculate the polarization vectors $\mathbf{U}^{(m)0}$ and the phase velocities $\mathcal{C}^{(m)0}$ in perfectly elastic anisotropic media are well known, see, e.g., Červený (2001, p. 152).

All quantities in eq. (48) are real valued, except $\Delta\Gamma_{ij}^{(1)}$, which is imaginary valued, see eq. (25) with eqs (17) and (28). Thus the first-order perturbations of the polarization vectors $\mathbf{U}^{(1)}$, $\mathbf{U}^{(2)}$ and $\mathbf{U}^{(3)}$ are imaginary valued for all three plane waves propagating in viscoelastic anisotropic media. In this way, the perturbation (17) (or (18)) causes nonlinear polarization of plane waves in viscoelastic media, no matter if the waves are homogeneous ($D = 0$) or inhomogeneous ($D \neq 0$). Another consequence of the fact that $\Delta\mathbf{U}^{(1)}$ is purely imaginary is equality of the perturbed and unperturbed time-harmonic energy-velocity vector $\boldsymbol{\mathcal{U}} = \boldsymbol{\mathcal{U}}^0$, see Červený & Pšenčík (2006a).

The final equation for the polarization vector $\mathbf{U}^{(1)}$ in the perturbed medium reads:

$$U_k^{(1)} = U_k^{(1)0} + \frac{\Delta\Gamma_{ij}^{(1)} U_i^{(2)0} U_j^{(1)0}}{1 - (\mathcal{C}^{(2)0}/\mathcal{C}^{(1)0})^2} U_k^{(2)0} + \frac{\Delta\Gamma_{ij}^{(1)} U_i^{(3)0} U_j^{(1)0}}{1 - (\mathcal{C}^{(3)0}/\mathcal{C}^{(1)0})^2} U_k^{(3)0} \ . \tag{49}$$

As the first term of eq. (49) is real valued and the second and third are imaginary valued, the expression for $\mathbf{U}^{(1)}$ is complex valued. Consequently, the polarization vector $\mathbf{U}^{(1)}$ is a unit, complex-valued vector. For $D$ zero, the polarization vectors $\mathbf{U}^{(1)}$, $\mathbf{U}^{(2)}$ and $\mathbf{U}^{(3)}$ are mutually perpendicular, for $D$ nonzero, they are not.

Equation (49) becomes inaccurate if $\mathcal{C}^{(1)0}$ approaches $\mathcal{C}^{(2)0}$ or $\mathcal{C}^{(3)0}$, and completely useless if $\mathcal{C}^{(1)0}$ equals either $\mathcal{C}^{(2)0}$ or $\mathcal{C}^{(3)0}$. This may happen, for example, in the case of shear waves in a vicinity of shear-wave singularities. In such a case, referred to as degenerate, it is necessary to use a more sophisticated method to determine $\Delta\mathbf{U}$, see, for example, Jech & Pšenčík (1989).

# 4 Numerical examples

In this section we present and discuss comparison of results based on the mixed specification of the slowness vector given by eq. (8) with $\sigma$ determined exactly by the numerical solution of eq. (9) and approximately from eqs (23) and (28). We consider both homogeneous and inhomogeneous plane waves propagating in a model of an anisotropic viscoelastic medium described below. The model has been proposed by Jakobsen et al. (2003) and used by Červený & Pšenčík (2005b, 2006a,b). As in the previous papers, we choose again the model corresponding to the frequency of approximately 35 Hz. For simplicity, we consider the density of 1000 kg/m$^3$. The $6 \times 6$ matrix of complex-valued, density-normalized viscoelastic moduli $\mathbf{A_C}$ in the Voigt notation, measured in (km/s)$^2$ reads:

$$\mathbf{A_C} = \mathbf{A_1} - \mathrm{i}\mathbf{A_2}, \tag{50}$$

where

$$\mathbf{A_1} = \begin{pmatrix} 46.631 & 5.983 & 4.278 & 0. & 0. & 0. \\ & 46.631 & 4.278 & 0. & 0. & 0. \\ & & 19.931 & 0. & 0. & 0. \\ & & & 13.444 & 0. & 0. \\ & & & & 13.444 & 0. \\ & & & & & 20.324 \end{pmatrix}$$

and

$$\mathbf{A_2} = \begin{pmatrix} 0.033 & 0.022 & 0.156 & 0. & 0. & 0. \\ & 0.033 & 0.156 & 0. & 0. & 0. \\ & & 1.312 & 0. & 0. & 0. \\ & & & 0.055 & 0. & 0. \\ & & & & 0.055 & 0. \\ & & & & & 0.005 \end{pmatrix}.$$

Note that both matrices $\mathbf{A_1}$ and $\mathbf{A_2}$ are positive definite. The matrix $\mathbf{A_1}$ corresponds to an anisotropic medium of hexagonal symmetry with vertical axis of symmetry, which represents the direction of the kiss singularity.

The most important feature of the matrix $\mathbf{A_2}$ is the extremely great value of $(\mathbf{A_2})_{33}$ and small values of $(\mathbf{A_2})_{44}$, $(\mathbf{A_2})_{55}$ and $(\mathbf{A_2})_{66}$. They indicate that the attenuation of SH waves is considerably smaller than the attenuation of P and SV waves.

In the following, we compare the quantity $\mathrm{Im}\sigma$, i.e., the projection of the attenuation vector $\mathbf{A}$ into the vector $\mathbf{n}$, and the polarization vectors $\mathbf{U}$ calculated by exact and approximate formulae, presented and derived in the preceding sections. As in previous studies, we concentrate on P, S1 and S2 plane waves propagating in the plane of symmetry of the medium specified in (50). We study first $\mathrm{Im}\sigma$ as a function of direction $\mathbf{n}$ for several fixed values of the inhomogeneity parameter $D$, which is measured in s km$^{-1}$. Then we study $\mathrm{Im}\sigma$ as a function of $D$, for several directions of $\mathbf{n}$. Finally we study polarization as a function of $\mathbf{n}$ for several values of $D$. The polar diagrams presented in this section are related to the direction of the unit vector $\mathbf{n}$, perpendicular to the wavefront. The vector $\mathbf{n}$ is parameterized by the propagation angle $i$; $\mathbf{n} \equiv (\sin i, 0, \cos i)$, with $i = 0^0$ upwards, and $i = 90^0$ to the right.

## 4.1 Approximate and exact $\mathrm{Im}\sigma$

In Fig.1, we present P-wave (fastest wave) polar diagrams of $\mathrm{Im}\sigma = \mathbf{A} \cdot \mathbf{n}$ for four values of $D$: $D = 0$ (homogeneous wave), $D = 0.01$, 0.02 and 0.05. Red colour is reserved for the approximate, first-order, results obtained from eqs (23) and (28). Violet is used to denote the exact results, obtained by solving numerically the algebraic equation (9). The approximate results are plotted over the exact ones, therefore, due to the perfect fit of approximate and the exact results for smaller values of $D$, the red colour is prevailing colour of the plots. Slight differences between exact and approximate results can be observed only for $D = 0.05$. In all plots we can observe strong directivity of $\mathrm{Im}\sigma$, which could be observed for several other quantities related to $\mathrm{Im}\sigma$, including the intrinsic attenuation factor $\mathcal{A}^{in}$ ($\mathcal{A}^{in} = 2[\mathcal{C}^0 \mathrm{Im}\sigma + D(\boldsymbol{\mathcal{U}}^0 \cdot \mathbf{m})]$). For more details on behaviour of $\mathrm{Im}\sigma$, see Červený & Pšenčík (2005b).

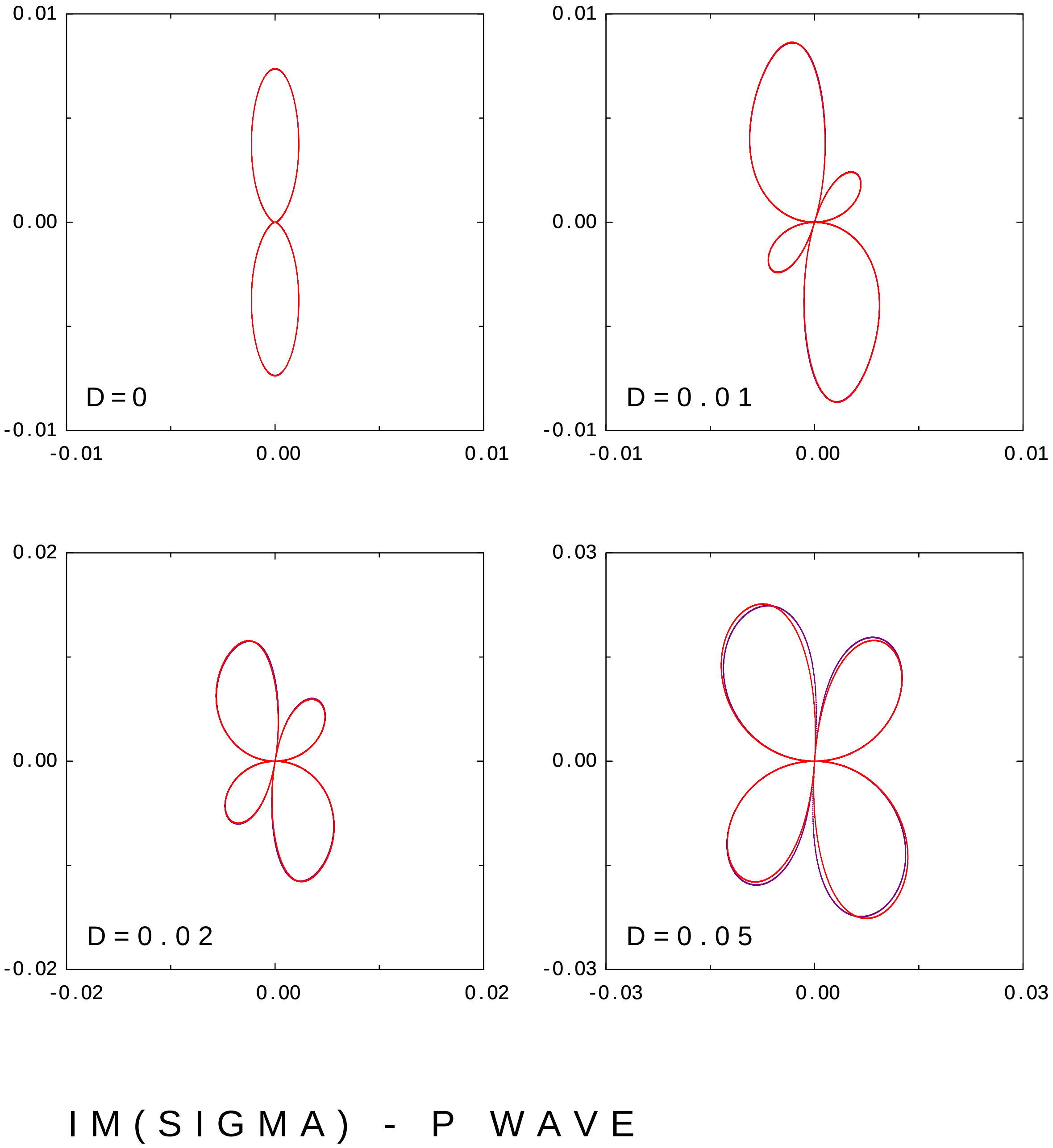



Fig.1 Polar diagrams of Im$\sigma$ for the fastest plane wave (P wave) in a plane of symmetry of the medium (eq. 50) for $D = 0$ (homogeneous wave), 0.01, 0.02 and 0.05. Violet: exact computations, solution of eq. (9); red: approximate computations, eqs (23) and (28). Observable differences only for $D = 0.05$.

Fig.2 shows the same as Fig.1 but for S waves. Both S waves are denoted by the same colour, blue for the approximate results and green for the exact ones. Again, the approximate results are plotted over the exact ones. Therefore, due to a perfect fit of the approximate and exact results, blue is the prevailing colour. For $D = 0.01$ and $D = 0.02$, each of the two S waves has four lobes. For $D = 0.05$, the number of lobes of the SH wave remains four, but of the SV wave is even eight. As explained by Červený & Pšenčík (2005b), the increased number of lobes is related to the intersection of the phase-velocity sheets of both S waves. The plot for $D = 0.05$ in Fig.2 is complicated even more by differences of the plots of exact and approximate results. In contrast to P waves, slight differences of approximate and exact results can be observed already for $D = 0.02$. For more details on behaviour of Im$\sigma$, see again Červený & Pšenčík (2005b).

From Figs 1 and 2, we can conclude that the approximate formula for Im$\sigma$ yields, upto $|D|$=0.02, results, which are nearly identical with exact ones. The range of the parameter $|D|$, for which the fit is perfect, can be even extended to $|D|$=0.05 if we concentrate on angles of propagation different from $0^0$, which corresponds to the kiss singularity of $\mathbf{A_1}$.

The latter observation is confirmed by results shown in Fig.3, where the dependence of Im$\sigma$ on $D$ is shown for three propagation angles, $25^0$, $45^0$ and $65^0$. The colours have the same meaning as in Figs 1 and 2. We can see that the fit of the approximate and exact results improves with increasing value of the angle of propagation. For angles of propagation close to $0^0$ (not shown here), i.e., for directions close to the direction of the kiss singularity of the corresponding perfectly elastic medium (see $\mathbf{A}_1$ in (50)), the interval of $|D|$ with perfect fit shrinks. Note that the approximate and exact curves corresponding to one of the S waves (specifically SH) coincide in the whole interval of $D$ shown in Fig.3.

## 4.2 Intrinsic attenuation factor

Fig.4 shows the intrinsic attenuation factor $\mathcal{A}^{in}$ of all three waves considered in the previous figures, in the plane of symmetry of the medium specified in (50). Red colour denotes the fastest wave, i.e., the P wave, blue colour faster of the S waves, (SV wave); black colour: slower of the S waves (SH-wave). Let us emphasize that Fig.4 holds for homogeneous as well as weakly inhomogeneous plane waves. We can observe significant similarity with the plots for $D = 0$ in Figs 1 and 2. This similarity follows from the above-mentioned relation ($\mathcal{A}^{in} = 2[\mathcal{C}^0\mathrm{Im}\sigma + D(\boldsymbol{\mathcal{U}}^0 \cdot \mathbf{m})]$). Since Fig.4 is the same for any small value of $D$, including $D$=0, we can see that, for $D$=0, $\mathcal{A}^{in}$ differs from Im$\sigma$ only by the factor $2\mathcal{C}^0$. Note that the angular variation of $\mathcal{C}^0$ is only moderate, see Červený & Pšenčík (2005b), which means that the factor $2\mathcal{C}^0$ is nearly constant. Fig. 4 clearly shows that in the plane of symmetry of the medium (50), the intrinsic attenuation of the P wave is maximum along the axis of symmetry and minimum in the direction perpendicular to it. Note that the maximum value of the P-wave intrinsic attenuation factor $\mathcal{A}^{in}$ is greater than the maximum value of $\mathcal{A}^{in}$ of the SV wave. The SH-wave attenuation behaves like the P-wave attenuation, but it is considerably weaker. The SV-wave attenuation is minimum along the axis of symmetry and in the direction perpendicular to it. It is maximum in between. The above propoerties of $\mathcal{A}^{in}$ are consequence of great value of $(\mathbf{A_2})_{33}$ and small values of $(\mathbf{A_2})_{44}$, $(\mathbf{A_2})_{55}$ and $(\mathbf{A_2})_{66}$.

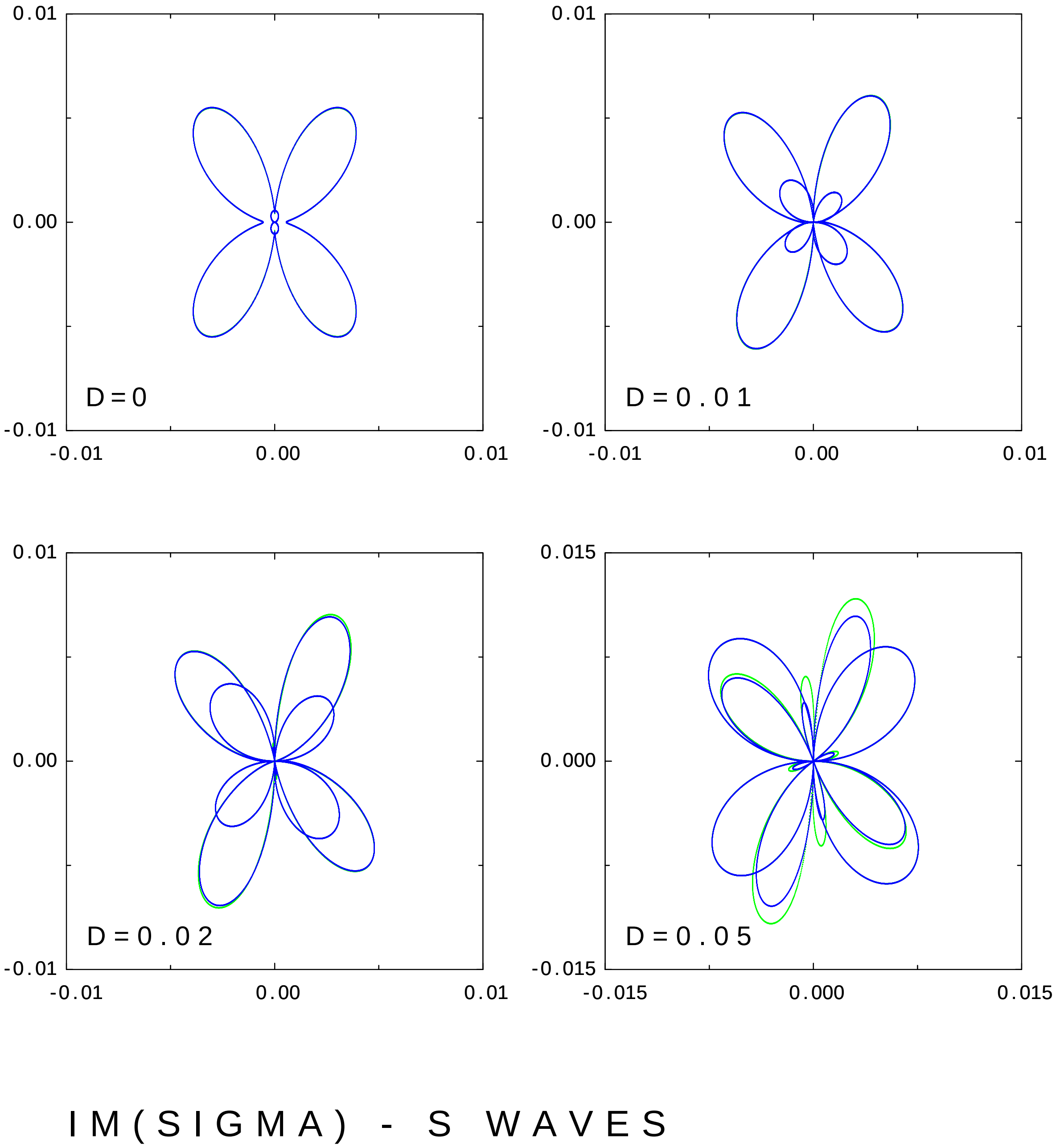



Fig.2 Polar diagrams of Im$\sigma$ for the slower plane waves (S waves) in a plane of symmetry of the medium (eq. 50) for $D = 0$ (homogeneous wave), 0.01, 0.02 and 0.05. Green: exact computations, solution of eq. (9); blue: approximate computations, eqs (23) and (28). Observable differences slight for $D = 0.02$, more significant for $D = 0.05$. Note the complicated diagram for $D = 0.05$, caused by eight lobes of the SV wave (in contrast to four lobes in the remaining plots) and four lobes of the SH wave.

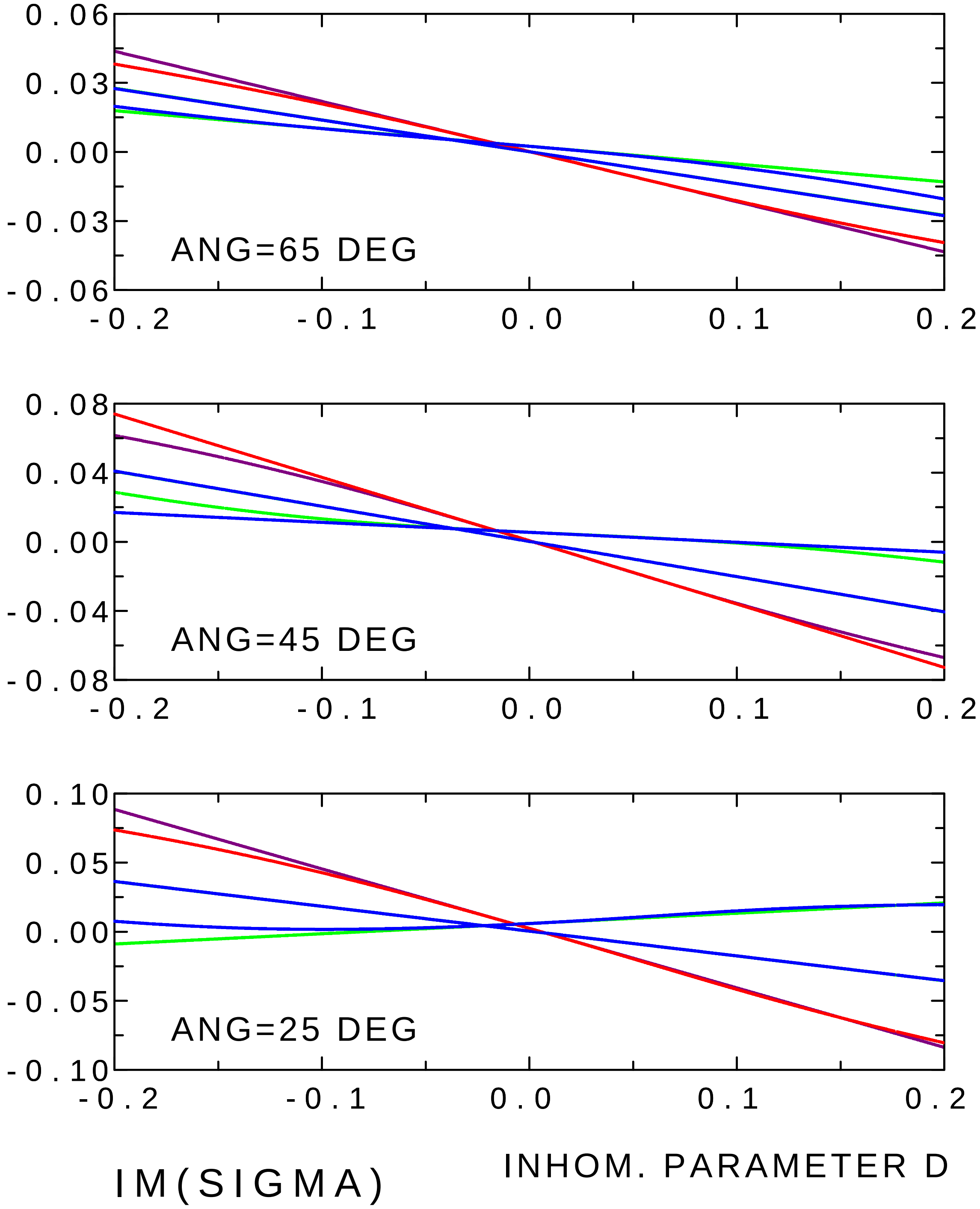



Fig.3 Variations with $D$ of Im$\sigma$ of plane P and S waves in a plane of symmetry of the medium (eq. 50) for the propagation angles $i = 25^0$ (bottom), $45^0$ (middle) and $65^0$ (top). Violet and green: exact computations for P waves and S waves, respectively, solution of eq. (9); red and blue: approximate computations for P and S waves, respectively, eqs (23) and (28).

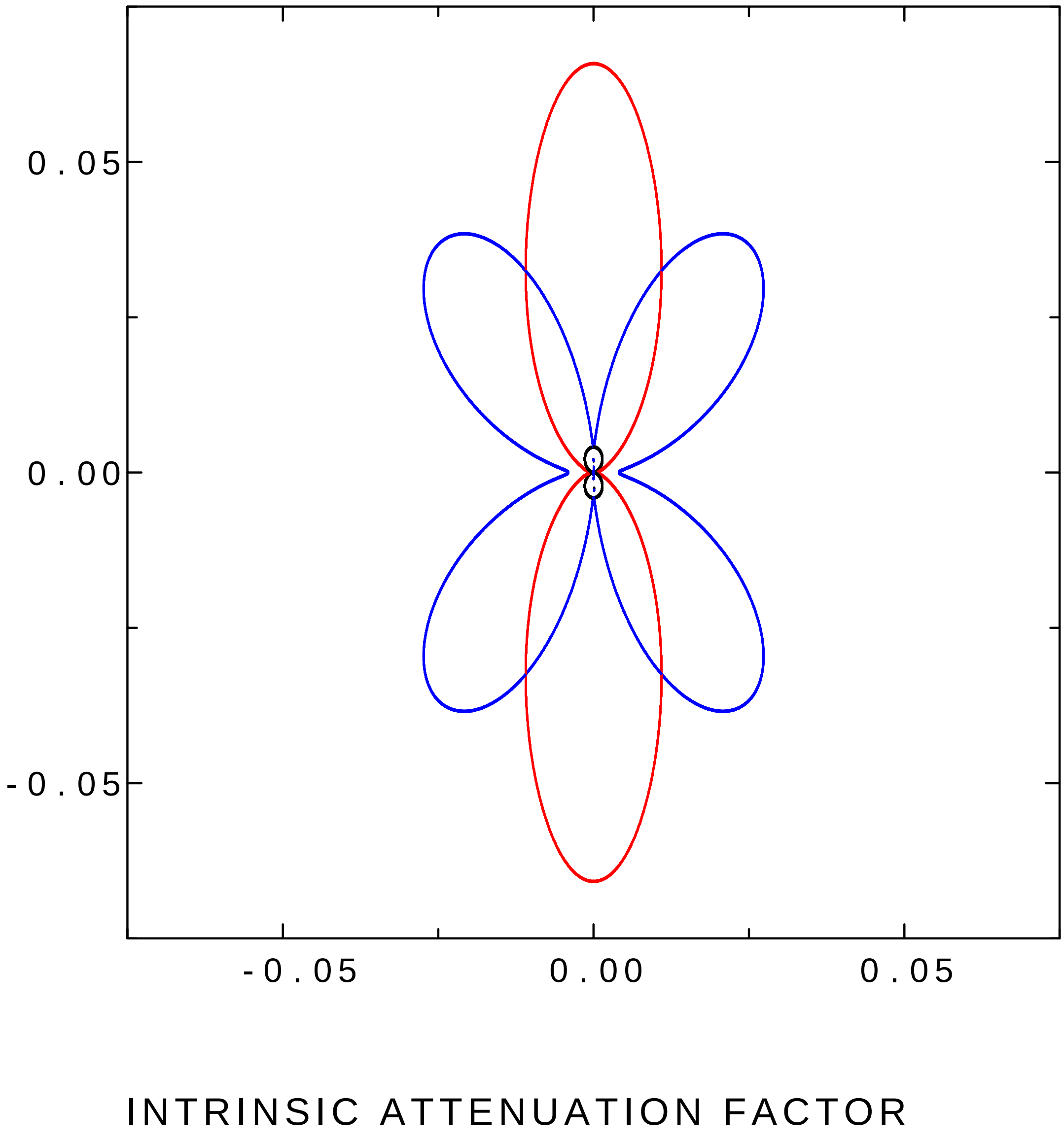



Fig.4 Intrinsic attenuation factor $\mathcal{A}^{in}$. Red colour: fastest wave (P wave); blue colour: faster of S waves (SV wave); black colour: slower of S waves (SH-wave).

### 4.3 Approximate and exact polarization vectors

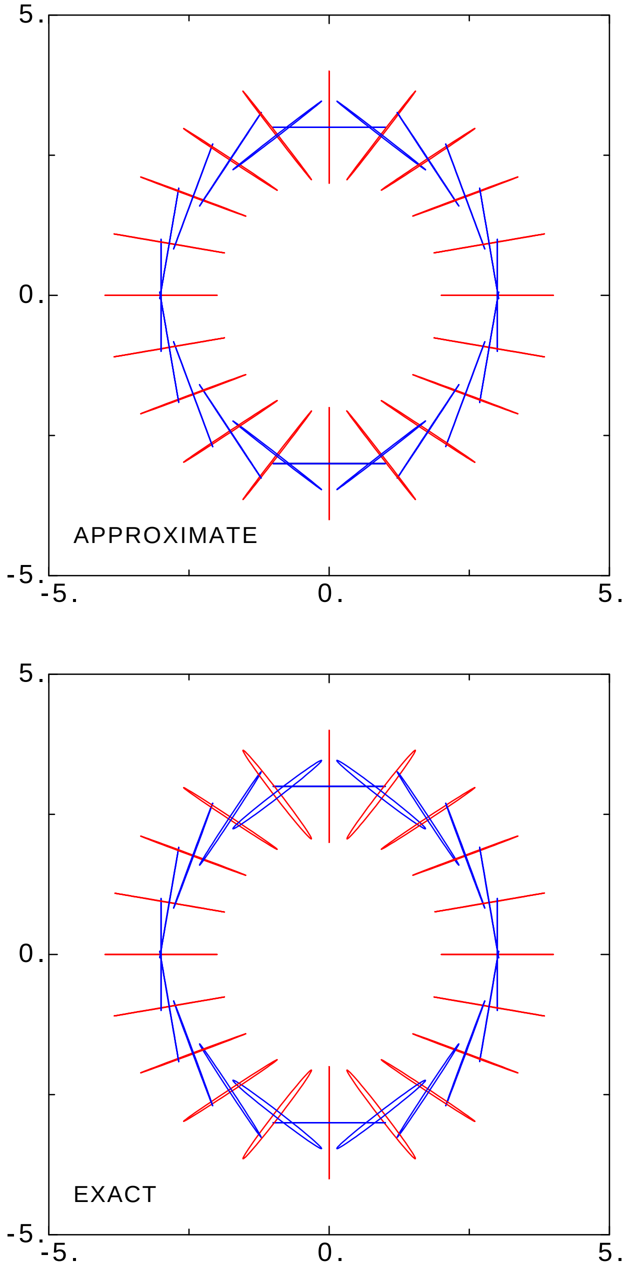
Figs 5 and 6 show comparison of approximate (top) and exact (bottom) particle motion diagrams corresponding to homogeneous waves ($D$=0) and inhomogeneous waves ($D$=0.05). Red colour is used to denote P waves, blue the slower of the S waves, mostly SV waves and black faster of the S waves, mostly SH waves. Note that for propagation directions close to the vertical, SV wave becomes faster than the SH wave, see Červený & Pšenčík (2005b). SH waves, which are polarized linearly, have their particle motion perpendicular to the plane of the plots. We can see a good overall coincidence of approximate and exact results. Nevertheless, some small differences can be found even for homogeneous waves, see Fig.5. Ellipticity of the polarization calculated approximately is slightly smaller than in the exact case. The situation is opposite for inhomogeneous waves, see Fig. 6. The nonsymmetry of plots with respect to vertical is less pronounced in the approximate results than in the exact ones. For more details about polarization of plane waves in viscoelastic anisotropic media, see, e.g. Červený & Pšenčík (2006b). Note strong elliptical polarization for $D = 0.05$, which indicates that already for such a small value of $D$, behaviour of the corresponding waves deviates rather strongly from behaviour of a homogeneous wave. This is why we did not consider higher values of $D$ than 0.05 in the numerical tests and why we can consider as weakly inhomogeneous waves those whose value of the inhomogeneity parameter $|D|$ is less than 0.05.

# 5 Conclusions

Perturbation method was used to derive approximate expressions for the slowness and polarization vectors of P, S1 and S2 weakly inhomogeneous plane waves propagating in a generally anisotropic, weakly attenuating medium. As an unperturbed case, homogeneous plane waves with a real-valued slowness vector, propagating in a generally anisotropic, perfectly elastic medium were considered. It was found that in the first-order approximation, weak inhomogeneity of the considered waves and weak intrinsic dissipation of the medium have no effect on the propagation vector. The attenuation vector, on the contrary, depends, in the first-order approximation, both on intrinsic dissipation and on inhomogeneity of the considered wave.

It was shown that the projection of the attenuation vector on the direction of the energy-velocity vector yields a quantity, which is independent of the inhomogeneity of the wave. We introduced the so-called intrinsic attenuation factor $\mathcal{A}^{in}$, which is proportional to this quantity and since it depends only on intrinsic dissipation, it can be used for characterization of material properties of the medium through which the waves propagate. The intrinsic attenuation factor can be determined by simply computing the amplitude decay along the direction of the energy-velocity vector, i.e. along the direction of the ray. As the results of numerical test in Fig.4 indicate, the intrinsic attenuation factor displays a strong directivity, property, which can be utilized in the inversion.

The approximate formulae for the slowness and polarization vectors were tested by comparing approximate and exact results. The tests have shown that the approximate results are nearly identical with exact ones for the values of the inhomogeneity parameter




D = 0

Fig.5 Comparison of exact (bottom) and approximate (top) polar diagrams of the particle motion in a plane of symmetry of the medium (eq. 50) for $D = 0$ (homogeneous wave). Red colour: fastest wave (P wave); blue colour: faster of S waves (SV wave); SH-wave polarization is perpendicular to the plots.

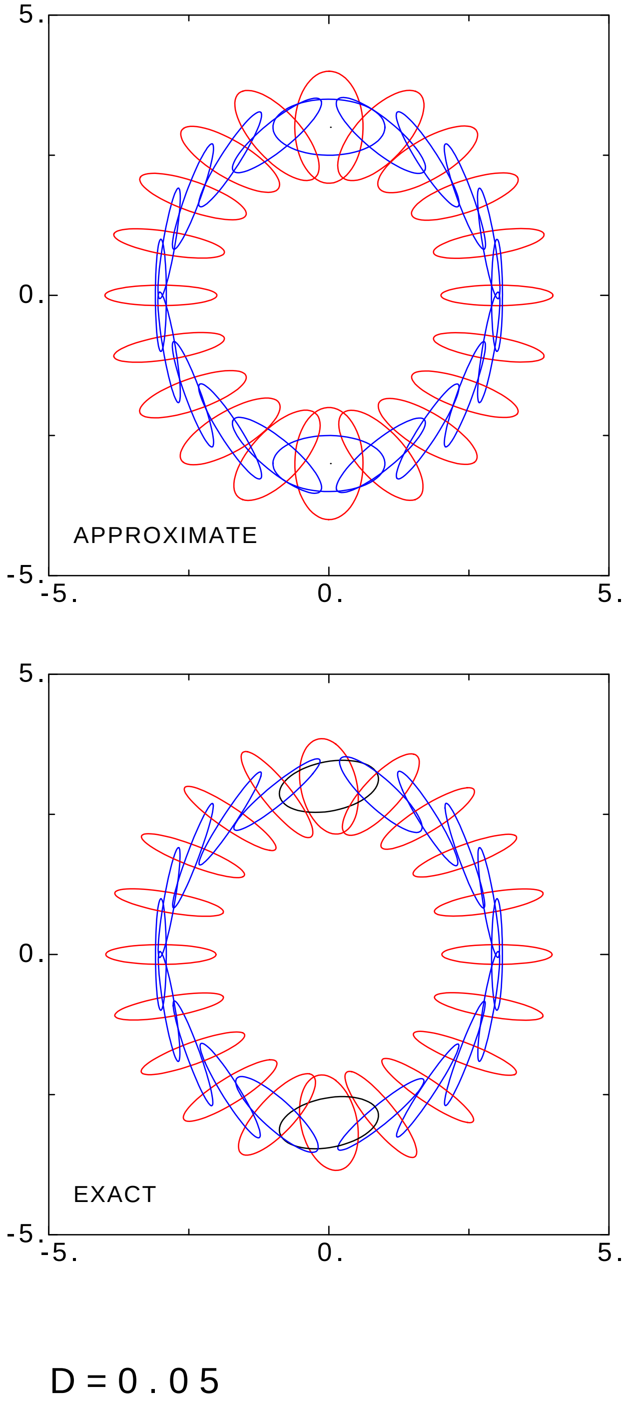



Fig.6  Comparison of exact (bottom) and approximate (top) polar diagrams of the particle motion in a plane of symmetry of the medium (eq. 50) for $D = 0.05$. Red colour: fastest wave (P wave); blue colour: faster of S waves (mostly SV wave); Black colour: slower of S waves (mostly SH wave polarized perpendicularly to the plots).

$D$ from the interval (-0.02, 0.02), sometimes even (-0.05, 0.05) s km$^{-1}$. Since as weakly inhomogeneous, we consider waves whose $D$ is less than 0.05 s km$^{-1}$, see Sec. 4.3, this means that the approximate formulae can be used instead of the exact ones for weakly inhomogeneous waves without limitations.

In isotropic viscoelastic media, the intrinsic attenuation factor is closely related to the well-known quality factor $Q$. In fact, $\mathcal{A}^{in} = Q^{-1}$. This offers a straightforward extension of the definition of the quality factor for anisotropic viscoelastic media. The $Q$ factor, defined in this way, is independent of inhomogeneity of waves and describes only intrinsic material properties. This distinguishes it from, for example, the definition of the $Q$ factor introduced by Krebes & Le (1994), in which $Q$ depends also on inhomogeneity of a considered wave. For inhomogeneous waves, such a definition of $Q$ leads to irregular behaviour described by Krebes & Le (1994). The definition of $Q$ as $Q^{-1} = \mathcal{A}^{in}$, where $\mathcal{A}^{in}$ is given by (32) removes the mentioned problems. This definition may also represent an alternative to other definitions of the $Q$ factor for viscoelastic anisotropic media, which appear in the literature, see, e.g., Zhu & Tsvankin (2006, 2007), Zhu et al., (2007). A detailed study of behaviour of the $Q$ factor defined in the above way will be published elsewhere.

Another desirable future step is to generalize the above results for elementary high-frequency inhomogeneous waves propagating in heterogeneous anisotropic, weakly viscoelastic media. Great advantage of such an approach will be that it will not require time-consuming and complicated complex-valued ray tracing, necessary in inhomogeneous viscoelastic anisotropic media, see, e.g., Thomson (1997). Only standard real-valued ray tracing in the reference, perfectly elastic anisotropic medium, will be sufficient.

The perturbation formula for the slowness vector presented in this paper may also find applications in the solution of the R/T problems, specifically in the determination of slowness vectors of R/T waves generated by incidence of a plane wave at a plane interface separating two homogeneous, viscoelastic, isotropic or anisotropic media, see Červený (2007).

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## References

Aki, K. & Richards, P., 1980. *Quantitative seismology. Theory and methods.* Freeman, San Francisco.

Carcione, J.M., 2001. *Wave Fields in Real Media: Wave Propagation in Anisotropic, Anelastic and Porous Media.* Pergamon, Amsterdam.

Carcione, J.M., 2007. *Wave Fields in Real Media: Wave Propagation in Anisotropic, Anelastic, Porous and electromagnetic Media.* Elsevier Science Ltd, Amsterdam.

Červený, V., 1982. Direct and inverse kinematic problems for inhomogeneous anisotropic media - Linearization approach. *Contrib. Geophys. Inst. Slov. Acad. Sci.*, **13**, 127–133.

Červený, V., 2001. *Seismic Ray Theory.* Cambridge Univ. Press, Cambridge.

Červený, V., 2004. Inhomogeneous harmonic plane waves in viscoelastic anisotropic media. *Stud. Geophys. Geod.*, **48**, 167–186.

Červený, V., 2007. Reflection/transmission laws for slowness vectors in viscoelastic anisotropic media. *Stud. Geophys. Geod.*, **51**, in print.

Červený, V. & Jech, J., 1982. Linearized solutions of kinematic problems of seismic body waves in inhomogeneous slightly anisotropic media. *J. Geophys.*, **51**, 96–104.

Červený, V. & Pšenčík, I. 2005a. Plane waves in viscoelastic anisotropic media. I. Theory. *Geophys. J. Int.*, **161**, 197–212.

Červený, V. & Pšenčík, I. 2005b. Plane waves in viscoelastic anisotropic media. II. Numerical examples. *Geophys. J. Int.*, **161**, 213–229.

Červený, V. & Pšenčík, I., 2006a. Energy flux in viscoelastic anisotropic media. *Geophys. J. Int.*, **166**, 1299–1317.

Červený, V. & Pšenčík, I.,2006b. Particle motion of plane waves in viscoelastic anisotropic media. *Russian Geology and Geophysics*, 47, 551–562.

Chapman, C.H. & Pratt, R.G., 1992. Traveltime tomography in anisotropic media: I. Theory. *Geophys. J. Int.*, **109**, 1–19.

Declercq, N.F., Briers, R. Degrieck, J. & O. Leroy, 2005. The history and properties of ultrasonic inhomogeneous waves. *IEEE Transactions on Ultrasonics, Ferroelectrics, and Frequency Control*, **52**, 776–791.

Deschamps, M. & Assouline, P., 2000. Attenuation along the Poynting vector direction of inhomogeneous plane waves in absorbing and anisotropic solids, *Acustica*, **86**, 295–302.

Deschamps, M. & Hosten, B., 1989. Génération de l'onde hétérogène de volume dans un liquide non absorbant. *Acustica*, **68**, 92–96.

Farra, V., 1989. Ray perturbation theory for heterogeneous hexagonal anisotropic medium. *Geophys. J. Int.*, **99**, 723–737.

Farra, V., 1999. Computation of second-order traveltime perturbation by Hamiltonian ray theory. *Geophys. J. Int.*, **136**, 205–217.

Fedorov, F.I., 1968. *Theory of Elastic Waves in Crystals*, Plenum, New York.

Gajewski, D. & Pšenčík, I., 1992. Vector wavefield for weakly attenuating anisotropic media by the ray method. *Geophysics*, **57**, 27–38.

Hayes, M.A. & R.S. Rivlin, 1974, Plane wave in linear viscoelastic materials. *Quart. Appl. Math.*, **32**, 113–121.

Hanyga, A., 1982. The kinematic inverse problem for weakly laterally inhomogeneous anisotropic media. *Tectonophysics*, **90**, 253–262.

Helbig, K., 1994. *Foundations of Anisotropy for Exploration Seismics.* Pergamon, Oxford.

Jakobsen,M., Johansen,T.A. & McCann,C., 2003. The acoustic signature of fluid flow in complex porous media. *J.Appl.Geophys.* **54**, 219–246.

Jech, J. & Pšenčík, I., 1989. First-order perturbation method for anisotropic media. *Geophys. J. Int.*, **99**, 369–376.

Klimeš, L., 2002. Second-order and higher-order perturbations of travel time in isotropic and anisotropic media. *Stud. Geophys. Geod.*, **46**, 213–248.

Krebes, E.S., and Le, L.H.T., 1994. Inhomogeneous plane waves and cylindrical waves in anisotropic anelastic media. *J. Geophys. Res.*, **99**, No. B12, 23899–23919.

Musgrave, M.I.P., 1970. *Crystal acoustics.*Holden Day, San Francisco.

Nowack, R.L., & Pšenčík, I., 1991. Perturbation from isotropic to anisotropic heterogeneous media in the ray approximation. *Geophys. J. Int.*, **106**, 1–10.

Pšenčík, I. & V. Farra, 2005. First-order ray tracing for qP waves in inhomogeneous weakly anisotropic media. *Geophysics*, **70**, D65–D75.

Pšenčík,I., and Farra,V., 2007. First-order P-wave ray synthetic seismograms in inhomogeneous weakly anisotropic media. *Geophys.J.Int.*, in print.

Thomson, C.J., 1997. Complex rays and wave packets for decaying signals in inhomogeneous, anisotropic and anelastic media. *Stud. Geophys. Geod.*, **41**, 345–381.

Zhu, Y.P. & Tsvankin, I., 2006, Plane-wave propagation in attenuative transversely isotropic media: Geophysics, **71**, T17-T30.

Zhu, Y.P. & Tsvankin, I., 2007, Plane-wave attenuation anisotropy in orthorhombic media: Geophysics, **72**, D9-D19.

Zhu, Y.P., Tsvankin, I. & Dewangan, P., 2007, Physical modeling and analysis of P-wave attenuation anisotropy in transversely isotropic media: Geophysics, **72**, D1-D7.