First-order P-wave ray synthetic seismograms in inhomogeneous weakly anisotropic media

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Summary

We propose approximate equations for P-wave ray-theory Green function for smooth inhomogeneous weakly anisotropic media. Equations are based on perturbation theory, in which deviations of anisotropy from isotropy are considered to be the first-order quantities. For evaluation of the approximate Green function, earlier derived first-order ray-tracing equations and in this paper derived first-order dynamic-ray-tracing equations are used.

The first-order ray-theory P-wave Green function for inhomogeneous, weakly anisotropic media of arbitrary symmetry depends, at most, on 15 weak-anisotropy parameters. For anisotropic media of higher-symmetry than monoclinic, all equations involved differ only slightly from the corresponding equations for isotropic media. For vanishing anisotropy, the equations reduce to equations for computation of standard ray-theory Green function for isotropic media. These properties make the proposed approximate Green function an easy and natural substitute of traditional Green function for isotropic media.

Numerical tests for configuration and models used in seismic prospecting indicate negligible dependence of accuracy of the approximate Green function on inhomogeneity of the medium. Accuracy depends more strongly on strength of anisotropy in general and on angular variation of phase velocity due to anisotropy in particular. For example, for anisotropy of about 8%, considered in the examples presented, the relative errors of the geometrical spreading are usually under 1%; for anisotropy of about 20%, however, they may locally reach as much as 20%.

Keywords: inhomogeneous media, perturbation methods, P waves, seismic anisotropy, seismic ray theory, synthetic seismograms.

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1 Introduction

We extend the applicability of the procedure for an approximate P-wave ray tracing and ray-theory travel-time computations in smooth laterally inhomogeneous, weakly anisotropic media, proposed by Pšenčík & Farra (2005) and called first-order ray tracing (FORT), to computations of P-wave ray amplitudes and ray-theory Green function. In this way, approximate P-wave ray-theory synthetic seismograms in smooth inhomogeneous weakly anisotropic media can be computed very efficiently. The extension depends basically on the first-order dynamic ray tracing (FODRT). The FODRT equations depend, as the FORT equations do, on only 15 weak-anisotropy (WA) parameters, which depend on all 21 elastic moduli, or some of their combinations. The WA parameters represent a linearized generalization, to weakly anisotropic media of arbitrary symmetry, of Thomsen's (1986) parameters for transversely isotropic media, see Appendix A and Pšenčík & Gajewski (1998) or Farra & Pšenčík (2003).

Presented results are based on perturbation theory, in which small deviations of anisotropy from isotropy are considered to be of the first order. As in the case of the kinematic ray tracing, the approximate dynamic ray-tracing equations are derived from the exact ones so that the exact eigenvalue of the Christoffel matrix, appearing in them, is substituted by its first-order approximation. This is why we speak about first-order ray tracing and first-order dynamic ray tracing. FORT and FODRT do not require calculation of reference rays in a reference medium as is common in most perturbation approaches. FORT and FODRT are used just for the given weakly anisotropic medium. They provide directly the first-order rays, the first-order or second-order travel times and first-order ray amplitudes along them. FORT and FODRT are very simple and transparent. They provide simplified relations between ray attributes and model parameters. Another useful property is that FORT and FODRT provide description of P-wave propagation decoupled from S-wave propagation. For anisotropic media of higher symmetry than monoclinic, FORT and FODRT differ only slightly from their isotropic counterparts used in many routine applications. This makes FORT and FODRT their natural potential substitutes.

A brief review of first-order ray tracing in the next section is followed by a section, in which formulae for the first-order ray-theory Green function and related first-order quantities, on which it depends, are presented. Basic role is played by FODRT. The final sections are devoted to numerical examples illustrating the accuracy of the proposed approach and to conclusions. Appendix A contains definitions of the P-wave WA parameters. Appendix B contains explicit expressions for the first and second derivatives of the first-order P-wave eigenvalue with respect to spatial coordinates and with respect to components of the slowness vector. The expressions are valid for inhomogeneous media of weak, but arbitrary anisotropy.

Three quantities with dimension of wave velocity are used in the text: c, α and V_P. The symbol c denotes spatially and angularly varying P-wave phase velocity. The symbol α denotes a P-wave velocity of a reference isotropic medium. The velocity α is an arbitrary constant; it is used for the definition of the weak-anisotropy parameters, see Appendix A. If elastic parameters were used instead of weak-anisotropy parameters, the symbol α would disappear from the RT and DRT equations. Finally, the velocity V_P is the P-wave velocity of a reference istropic medium varying along a ray, along which the evaluation of the second-order travel-time correction is performed.

In the following, component notation is mostly used. Einstein summation convention is used for the repeated subscripts. Subscripts behind a comma denote differentiation with respect to the corresponding Cartesian coordinates. Lower-case indices run from 1 to 3, the upper-case indices from 1 to 2.

2 Formulae for the P-wave first-order ray tracing

Explicit expressions of the P-wave first-order ray-tracing equations for inhomogeneous media of weak, but arbitrary anisotropy can be found in Pšenčík & Farra (2005), where also specifications for selected media of higher-order symmetry than monoclinic are given. Formally, FORT equations can be written as follows:

$$\frac{dx_i}{d\tau}=\frac{1}{2}\frac{\partial G}{\partial p_i}, \qquad \frac{dp_i}{d\tau}=-\frac{1}{2}\frac{\partial G}{\partial x_i}, \tag{1}$$

Here x_i are Cartesian coordinates of the trajectory of the first-order ray, and p_i are components of the first-order slowness vectors at corresponding points of the ray, $p_i = n_i/c$. The vector $\mathbf{n}$ is the unit normal to the wavefront (the unit vector parallel to the slowness vector $\mathbf{p}$) and $c = c(x_m, n_m)$ is the first-order phase velocity. The parameter along the ray, τ, in equations (1) is the first-order travel time. The symbol $G = G(x_m, p_m)$ denotes the greatest of the three first-order eigenvalues of the generalized Christoffel matrix:

$$\Gamma_{ik}(x_m,p_m)=a_{ijkl}(x_m)p_j\ p_l. \tag{2}$$

The matrix Γ_{ik} is called generalized since it depends on p_i instead on n_i, which appear in the classical Christoffel matrix $\bar{\Gamma}_{ik} = a_{ijkl}n_j n_l$. For simplicity, we call Γ_{ik} from eq. (2) the Christoffel matrix in the following text. The symbol a_{ijkl} in eq. (2) denotes a tensor of density-normalized elastic moduli.

The slowness vectors determined from equations (1) have to satisfy, at each point of the ray, the first-order eikonal equation:

$$G(x_m,p_m)=\Gamma_{ik}(x_m,p_m)n_i\ n_k=c^{-2}G(x_m,n_m)=1, \tag{3}$$

where c denotes again the first-order phase velocity of the considered P wave. The first-order P-wave eigenvalue $G(x_m, p_m)$ of the Christoffel matrix is given by an explicit formula (Pšenčík & Farra, 2005):

$$\begin{aligned}G(x_m,p_m)=\alpha^2\Big(&p_kp_k+2(\epsilon_x p_1^2+\epsilon_y p_2^2+\epsilon_z p_3^2)+2(p_kp_k)^{-1}[\eta_x p_2^2p_3^2+\eta_y p_1^2p_3^2+\eta_z p_1^2p_2^2\\&+2(\epsilon_{16}p_2+\epsilon_{15}p_3)p_1^3+2(\epsilon_{24}p_3+\epsilon_{26}p_1)p_2^3+2(\epsilon_{35}p_1+\epsilon_{34}p_2)p_3^3\\&+2(\chi_x p_1+\chi_y p_2+\chi_z p_3)p_1p_2p_3]\Big).\end{aligned} \tag{4}$$

The definition of the WA parameters acting as coefficients in eq. (4) is given in eqs (A-1). The parameters η_x, η_y and η_z are defined in eqs (A-2). The quantity α in eq.

(4) is a reference velocity used in the definition of weak anisotropy (WA) parameters, see equation (A-1). By inserting (A-1) into (4), it can be easily shown that $G(x_m, p_m)$ and its derivatives are independent of α, see Pšenčík & Farra (2005). The reference velocity α can thus be chosen as an arbitrary non-zero constant ($\alpha \neq 0$), which can be chosen, for example, so that it makes the WA parameters as small as possible.

The explicit expressions for the first-order ray tracing in an inhomogeneous weakly anisotropic medium as well as for several weakly anisotropic media of higher symmetry than monoclinic can be found in Pšenčík & Farra (2005). The expressions for the derivatives $\partial G/\partial x_i$ and $\partial G/\partial p_i$ appearing in eqs (1) can be found in Appendix B, see eq. (B-1).

The initial conditions at the source, for $\tau = 0$, read:

$$x_i(0) = x_i^0, \qquad p_i(0) = p_i^0. \tag{5}$$

Here x_i^0 are the coordinates of the source point and p_i^0 are the components of the first-order slowness vector, specified as $p_i^0 = n_i^0/c_0$. The symbol c_0 denotes the first-order approximation of the phase velocity in the direction specified by the vector $\mathbf{n^0}$ at the point with coordinates x_i^0. The phase velocity c_0 is given by the square root of $G(x_m^0, n_m^0)$, see eqs (3) and (5). The vector $\mathbf{n^0}$ is specified by two ray parameters $\gamma^{(J)}$, chosen as two take-off angles, ϕ_0 and δ_0:

$$n_1^0 = \cos\phi_0 \cos\delta_0, \quad n_2^0 = \sin\phi_0 \cos\delta_0, \quad n_3^0 = \sin\delta_0. \tag{6}$$

Pšenčík & Farra (2005) have shown that accuracy of travel-time computations can be increased by calculating a second-order travel-time correction along the first-order ray Ω_0:

$$\Delta\tau = -\frac{1}{2}\int_{\Omega_0} c^{-2}(x_m, n_m)\frac{B_{13}^2(x_m, n_m) + B_{23}^2(x_m, n_m)}{V_P^2 - V_S^2}d\tau. \tag{7}$$

It can be easily evaluated by numerical quadratures along the FORT ray; c^{-2} can be evaluated as $c^{-2} = p_k p_k$. V_P and V_S are P- and S-wave velocities of a reference isotropic medium closely approximating the studied weakly anisotropic medium. The velocity V_P has nothing to do with the velocity α used for the definition of the WA parameters, see equations (A-1). Thus, the choice of V_P is completely independent of the choice of α. Optimal choice of V_P is $V_P^2 = (p_k p_k)^{-1}$, where p_k are the components of the slowness vectors obtained by integration of the FORT system (1). For the determination of V_S, we can use the relation $V_S^2 = V_P^2/3$. Thus, V_P and V_S may vary along the ray. The quantities B_{13} and B_{23} are elements of the matrix

$$B_{ij}(x_m, n_m) = \Gamma_{kl}(x_m, n_m)e_k^i e_l^j, \tag{8}$$

see Farra & Pšenčík (2003) and Pšenčík & Farra (2005). For elements B_{13} and B_{23} see Appendix A, eq. (A-3). The symbols e_m^j denote components of three mutually perpendicular unit vectors $\mathbf{e^1}$, $\mathbf{e^2}$ and $\mathbf{e^3}$. The vector $\mathbf{e^3}$ is identical to $\mathbf{n}$, $\mathbf{e^3} = \mathbf{n}$. The matrix B_{mn}, as $G(x_m, p_m)$, is independent of the choice of the reference velocity α. Moreover, the expression $B_{13}^2 + B_{23}^2$ is also independent of the choice of the vectors $\mathbf{e^1}$ and $\mathbf{e^2}$ in the plane perpendicular to $\mathbf{e_i}$.

3 First-order P-wave ray-theory Green function

Basic step in the computation of ray synthetic seismograms is the computation of the ray-theory Green function. The ray-theory Green function $G_{in}(R,t,S,t_0)$ represents the i-th component of the displacement vector at a receiver R and time t due to a single-force point source acting at S and oriented along the n-th Cartesian coordinate axis, with the time dependence $\delta(t-t_0)$. The ray-theory Green function belongs to a selected elementary wave, in our case, to the P wave. The Fourier transform of the ray-theory Green function $G_{in}(R,t,S,t_0)$ with respect to $t-t_0$ reads (see e.g. Červený, 2001; Červený et al., 2007):

$$G_{in}(R,S,\omega) = \frac{g_n(S)g_i(R)\exp[\mathrm{i}T^G(R,S) + \mathrm{i}\omega\tau(R,S)]}{4\pi[\rho(S)\rho(R)c(S)c(R)]^{1/2}\mathcal{L}(R,S)}. \tag{9}$$

The first-order ray-theory Green function is described by formally the same expression as in eq. (9). The first-order ray-theory Green function is, however, calculated along first-order rays and some of quantities, on which it depends are of the first order. In this way, in eq. (9), g_i are components of the first-order polarization vectors, symbols c denote the first-order phase velocities. The symbol $\tau(R,S)$ denotes the first-order travel time from S to R calculated along a ray determined by solving the FORT system (1). The first-order travel time can be substituted by more accurate second-order travel time including the travel-time correction (7). The symbol $T^G(R,S)$ is the complete phase shift due to caustics along the ray from S to R and ρ denotes the density.

The first-order or second-order travel time $\tau(R,S)$, the first-order slowness vector components, p_i, and phase velocities c can be obtained as a solution of the FORT equations (1). The first-order P-wave polarization vector components, g_i, can be determined from the formula (see Farra & Pšenčík, 2003):

$$g_i = n_i + \frac{B_{13}(x_m,n_m)e_i^1 + B_{23}(x_m,n_m)e_i^2}{V_P^2 - V_S^2}. \tag{10}$$

All the quantities appearing in eq. (10) have been described after eq. (7) for the second-order travel-time correction. Note that the components g_i constitute a unit vector only approximately.

The symbol $\mathcal{L}(R,S)$ denotes the first-order relative geometrical spreading (basic quantity affecting ray amplitudes)

$$\mathcal{L}(R,S) = |\mathbf{X}^{(1)} \times \mathbf{X}^{(2)}|^{1/2} \tag{11}$$

along the ray from S to R. Eq. (11) holds for the vectors $\mathbf{X}^{(I)}$ and other important vectors $\mathbf{Y}^{(I)}$, defined below, determined from the first-order dynamic ray-tracing system with especially chosen initial conditions, given below, see also Pšenčík & Teles (1996). For the sake of brevity, we call $\mathcal{L}(R,S)$ the geometrical spreading in the following. If we denote the ray parameters specifying the FORT ray (i.e., the initial angles, ϕ_0 or δ_0) by $\gamma^{(I)}$, the components of the vectors $\mathbf{X}^{(I)}$ and $\mathbf{Y}^{(I)}$ can be expressed as

$$X_i^{(I)} = (\frac{\partial x_i}{\partial\gamma^{(I)}})_{\tau=const}, \quad Y_i^{(I)} = (\frac{\partial p_i}{\partial\gamma^{(I)}})_{\tau=const}. \tag{12}$$

The quantities $X_i^{(I)}$ and $Y_i^{(I)}$ describe variations along the wavefront of the coordinates x_i and of the components p_i of the slowness vector due to the variations of the parameters $\gamma^{(I)}$. The values of $X_i^{(I)}$ and $Y_i^{(I)}$ can be found from a system of linear differential equations obtained by the differentiation of the FORT equations (1) with respect to $\gamma^{(I)}$:

$$\frac{dX_i^{(I)}}{d\tau} = \frac{1}{2}\Big(\frac{\partial^2 G(x_m,p_m)}{\partial p_i \partial x_j} X_j^{(I)} + \frac{\partial^2 G(x_m,p_m)}{\partial p_i \partial p_j} Y_j^{(I)}\Big),$$

$$\frac{dY_i^{(I)}}{d\tau} = -\frac{1}{2}\Big(\frac{\partial^2 G(x_m,p_m)}{\partial x_i \partial x_j} X_j^{(I)} + \frac{\partial^2 G(x_m,p_m)}{\partial x_i \partial p_j} Y_j^{(I)}\Big). \qquad (13)$$

The system of equations (13) is the first-order dynamic ray-tracing (FODRT) system. Explicit expressions for the second derivatives of the P-wave first-order eigenvalue $G(x_m, p_m)$ of the Christoffel matrix are given in Appendix B. For vanishing anisotropy, eqs (13) reduce to the corresponding exact equations for isotropic media. We obtain the quantities $X_i^{(J)}$ required for the calculation of the relative geometrical spreading, see eq. (11), by specifying the initial conditions for the FODRT equations (13) in the following way:

$$X_i^{(J)} = 0, \quad Y_i^{(J)} = c_0^{-1}(Z_{iJ} - p_i^0 v_{0k} Z_{kJ}), \qquad (14)$$

where

$$Z_{11} = -\sin\phi_0 \cos\delta_0, \quad Z_{21} = \cos\phi_0 \cos\delta_0, \quad Z_{31} = 0,$$

$$Z_{12} = -\cos\phi_0 \sin\delta_0, \quad Z_{22} = -\sin\phi_0 \sin\delta_0, \quad Z_{32} = \cos\delta_0, \qquad (15)$$

see Pšenčík & Teles (1996). In eqs (14), c_0 denotes, as before, the first-order phase velocity at the initial point S, v_{0i} denotes components of the first-order ray-velocity vector, $v_i = dx_i/d\tau$, at the same point. The symbols ϕ_0 or δ_0 in eqs (15) denote again the take-off angles introduced in eqs (6).

The solution of FODRT equations (13) provides a tool for the determination of the complete phase shift due to caustics $T^G(R,S)$, see, e.g., Červený et al. (2007). Moreover, the solution of FODRT equations (13) can be also effectively used for the two-point ray tracing based on shooting. The quantities $X_i^{(J)}$ are then used for estimation of the ray parameters $\gamma^{(J)}$ (take-off angles ϕ_0 and δ_0) from the distance between termination points of rays shot at a receiver surface (surface containing receivers) and the receiver, at which the two-point ray should terminate.

4 Numerical examples

In the following we show examples illustrating the accuracy of the proposed method. We consider the same VSP configuration and models as Pšenčík & Farra (2005), in which only accuracy of the computed first- and second-order travel times was studied. Here, we study accuracy of the geometrical spreading and of the first-order ray synthetic seismograms. The VSP configuration used in the tests is shown in Fig. 1.

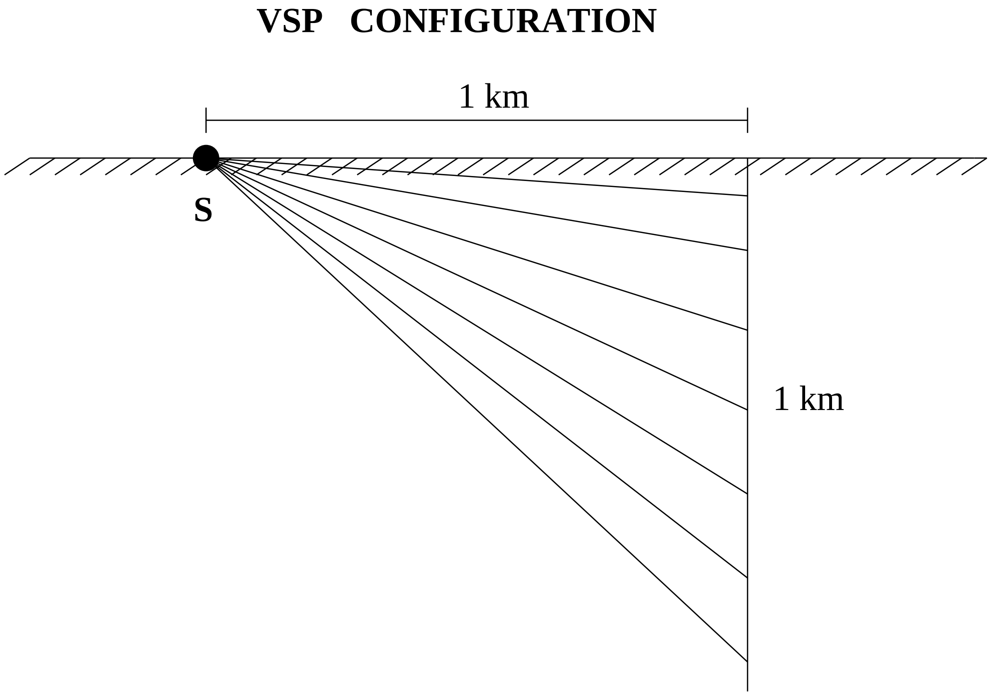

Figure 1: Schematic configuration of experiments.

The source and the borehole are situated in a vertical plane (x, z). The borehole lies along the z axis, the vertical single-force source is located on the surface at $z = 0$ km, at a distance of 1 km from the borehole. The source time function is a windowed symmetric Gabor wavelet $\exp[-(2\pi f/\gamma)^2 t^2] cos 2\pi f t$ with the dominant frequency $f = 25$ Hz and $\gamma = 4.44$. There are 24 receivers in the borehole, distributed with a uniform step size of 0.04 km, with receiver depths ranging from 0.04 km to 0.96 km. The 3-component receivers record the vertical (positive downwards), transverse, and radial (along the line source - top of the borehole; positive away from the source) components of the wave field. The recording system is right-handed. All calculated seismograms are shown with no differential scaling between components and traces, so true relative amplitudes can be seen. We consider two types of models: one transversely isotropic (models "TI") and one orthorhombic (models "ORTHO"). If we use common measure of P-wave anisotropy (maximum of $|\epsilon| \times$ 100%, where ϵ stands for ϵ_x, ϵ_y or ϵ_z), the anisotropy of the P wave in the models "TI" is about 8%, in the models "ORTHO" about 20%. As in Pšenčík & Farra (2005), we consider, for simplicity, only vertical variation of parameters of the medium. The procedure can, however, be applied to arbitrary laterally varying weakly anisotropic structures. The stiffness matrices are specified at the top of the model, at $z = 0$ km, and at a depth of $z = 3$ km. The values of the elements of the stiffness matrices are interpolated linearly between the two levels.

In the following, we measure the accuracy of the FORT and FODRT in the two types of models by plotting the relative differences of the geometrical spreading

$$\frac{\mathcal{L}_{FORT} - \mathcal{L}_{EXACT}}{\mathcal{L}_{EXACT}} \times 100\%. \tag{16}$$

The geometrical spreading values $\mathcal{L}_{FORT}$ are obtained by the approach described in the preceding sections, i.e., by solving the FODRT equations (13). The geometrical spreading values $\mathcal{L}_{EXACT}$ are computed by a standard ray tracer for anisotropic media – a modified version of the program package ANRAY (Gajewski & Pšenčík, 1990).

In addition to the plots of relative differences of the geometrical spreading, we also present comparison of resulting ray synthetic seismograms. All seismograms are shifted by 0.03 s in order to shift the travel times from the maxima of the Gabor wavelet to, approximately, their onsets. FORT seismograms are in red, exact ray synthetic seismograms are in black. First-order travel times are used in the seismograms for the "TI" models, second-order travel-time correction (7) is used in the seismograms for the "ORTHO" models. In (7), we use $V_S^2 = V_P^2/3$ and $V_P^2 = (p_k p_k)^{-1}$, where p_k are the components of the first-order slowness vector obtained during the integration of the FORT equations (1). The values of V_P and V_S determined in the above way yield the best results. Dependence of the results on the V_P/V_S ratio is negligible.

4.1 "TI" models

The density-normalized stiffness matrices of TI models with elements, measured in (km/s)^2, are:

$$\begin{pmatrix} 15.71 & 5.05 & 4.46 & 0. & 0. & 0. \\ & 15.71 & 4.46 & 0. & 0. & 0. \\ & & 13.39 & 0. & 0. & 0. \\ & & & 4.98 & 0. & 0. \\ & & & & 4.98 & 0. \\ & & & & & 5.33 \end{pmatrix}$$

at $z = 0$ km, and

$$\begin{pmatrix} 35.35 & 11.36 & 10.04 & 0. & 0. & 0. \\ & 35.35 & 10.04 & 0. & 0. & 0. \\ & & 30.13 & 0. & 0. & 0. \\ & & & 11.21 & 0. & 0. \\ & & & & 11.21 & 0. \\ & & & & & 11.99 \end{pmatrix} \tag{17}$$

at $z = 3$ km. The stiffness matrices in eq. (17) are based on the Voigt notation. The variations of the P-wave phase velocities in a vertical plane from horizontal (0^0) to vertical (90^0) direction of the normal to the wavefront are shown in Fig. 2, for $z = 0$ and $z = 3$ km, respectively. Numerical experiments were performed for three specific models: "HTI", "ISO" and "HTI-ROT", described below.

Fig. 3 shows the relative differences of the geometrical spreading, see eq. (16), for the models "HTI" (black), "ISO" (blue) and "HTI-ROT" (red), respectively. The model "HTI" is obtained from the model specified in equation (17) by rotating the axes of symmetry at the top and the bottom of the model so that they are parallel to the x axis, i.e., to the line connecting the source and the top of the borehole. In contrast to behaviour of the travel time, see Fig.3 of Pšenčík & Farra (2005), the first-order geometrical spreading is less accurate for shallow receivers, the relative error reaching nearly 1%. With increasing depth of receivers the error decreases to effectively 0% at $z = 1$ km. The model "ISO" is obtained from the model given by equation (17) by orienting the axes of symmetry at the plane $z = 0$ and $z = 3$ km perpendicularly to the vertical (x, z) plane containing the source and the borehole. In this case the relative error, see eq. (16), slightly increases with depth, from about 0.5% to nearly 1%. In both mentioned cases, the first-order geometrical spreading $\mathcal{L}_{FORT}$ slightly exceeds the value of $\mathcal{L}_{EXACT}$. This is different in the "HTI-ROT" model. In this model, the axis of symmetry deviates by 45^o from the x axis in the horizontal plane $z = 0$ km. At $z = 3$ km, it is oriented along the x axis. In between, as depth varies, the axis of symmetry is smoothly rotated, remaining horizontal. The relative errors are least in this case, they vary from about -0.4% at $z = 0$ km to nearly 0% at $z = 1$ km.

In Fig. 4, we show comparison of exact (black) and first-order (red) ray synthetic seismograms for the model "HTI". Only radial and vertical components are shown since the transverse component is identically zero (the plane (x, z) coincides with the plane of symmetry and the source is a vertical force). First-order travel times obtained by integration of the FORT equations (1) are used. Since maximum relative travel-time errors are less than 0.15% and maximum relative geometrical spreading errors are less than 1% in this case, see Fig.3 of Pšenčík & Farra (2005) and Fig.3 here, respectively, the fit of the seismograms is perfect.

4.2 "ORTHO" models

As a next model, we consider the model of Schoenberg & Helbig (1997). It has anisotropy substantially larger than the TI models previously considered. The anisotropy is about 20%. The density-normalized stiffness matrices of the "ORTHO" models with elements measured in (km/s)^2, are at the top, $z = 0$ km:

$$\begin{pmatrix} 9.00 & 3.60 & 2.25 & 0. & 0. & 0. \\ & 9.84 & 2.40 & 0. & 0. & 0. \\ & & 5.94 & 0. & 0. & 0. \\ & & & 2.00 & 0. & 0. \\ & & & & 1.60 & 0. \\ & & & & & 2.18 \end{pmatrix}$$

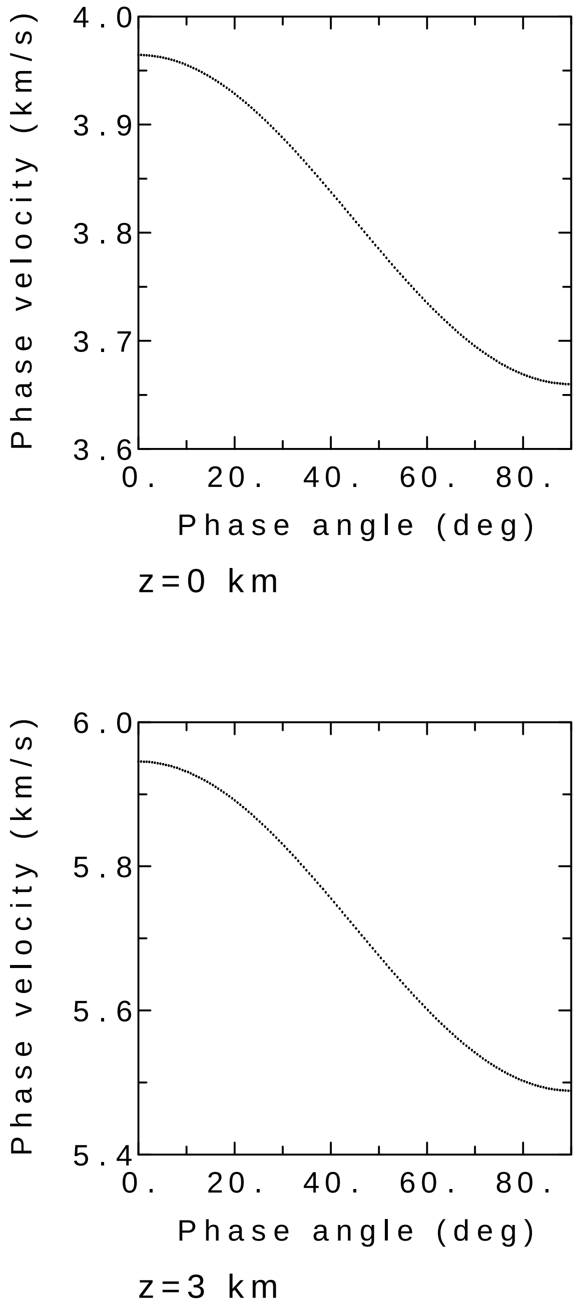

Figure 2:P-wave phase-velocity variations corresponding to stiffness matrices for the TI model specified in eq. (17) from horizontal (0^0) to vertical (90^0) direction of the normal to the wavefront.

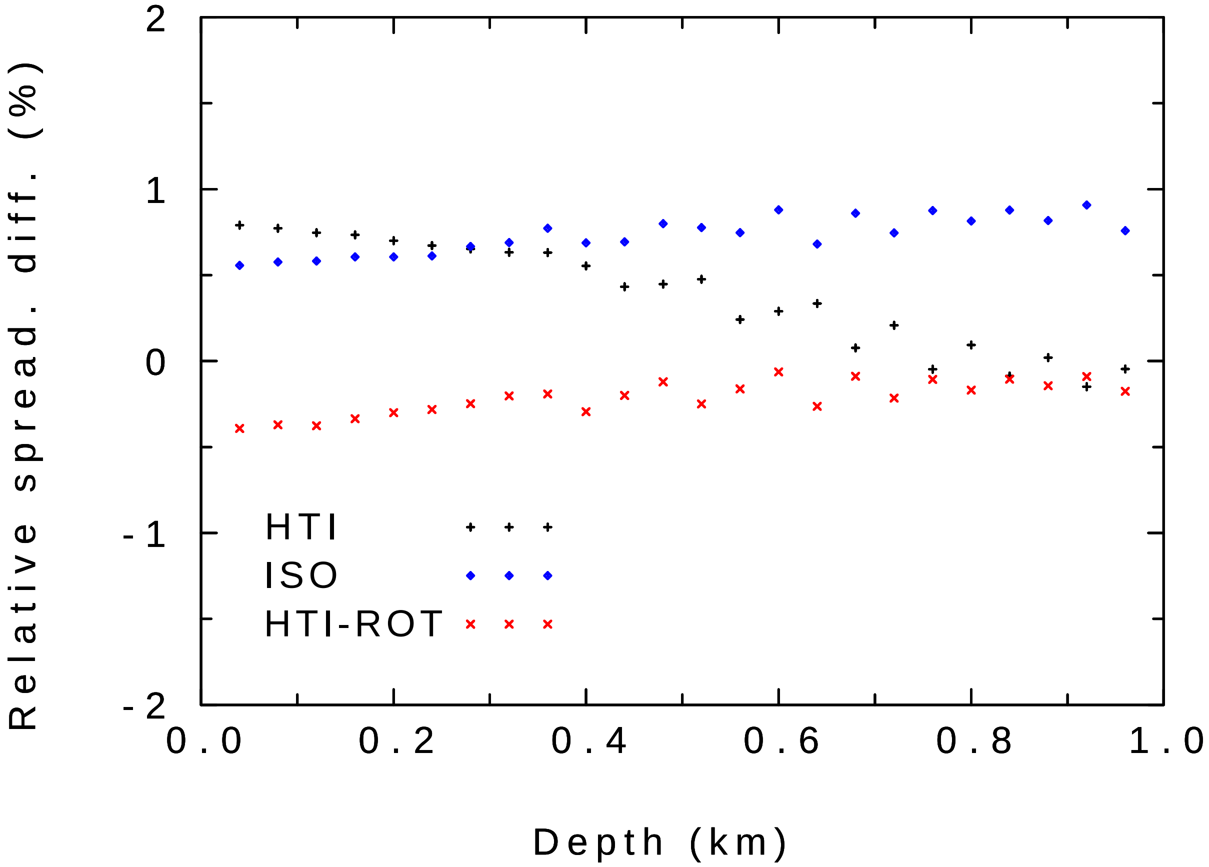

Figure 3: Relative geometrical-spreading differences, see eq. (16), for the models "HTI" (black), "ISO" (blue) and "HTI-ROT" (red).

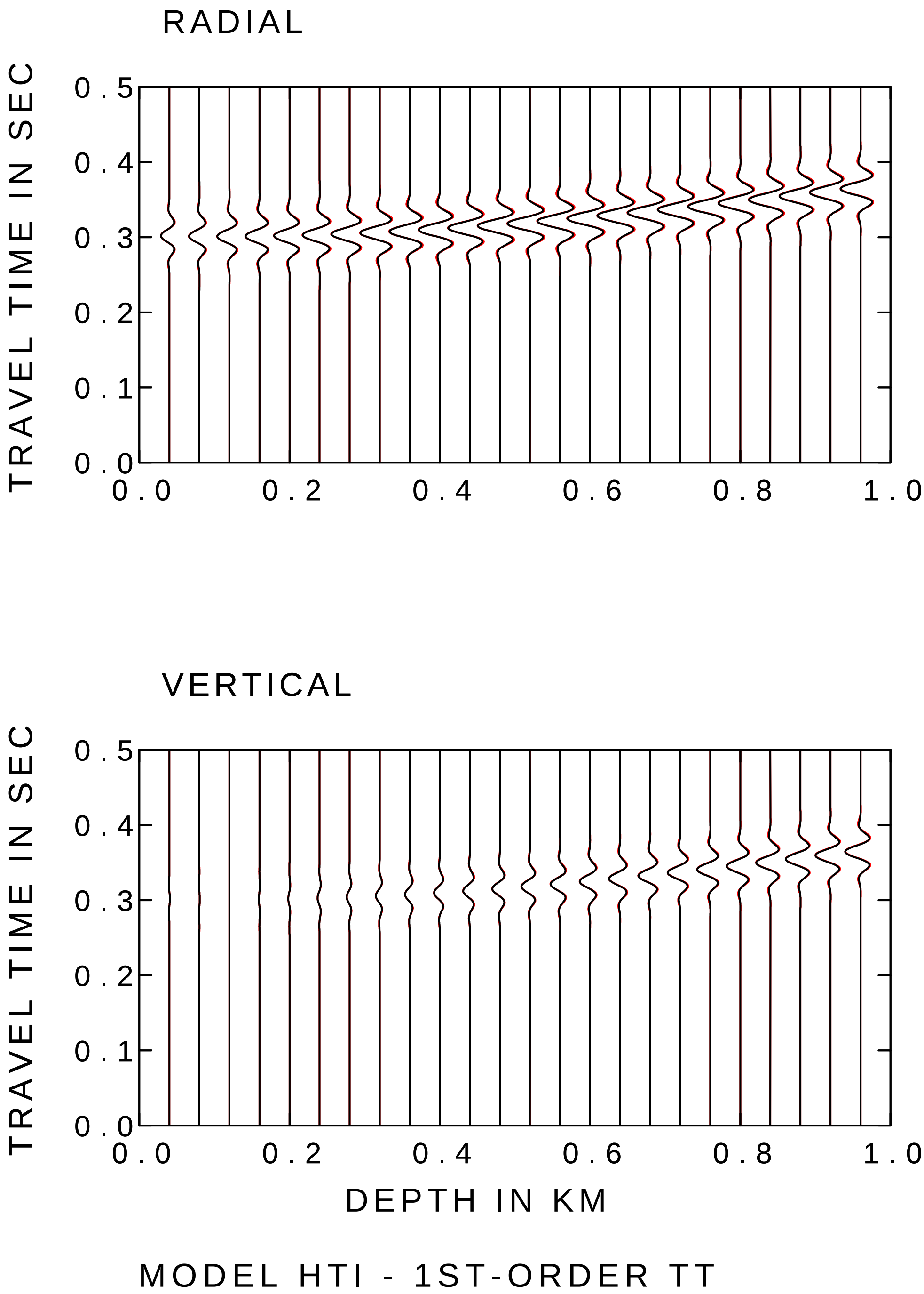

Figure 4: Comparison of the first-order (red) and exact (black) ray synthetic seismograms for the vertical single-force source in the "HTI" model. First-order travel times obtained by solving eqs (1) are used.

and at the bottom, $z = 3$ km:

$$\begin{pmatrix} 19.80 & 7.92 & 4.95 & 0. & 0. & 0. \\ & 21.65 & 5.28 & 0. & 0. & 0. \\ & & 13.07 & 0. & 0. & 0. \\ & & & 4.40 & 0. & 0. \\ & & & & 3.52 & 0. \\ & & & & & 4.80 \end{pmatrix}. \tag{18}$$

Variations of the P-wave phase velocities in the vertical (x, z) plane (left column) and the vertical (y, z) plane (right column) are shown in Fig. 5 from horizontal (0^0) to vertical (90^0) direction of the normal to the wavefront, for the top and the bottom of the model. Numerical experiments were performed for two specific models: ORT and ORT-ROT, described below.

Fig. 6 shows relative differences, see eq. (16), of the geometrical spreading (black) and of vertical component of the polarization vector at the source (red) for the model "ORT", i.e. for the (x, z) plane of the model specified in eq. (18). The mentioned two quantities show most significant differences and have most pronounced effects on accuracy of the first-order synthetic seismograms shown in Fig.7. In comparison with these two quantities, differences of horizontal components of the polarization vectors at receivers as well as of phase velocities at the source and at receivers are negligible. The differences of vertical components of the polarization vectors at receivers, which are comparable with those of vertical components of the polarization vectors at the source, are not shown. The differences in Fig.6 have an oscillating character. Due to the stronger anisotropy the maximum difference of the geometrical spreading reaches nearly -20% around $z = 0$ km, and slightly exceeds 20% for depths around 0.2 km. The maximum difference of vertical components of the polarization vector at the source reaches 15% for depths around 0.1 km. Note that the error of the first-order travel time was also relatively large, more than 1.5% for this model. As explained in Pšenčík & Farra (2005), the differences are caused by errors of the first-order approximation $G^{(1)}$ (specifically of the curvature of the surface $G^{(1)} = 1$) in the direction from the source towards the shallow receivers. With increasing depth, the differences decrease and become slightly negative for deepest receivers.

Fig. 7 shows a comparison of the ray synthetic seismograms generated by the vertical force in the "ORT" model. Exact seismograms and their first-order approximations are shown in black and red, respectively. Only radial and vertical components are shown since the transverse component is again identically zero (the plane (x, z) coincides with the plane of symmetry and the source is a vertical force). This time, second-order travel-time correction (7) is used because of non-negligible first-order travel-time errors, see Fig.7 of Pšenčík & Farra (2005). In contrast to Fig.4, we can see some differences in Fig.7. They concentrate at depths between 0 and 0.2 km on radial component (on vertical component they are supressed by small amplitudes). The differences on the radial component are a result of combined effects of the errors of the approximation of the geometrical spreading and of vertical component of the polarization vector at the source, see Fig.6. At the shallowest receiver, errors of both quantites lead to an increase of amplitudes of the first-order seismograms. At the second receiver, the increase of amplitudes is caused, practicaly solely, by the error of the first-order polarization vector, see Fig.6. At deeper receivers, errors of both quantities approximately compensate each other.

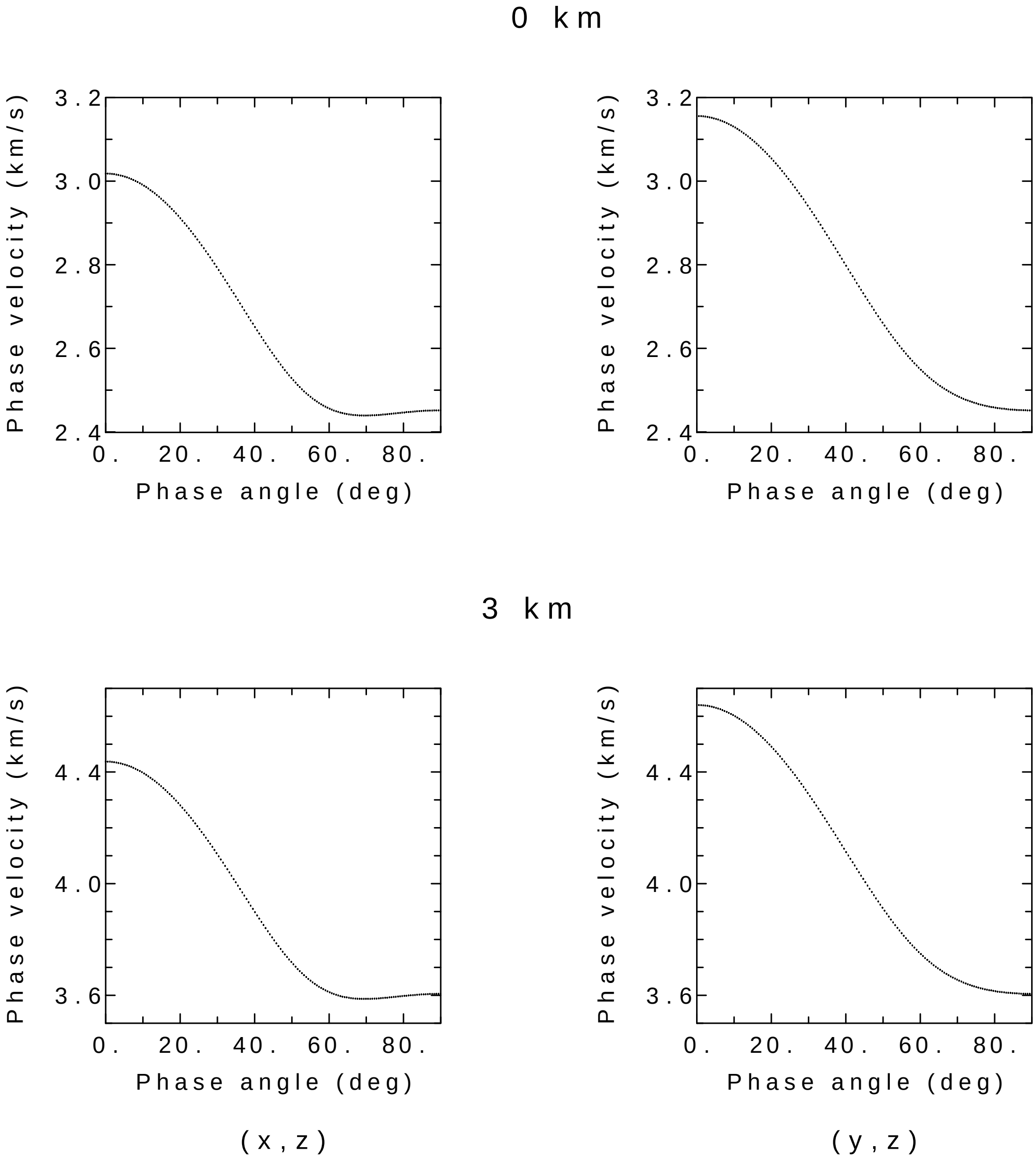

Figure 5: P-wave phase-velocity variations corresponding to stiffness matrices for the ORTHO model specified in eq. (18) from horizontal (0^0) to vertical (90^0) direction of the normal to the wavefront. The upper plots correspond to $z = 0$ km, the lower plots to $z = 3$ km, the left column to the (x, z) plane, the right column to the (y, z) plane.

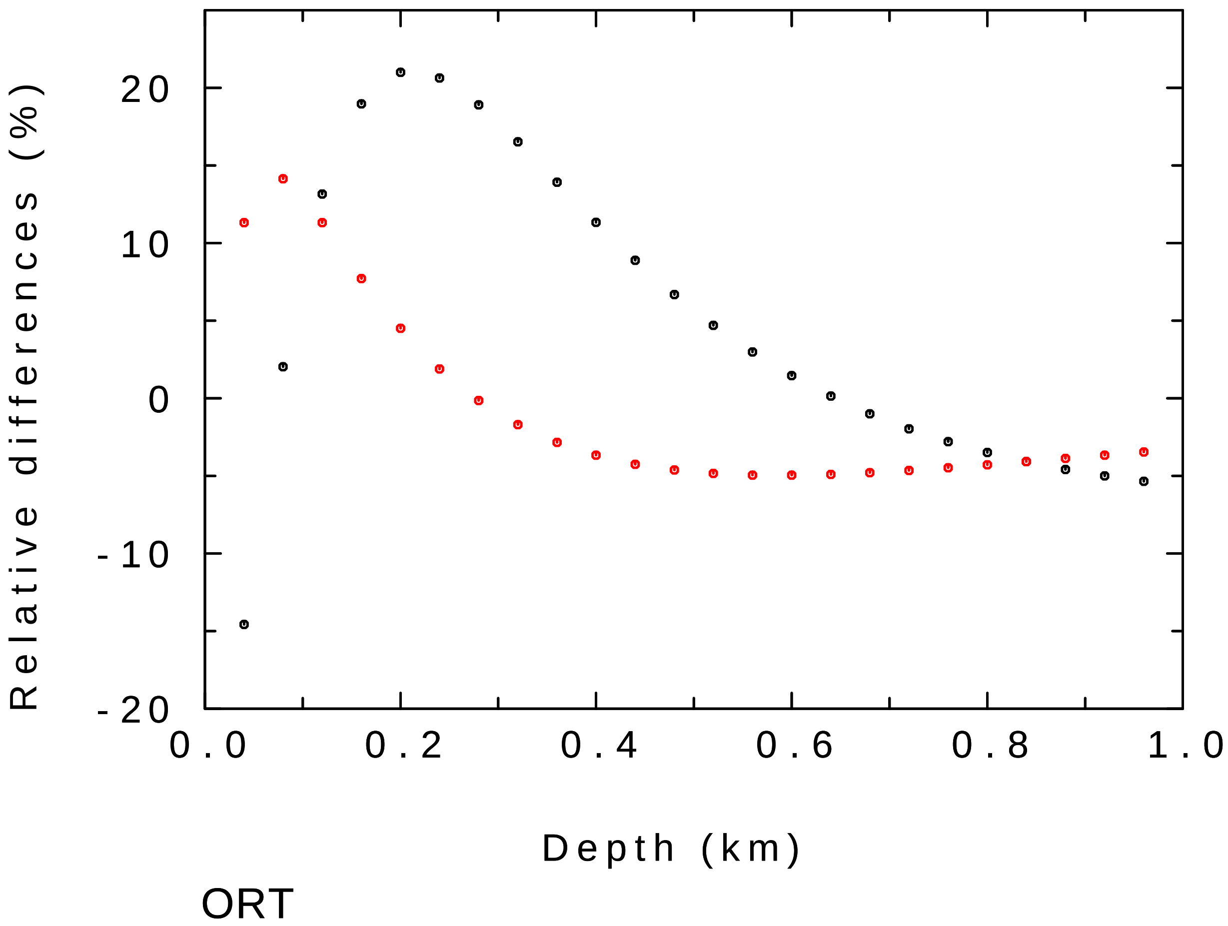

Figure 6: Relative differences, see eq. (16), of the geometrical spreading (black), see eq. (11), and of vertical components of the polarization vectors at the source (red), see eq. (10), for the model “ORT”.

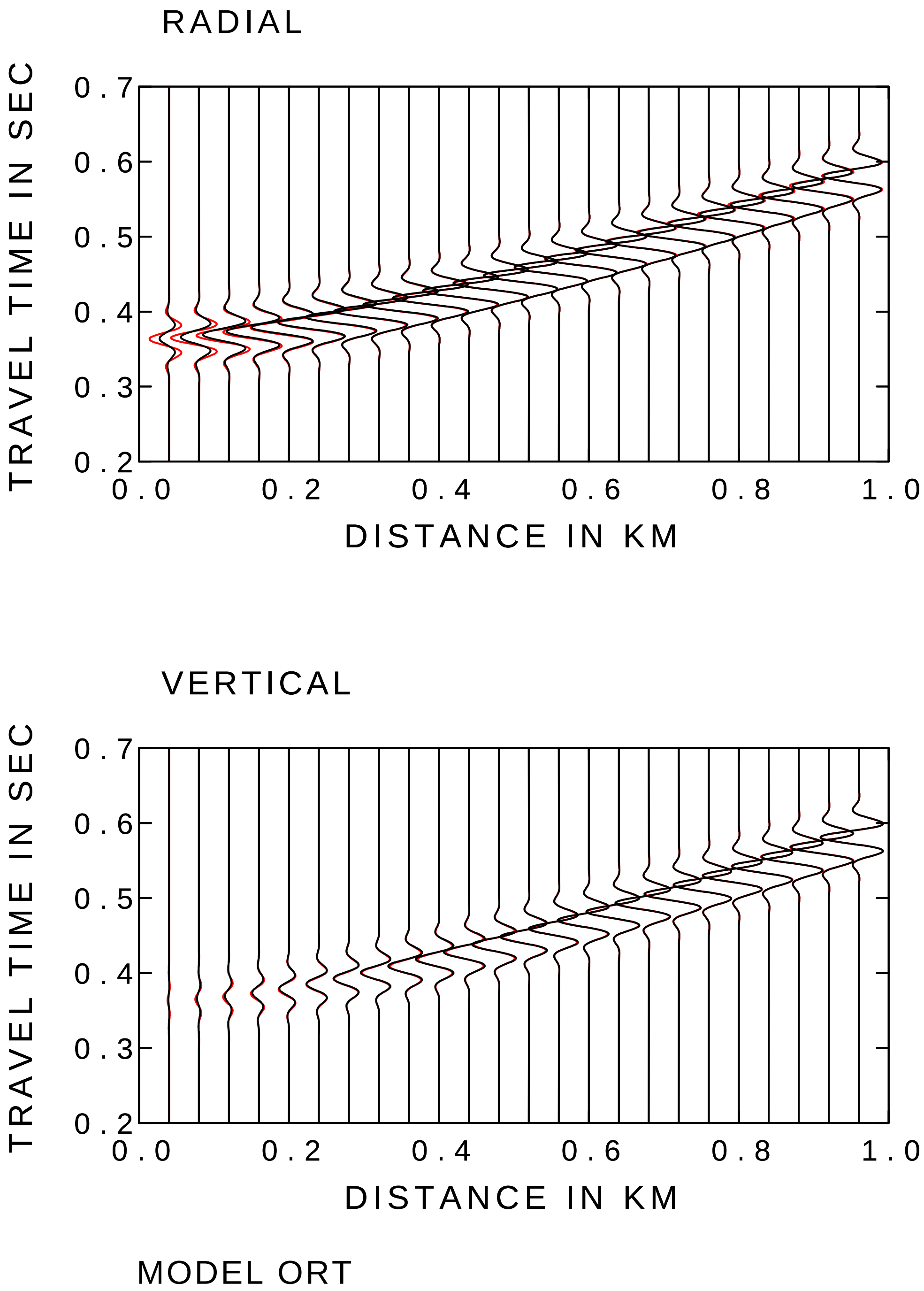

Figure 7: Comparison of the first-order (red) and exact (black) ray synthetic seismograms for the vertical single-force source in the "ORT" model. Second-order travel-time correction (7) used.

As Pšenčík & Farra (2005), we also consider a model "ORT-ROT", which results from a different rotation of matrices in eq. (18). The upper matrix of the model in eq. (18) is first rotated by 45^o around z axis, and then the z axis is rotated by 45^o around the new y axis. The matrix corresponding to a depth of z =3 km is kept unrotated. Similarly as the relative travel-time differences, see Fig.8 of Pšenčík & Farra (2005), also the relative geometrical spreading differences, see eq. (16), are significantly smaller in this case than for the "ORT" model, see Fig. 8. They are about 1% for the shallowest receivers, decrease to 0% and then decrease to nearly -3%. Because of the small deviations, the comparison of ray synthetic seismograms (black) with their first-order approximation (red) in Fig. 9 gives a nearly perfect fit. In the first-order seismograms in Fig.9, the second-order travel-time correction (7) is used.

5 Conclusions

We have extended the applicability of the FORT approach for P waves propagating in smooth, laterally inhomogeneous, weakly anisotropic media of arbitrary symmetry, proposed by Pšenčík & Farra (2005), to the evaluation of the first-order ray amplitudes and, in this way, also the first-order ray synthetic seismograms. As shown in this paper, the relative errors of amplitudes (specifically of the geometrical spreading) are generally larger than the relative travel-time errors. Nevertheless, in configurations as those considered in this paper, FORT and FODRT provide results with small deviations from exact ones, but with several important advantages:

1) FORT and FODRT equations are expressed in terms of the WA parameters, which represent a more natural description of a weakly anisotropic medium than standard elastic moduli.

2) FORT and FODRT equations have a considerably simpler structure than the exact ones, which leads to a significant reduction of the number of algebraic operations involved.

3) In the most general case, they depend only on the 15 P-wave WA parameters.

4) As the FORT and FODRT equations allow P waves to be treated separately from S waves, as in isotropic media, they can easily be used as substitutes for traditional "isotropic" ray and dynamic ray tracers in many routine applications such as prestack Kirchhoff depth migration.

5) The FORT and FODRT equations are applicable to inhomogeneous isotropic as well as anisotropic media. In isotropic media they yield exact rays and exact dynamic-ray-tracing result, in anisotropic media their first-order approximations.

As in the case of the FORT equations, the FODRT equations seem to show only very weak dependence of the accuracy of their solution on the strength of the inhomogeneity of the medium. The accuracy depends more on the strength of considered anisotropy, not so much on its type.

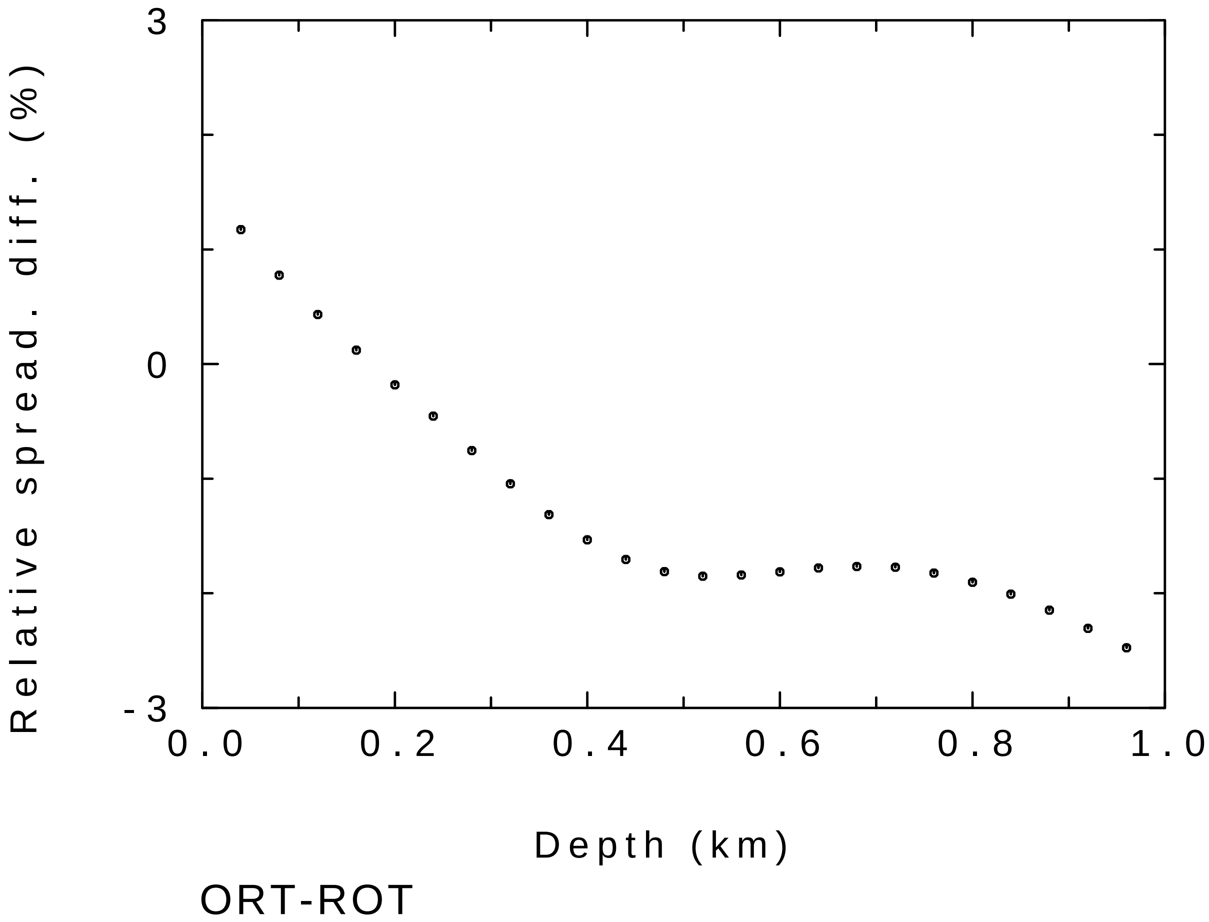

Figure 8: Relative geometrical-spreading differences, see eq. (16), for the model "ORT-ROT".

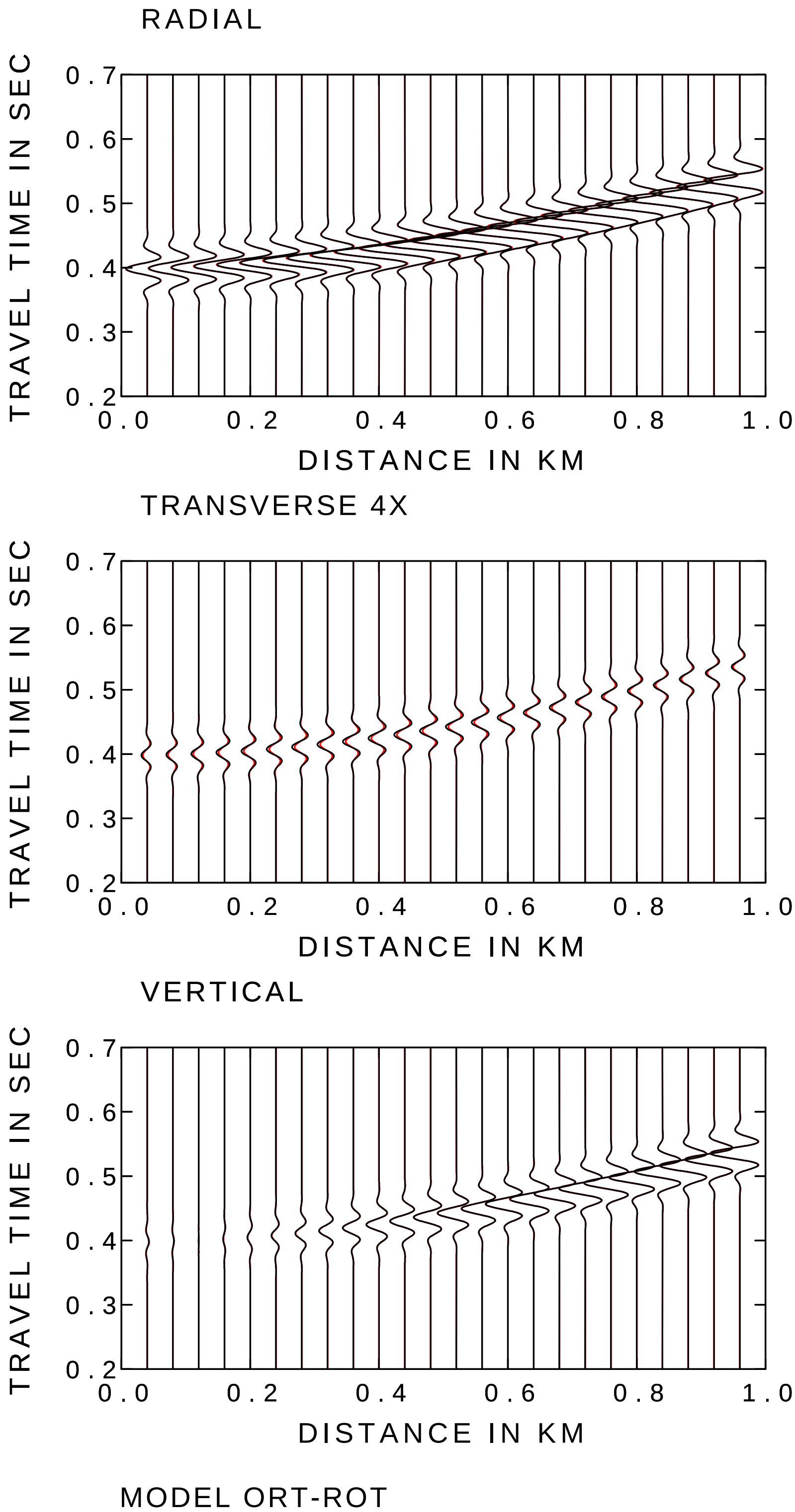

Figure 9: Comparison of the first-order (red) and exact (black) ray synthetic seismograms for the vertical single-force source in the "ORT-ROT" model. Second-order travel-time correction (7) used.

In this paper, the dynamic ray tracing was used only for the computation of the geometrical spreading and in the two-point ray-tracing procedure. Like the exact dynamic ray tracing, also FODRT has many additional important applications, see Červený (2001) for the extensive list of these applications. Use of the propagator technique, see, e.g., Červený (2001; Sec. 4.3), to solve equations (13) is straightforward.

Generalization of the described procedure for layered, laterally varying, weakly anisotropic media is under way. Extension to inhomogeneous, weakly anisotropic media with weak attenuation is also considered. Further plans include studies of S waves in inhomogeneous weakly anisotropic media (Farra, 2005). In this respect, one option is use of the quasi-isotropic approximation (Pšenčík, 1998) for S-waves. The FORT and FODRT formulae for media of higher-symmetry anisotropy will certainly find applications in procedures for ray tracing and dynamic ray tracing in structures with continuously varying orientation of symmetry axes or planes (Iversen & Pšenčík, 2007).

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Appendix A: P-wave WA parameters

If we denote by α the velocity of a reference isotropic medium, we can introduce 15 P-wave WA parameters in the following way:

$$\epsilon_x = \frac{A_{11} - \alpha^2}{2\alpha^2}, \quad \epsilon_y = \frac{A_{22} - \alpha^2}{2\alpha^2}, \quad \epsilon_z = \frac{A_{33} - \alpha^2}{2\alpha^2},$$

$$\delta_x = \frac{A_{13} + 2A_{55} - \alpha^2}{\alpha^2}, \quad \delta_y = \frac{A_{23} + 2A_{44} - \alpha^2}{\alpha^2}, \quad \delta_z = \frac{A_{12} + 2A_{66} - \alpha^2}{\alpha^2},$$

$$\chi_x = \frac{A_{14} + 2A_{56}}{\alpha^2}, \quad \chi_y = \frac{A_{25} + 2A_{46}}{\alpha^2}, \quad \chi_z = \frac{A_{36} + 2A_{45}}{\alpha^2},$$

$$\epsilon_{15} = \frac{A_{15}}{\alpha^2}, \quad \epsilon_{16} = \frac{A_{16}}{\alpha^2}, \quad \epsilon_{24} = \frac{A_{24}}{\alpha^2}, \quad \epsilon_{26} = \frac{A_{26}}{\alpha^2}, \quad \epsilon_{34} = \frac{A_{34}}{\alpha^2}, \quad \text{and} \quad \epsilon_{35} = \frac{A_{35}}{\alpha^2}. \qquad (A-1)$$

Here $A_{\alpha\beta}$ are the density-normalized elastic moduli in the Voigt notation.

The following variables have also been used:

$$\eta_x = \delta_y - \epsilon_y - \epsilon_z, \quad \eta_y = \delta_x - \epsilon_x - \epsilon_z, \quad \eta_z = \delta_z - \epsilon_x - \epsilon_y. \qquad (A-2)$$

The elements B_{13} and B_{23} of the matrix B_{mn}, see equation (8), can be expressed explicitely in the following way:

$$\begin{aligned}
B_{13} = \alpha^2 D^{-1}\{n_3^4(\epsilon_{34}n_2 + \epsilon_{35}n_1) + n_3^3[\eta_y n_1^2 + \eta_x n_2^2 + 2\chi_z n_1 n_2] \\
+n_3^2[(4\chi_x - 3\epsilon_{34})n_1^2 n_2 + (4\chi_y - 3\epsilon_{35})n_1 n_2^2 + (4\epsilon_{15} - 3\epsilon_{35})n_1^3 + (4\epsilon_{24} - 3\epsilon_{34})n_2^3] \\
+n_3[(2\eta_z - \eta_x - \eta_y)n_1^2 n_2^2 + 2(2\epsilon_{16} - \chi_z)n_1^3 n_2 + 2(2\epsilon_{26} - \chi_z)n_1 n_2^3 \\
-\eta_y n_1^4 - \eta_x n_2^4 + (\epsilon_x - \epsilon_z)n_1^2 + (\epsilon_y - \epsilon_z)n_2^2] - \chi_x n_1^2 n_2 - \chi_y n_1 n_2^2 - \epsilon_{15}n_1^3 - \epsilon_{24}n_2^3\},
\end{aligned}$$

$$\begin{aligned}
B_{23} = \alpha^2 D^{-1}\{n_3^3(\epsilon_{34}n_1 - \epsilon_{35}n_2) + n_3^2[(\eta_x - \eta_y)n_1 n_2 + \chi_z n_1^2 - \chi_z n_2^2] \\
+n_3[(2\chi_y - 3\epsilon_{15})n_1^2 n_2 - (2\chi_x - 3\epsilon_{24})n_1 n_2^2 + \chi_x n_1^3 - \chi_y n_2^3] \\
+\eta_z n_1^3 n_2 - \eta_z n_1 n_2^3 + 3(\epsilon_{26} - \epsilon_{16})n_1^2 n_2^2 + \epsilon_{16}n_1^4 - \epsilon_{26}n_2^4 + (\epsilon_y - \epsilon_x)n_1 n_2\}. \qquad (A-3)
\end{aligned}$$

The expressions in eq. (A-3) correspond to the following choice of vectors $\mathbf{e}^i$ in equation (8):

$$\mathbf{e}^1 \equiv D^{-1}(n_1 n_3, n_2 n_3, n_3^2 - 1), \quad \mathbf{e}^2 \equiv D^{-1}(-n_2, n_1, 0), \quad \mathbf{e}^3 = \mathbf{n} \equiv (n_1, n_2, n_3), \quad (A-4)$$

where

$$D = (n_1^2 + n_2^2)^{1/2}, \qquad n_1^2 + n_2^2 + n_3^2 = 1. \qquad (A-5)$$

The symbols n_i denote components of the unit normal to the wavefront.

Appendix B: Expressions for the first and second derivatives of the first-order P-wave eigenvalue G for an anisotropic medium of an arbitrary symmetry

The first derivatives of the first-order P-wave eigenvalue in eq. (4) appearing on the RHS of FORT equations (1) depend on x_m and p_m. They have the following explicit form:

$$\frac{\partial G}{\partial p_1} = 2\alpha^2\Big(p_1 + 2\epsilon_x p_1 + 2c^4[Ap_{23}p_1 + B(p_{23} - p_1^2) + C(3p_{23} + p_1^2)p_1^2 - Dp_1p_2p_3]\Big),$$

$$\frac{\partial G}{\partial p_2} = 2\alpha^2\Big(p_2 + 2\epsilon_y p_2 + 2c^4[Ep_{13}p_2 + F(p_{13} - p_2^2) + G(3p_{13} + p_2^2)p_2^2 - Hp_1p_2p_3]\Big),$$

$$\frac{\partial G}{\partial p_3} = 2\alpha^2\Big(p_3 + 2\epsilon_z p_3 + 2c^4[Pp_{12}p_3 + Q(p_{12} - p_3^2) + R(3p_{12} + p_3^2)p_3^2 - Sp_1p_2p_3]\Big),$$

$$\frac{\partial G}{\partial x_i} = 2\alpha^2\Big(T_{,i} + c^2[2R_{,i}p_3^3 + P_{,i}p_3^2 + 2Q_{,i}p_3 + S_{,i}p_1p_2]\Big). \qquad (B-1)$$

The second derivatives of the first-order P-wave eigenvalue in eq. (4) appearing on the RHS of FODRT equations (13) depend on x_m and p_m too. They have the following explicit form:

$$\frac{\partial^2 G}{\partial p_1^2} = 2\alpha^2\Big(1 + 2\epsilon_x + 2c^6[(Ap_{23} - Dp_2p_3)(p_{23} - 3p_1^2) + 2p_1(Cp_{23} - B)(3p_{23} - p_1^2)]\Big),$$

$$\frac{\partial^2 G}{\partial p_1 \partial p_2} = 4\alpha^2\Big(c^4[2(\chi_x p_3 + \eta_z p_2)p_1p_{23} + (\chi_z p_3^2 + 2\chi_y p_2p_3 + 3\epsilon_{26}p_2^2)(p_{23} - p_1^2)$$
$$+\epsilon_{16}(p_1^2 + 3p_{23})p_1^2 - (\eta_x p_3 + 4\epsilon_{24}p_2)p_1p_2p_3] - c^6[2A(p_{23} - p_1^2)p_1p_2 + 2B(p_{23} - 3p_1^2)p_2$$
$$+2C(3p_{23} - p_1^2)p_1^2p_2 + D(p_{13} - 3p_2^2)p_1p_3]\Big),$$

$$\frac{\partial^2 G}{\partial p_1 \partial p_3} = 4\alpha^2\Big(c^4[2(\chi_x p_2 + \eta_y p_3)p_1p_{23} + (\chi_y p_2^2 + 2\chi_z p_2p_3 + 3\epsilon_{35}p_3^2)(p_{23} - p_1^2)$$
$$+\epsilon_{15}(p_1^2 + 3p_{23})p_1^2 - (\eta_x p_2 + 4\epsilon_{34}p_3)p_1p_2p_3] - c^6[2A(p_{23} - p_1^2)p_1p_3 + 2B(p_{23} - 3p_1^2)p_3$$
$$+2C(3p_{23} - p_1^2)p_1^2p_3 + D(p_{12} - 3p_3^2)p_1p_2]\Big),$$

$$\frac{\partial^2 G}{\partial p_2^2} = 2\alpha^2\Big(1 + 2\epsilon_y + 2c^6[(Ep_{13} - Hp_1p_3)(p_{13} - 3p_2^2) + 2p_2(Gp_{13} - F)(3p_{13} - p_2^2)]\Big),$$

$$\frac{\partial^2 G}{\partial p_2 \partial p_3} = 4\alpha^2 \Big(c^4[2(\chi_y p_1 + \eta_x p_3)p_2 p_{13} + (\chi_x p_1^2 + 2\chi_z p_1 p_3 + 3\epsilon_{34} p_3^2)(p_{13} - p_2^2)$$
$$+\epsilon_{24}(p_2^2 + 3p_{13})p_2^2 - (\eta_y p_1 + 4\epsilon_{35} p_3)p_1 p_2 p_3] - c^6[2E(p_{13} - p_2^2)p_2 p_3 + 2F(p_{13} - 3p_2^2)p_3$$
$$+2G(3p_{13} - p_2^2)p_2^2 p_3 + H(p_{12} - 3p_3^2)p_1 p_2] \Big),$$

$$\frac{\partial^2 G}{\partial p_3^2} = 2\alpha^2 \Big(1 + 2\epsilon_z + 2c^6[(Pp_{12} - Sp_1 p_2)(p_{12} - 3p_3^2) + 2p_3(Rp_{12} - Q)(3p_{12} - p_3^2)] \Big),$$

$$\frac{\partial^2 G}{\partial x_i \partial p_1} = 4\alpha^2 \Big(\epsilon_{x,i} p_1 + c^4[A_{,i} p_{23} p_1 + B_{,i}(p_{23} - p_1^2) + C_{,i}(3p_{23} + p_1^2)p_1^2 - D_{,i} p_1 p_2 p_3] \Big),$$

$$\frac{\partial^2 G}{\partial x_i \partial p_2} = 4\alpha^2 \Big(\epsilon_{y,i} p_2 + c^4[E_{,i} p_{13} p_2 + F_{,i}(p_{13} - p_2^2) + G_{,i}(3p_{13} + p_2^2)p_2^2 - H_{,i} p_1 p_2 p_3] \Big),$$

$$\frac{\partial^2 G}{\partial x_i \partial p_3} = 4\alpha^2 \Big(\epsilon_{z,i} p_3 + c^4[P_{,i} p_{12} p_3 + Q_{,i}(p_{12} - p_3^2) + R_{,i}(3p_{12} + p_3^2)p_3^2 - S_{,i} p_1 p_2 p_3] \Big),$$

$$\frac{\partial^2 G}{\partial x_i \partial x_j} = 2\alpha^2 \Big(T_{,ij} + c^2[2R_{,ij} p_3^3 + P_{,ij} p_3^2 + 2Q_{,ij} p_3 + S_{,ij} p_1 p_2] \Big). \qquad (B-2)$$

In eqs (B-1) and (B-2),

$$c^2 = (p_k p_k)^{-1}, \qquad (B-3)$$

and the following notation is used:

$$p_{13} = p_1^2 + p_3^2, \quad p_{23} = p_2^2 + p_3^2, \quad p_{12} = p_1^2 + p_2^2,$$
$$A = \eta_y p_3^2 + 2\chi_x p_2 p_3 + \eta_z p_2^2, \quad B = \epsilon_{35} p_3^3 + \chi_z p_2 p_3^2 + \chi_y p_2^2 p_3 + \epsilon_{26} p_2^3,$$
$$C = \epsilon_{15} p_3 + \epsilon_{16} p_2, \quad D = 2\epsilon_{34} p_3^2 + \eta_x p_2 p_3 + 2\epsilon_{24} p_2^2,$$
$$E = \eta_x p_3^2 + 2\chi_y p_1 p_3 + \eta_z p_1^2, \quad F = \epsilon_{34} p_3^3 + \chi_z p_1 p_3^2 + \chi_x p_1^2 p_3 + \epsilon_{16} p_1^3,$$
$$G = \epsilon_{24} p_3 + \epsilon_{26} p_1, \quad H = 2\epsilon_{35} p_3^2 + \eta_y p_1 p_3 + 2\epsilon_{15} p_1^2,$$
$$P = \eta_y p_1^2 + 2\chi_z p_1 p_2 + \eta_x p_2^2, \quad Q = \epsilon_{15} p_1^3 + \chi_x p_1^2 p_2 + \chi_y p_1 p_2^2 + \epsilon_{24} p_2^3,$$
$$R = \epsilon_{34} p_2 + \epsilon_{35} p_1, \quad S = 2\epsilon_{16} p_1^2 + \eta_z p_1 p_2 + 2\epsilon_{26} p_2^2,$$
$$T = \epsilon_x p_1^2 + \epsilon_y p_2^2 + \epsilon_z p_3^2. \qquad (B-4)$$

Note that for isotropic media, for which

$$\epsilon_x = \epsilon_y = \epsilon_z = \frac{1}{2}\delta_x = \frac{1}{2}\delta_y = \frac{1}{2}\delta_z, \qquad \eta_x = \eta_y = \eta_z = 0, \qquad (B-5)$$

and all other WA parameters are zero, see eqs (16) and (26) of Pšenčík & Farra (2005), we get in (B-4): $A = B = ... = R = S = 0$. Only the term T remains non-zero in isotropic media. In this way, it is easy to check that all the expressions for the derivatives of $G(x_m, p_m)$ in eqs (B-1) and (B-2) reduce to exact ones for isotropic media.