Ray tracing for continuously rotated local coordinates belonging to a specified anisotropy

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(June 5, 2006)

ABSTRACT

Conventional ray tracing for arbitrarily anisotropic and heterogeneous media is expressed in terms of 21 elastic moduli belonging to a fixed, global, Cartesian coordinate system. Our principle objective is to obtain a new ray-tracing formulation, which takes advantage of the fact that the number of independent elastic moduli is often less than 21, and that the anisotropy thus has a simpler nature locally, as is the case for a medium possessing a transversely isotropic or orthorhombic symmetry locally. We have expressed the material properties and the ray-tracing quantities (e.g., the ray-velocity and slowness vectors) in a local anisotropy coordinate system with axes changing directions continuously within the model. In this manner, a transversely isotropic medium is described locally by five elastic moduli and two angles; an orthorhombic medium is described locally by nine elastic moduli and three angles. The new formulation is advantageous with respect to user-friendliness, efficiency, and memory usage. Concerning improved efficiency, a prominent role is played by the cross product of the ray-velocity and slowness vectors specified in the local coordinate system. Another important aspect is that the anisotropic symmetry properties are conserved when material properties are determined in arbitrary points by linear interpolation, spline function evaluation, or by other means.

Keywords: Ray tracing, rotated anisotropy, transverse isotropy, orthorhombic symmetry, TTI medium

Seismic Waves in Complex 3-D Structures, Report 16 (Department of Geophysics, Charles University, Prague 2006), pp.47-76

INTRODUCTION

In the last few years, there has been increased focus on modelling and imaging in the presence of highly complicated anisotropic structures with general orientation (Isaac and Lawton, 1999; Vestrum et al., 1999). The objective of this paper is to formulate a ray-tracing method applicable to such complicated structures, which takes advantage of the local anisotropic properties, so that modelling can be performed faster and with less memory usage compared to conventional ray tracing for general anisotropic media (Červený, 1972; Gajewski and Pšenčík, 1987).

The most general anisotropic medium, the triclinic medium, is characterized by 21 elastic moduli (stiffness parameters) that may all vary with the spatial coordinates. Although the need for modelling true triclinic anisotropy is probably very limited, consideration of general anisotropy is nevertheless useful for modelling of seismic waves. The reason is that a rotation of higher anisotropy symmetry, say, of the orthorhombic (OR) or transversely isotropic (TI) symmetries, parameterized by nine and five independent elastic moduli, generally leads to a medium with a greater number of non-zero and mutually dependent elastic moduli. Ray tracing for a medium of general anisotropy is known to be considerably slower than for non-rotated OR and TI media. In addition to slower computations, the large number of elastic moduli describing the general anisotropic medium makes ray tracing very demanding with respect to computer memory.

As far as we are aware of, only a few papers address the issue of formulating seismic modelling explicitly in terms of the local anisotropic properties of the medium. An early example is Jech (1983), who develop specific ray equations for the case that the rays are confined to a symmetry plane of a TI medium with a varying direction of the symmetry axis. Dickens (2004) describes a ray-tracing approach for tilted TI (TTI) media based on a model representation typically used by so-called "eikonal solvers", which are based on finite-differencing of the eikonal equation. The need for efficient traveltime computations for seismic imaging has made implementations based on eikonal solvers very attractive; as an example let us mention an algorithm for TTI media described by Kumar et al. (2004). Hokstad et al. (2002) formulate 3D finite-difference modelling in terms of local TI parameters, assuming a fixed orientation of the axis of symmetry. Santos et al. (2004) present a 2D finite-difference modelling approach where the direction of the symmetry axis is allowed to change continuously.

In this paper, we express the equations for ray tracing in terms of material properties and ray quantities belonging to a local Cartesian coordinate system. The axes of this local anisotropy coordinate system are required to vary continuously within each layer of the anisotropic velocity model. A fixed Cartesian coordinate system associated with the model is referred to as the global coordinate system. The local and global coordinate systems are both right-handed, and the third axis of the global coordinate system, the depth axis, is pointing down. The reformulated ray-tracing equations presented here are valid for any kind of anisotropy, including local anisotropy of the types TI and OR. A prominent role in the ray-tracing computations is played by the cross product of ray-velocity and slowness vectors specified in the local coordinate system. There are several reasons why we wanted a reformulation: enhanced user-friendliness, faster computations, lower memory requirements, and conservation of anisotropic symmetry properties when evaluating the material

parameters in an arbitrary point of the model. The latter is not guaranteed for ray tracing based on evaluation of 21 elastic moduli given in global coordinates. A basic assumption in our approach is that the local material properties and the parameters specifying the orientation of the local anisotropy coordinate system are known continuous functions of the global coordinates.

The paper is organized in five sections describing methodology, a section describing numerical examples, and finally, discussion and conclusions. The methodology starts with a review of ray tracing in global coordinates, followed by a definition of the local anisotropy coordinate system. Thereafter, ray tracing is expressed with respect to the local anisotropy coordinates. A rotation matrix is essential in the new formulation, and we describe a way to parameterize this rotation matrix by three Euler angles. In the last of the methodology sections, the "eta vector" occurring in the ray-tracing system is formulated explicitly with respect to the three angles, in order to gain further insight and more efficient computations.

RAY TRACING FOR ELASTIC MODULI IN GLOBAL COORDINATES

For an arbitrarily anisotropic and heterogeneous elastic medium described by density-normalized elastic moduli a_{ijkl}, a system of ray-tracing equations has the form

$$\frac{dx_n}{dT} = \frac{\partial \mathcal{H}}{\partial p_n} \ , \tag{1}$$

$$\frac{dp_n}{dT} = -\frac{\partial \mathcal{H}}{\partial x_n} \ . \tag{2}$$

Here, the quantities x_n and p_n are components of the position vector $\mathbf{x}$ and slowness vector $\mathbf{p}$ belonging to the ray, and the variable along the ray is denoted as T. As the Hamiltonian, $\mathcal{H}$, we take the function

$$\mathcal{H}(x_i, p_i) = \frac{1}{2}[G(x_i, p_i) - 1] \ , \tag{3}$$

for which the corresponding variable T along the ray is *traveltime*. The derivatives $v_n = dx_n/T$ and $\eta_n = dp_n/T$ are components of the ray-velocity vector $\mathbf{v}$ and "eta vector" $\boldsymbol{\eta}$.

In equation (3), G is one of the three eigenvalues of the Christoffel matrix,

$$G = a_{ijkl} p_j p_l g_i g_k \ , \tag{4}$$

where g_i and g_k are components of the corresponding eigenvector $\mathbf{g}$. Applying equation (3) in equations (1)-(2), we get

$$\frac{dx_n}{dT} = \frac{1}{2}\frac{\partial G}{\partial p_n} = v_n \ , \tag{5}$$

$$\frac{dp_n}{dT} = -\frac{1}{2}\frac{\partial G}{\partial x_n} = \eta_n \ . \tag{6}$$

The numerical integration of ray-tracing equations (5)-(6) is initialized by providing the slowness vector $\mathbf{p} = \mathbf{p}^0$ in the starting point $\mathbf{x} = \mathbf{x}^0$ of the ray. A component of the slowness vector $\mathbf{p}^0$ reads

$$p_n^0 = \frac{1}{c^0} n_n^0 \ , \tag{7}$$

where n_n^0 is a component of a unit vector, $\mathbf{n}^0$, normal to the wavefront at the initial point, $\mathbf{x}^0$, and c^0 is the phase velocity for the wave under consideration in the direction of $\mathbf{n}^0$.

LOCAL ANISOTROPY COORDINATE SYSTEM

The local anisotropy coordinate system ("LA coordinate system") is a Cartesian coordinate system with coordinate axes changing continuously with respect to the global coordinates. The LA coordinate system has coordinates x_1', x_2', x_3' and corresponding unit basis vectors $\mathbf{e}^1$, $\mathbf{e}^2$, $\mathbf{e}^3$. The three unit vectors form the columns of a 3×3 matrix $\mathbf{H}(\mathbf{x})$, which may be used for transformation of various quantities, e.g., position vectors, slowness vectors, ray-velocity vectors, elastic moduli, etc., from the LA coordinate system to the global coordinate system and vice versa. The transformations of position vector components are given by

$$x_i = H_{ij'} x_{j'}' \ , \qquad x_{i'}' = H_{ji'} x_j \ . \tag{8}$$

The elements H_{ij} of the transformation matrix $\mathbf{H}$ satisfy the relation,

$$H_{ji'} H_{jk'} = \delta_{i'k'} \ , \tag{9}$$

from which we immediately have

$$\frac{\partial H_{ji'}}{\partial x_n} H_{jk'} + H_{ji'} \frac{\partial H_{jk'}}{\partial x_n} = 0 \ . \tag{10}$$

The components of the slowness vector $\mathbf{p}$ and the eigenvector $\mathbf{g}$ transform according to

$$p_i = H_{ii'} p_{i'}' \ , \qquad g_i = H_{ii'} g_{i'}' \ . \tag{11}$$

Denoting a density-normalized elastic modulus in LA coordinates as $a_{i'j'k'l'}'$, the rotation of elastic moduli from LA to global coordinates is described by the relation

$$a_{ijkl} = H_{ii'} H_{jj'} H_{kk'} H_{ll'} a_{i'j'k'l'}' \ . \tag{12}$$

For a medium locally TI or OR, it is of great practical value to use the parameterizations of the elasticity introduced by Thomsen (1986) and Tsvankin (1997) rather than the local elastic moduli $a_{i'j'k'l'}'$. In Thomsen's parameterization, the TI medium

is described locally by the P-wave velocity V_P and the S-wave velocity V_S along the symmetry axis, and the three dimensionless parameters, ϵ, δ, and γ. We let one of the unit vectors, $\mathbf{e}^1$, $\mathbf{e}^2$, $\mathbf{e}^3$, define the orientation of the symmetry axis; commonly we use $\mathbf{e}^3$ for this purpose. The two other unit vectors are orthogonal to the symmetry axis. All three vectors must vary smoothly and remain orthonormal with variation of the position. Apart from these criteria, the specification of the two unit vectors orthogonal to the symmetry axis can be chosen freely. In the situation that the medium is locally orthorhombic, we let the three main coordinate planes of the LA coordinate system, i.e., $x'_1 = 0$, $x'_2 = 0$, and $x'_3 = 0$, coincide with the three mutually orthogonal symmetry planes of the OR medium. For the definitions of Thomsen's and Tsvankin's parameters in terms of the local elastic moduli, see the Appendices A and B.

RAY TRACING FOR ELASTIC MODULI IN LA COORDINATES

In this section we transform ray-tracing equations (5)-(6) into equations for a medium where the elastic moduli are specified in the LA coordinate system.

Ray-velocity vector

The derivation for the ray-velocity vector is fairly straightforward. Applying equation (12) in equation (4) yields

$$G = H_{ii'} H_{jj'} H_{kk'} H_{ll'} a'_{i'j'k'l'} p_j p_l g_i g_k \ . \tag{13}$$

Subsequent differentiation with respect to p_n and application of equations (9) and (11) leads to

$$\frac{\partial G}{\partial p_n} = 2 a'_{i'n'k'l'} H_{nn'} p'_{l'} g'_{i'} g'_{k'} + 2G \frac{\partial g_i}{\partial p_n} g_i \ . \tag{14}$$

Since the eigenvector $\mathbf{g}$ has fixed length ($g_i g_i = 1$), the second term in equation (14) is zero. Considering equation (5), the components of the ray-velocity vector are transformed by the equation

$$v_n = H_{nn'} v'_{n'} \ , \tag{15}$$

where the ray-velocity vector component $v'_{n'}$ in LA coordinates is given by

$$v'_{n'} = a'_{i'n'k'l'} p'_{l'} g'_{i'} g'_{k'} \ . \tag{16}$$

Eta vector

We now turn to expressing the eta-vector component in equation (6) in terms of quantities belonging to the LA coordinate system. Applying equations (9) and (11) in equation (13) yields

$$G = a'_{i'j'k'l'} p'_{j'} p'_{l'} g'_{i'} g'_{k'} \ , \tag{17}$$

which, when comparing to equation (4), shows that the eigenvalue G is invariant with respect to rotation between Cartesian coordinate systems. Differentiation of equation (17) with respect to x_n, with subsequent use of equation (11), leads to

$$\frac{\partial G}{\partial x_n} = \frac{\partial a'_{i'j'k'l'}}{\partial x_n} p'_{j'} p'_{l'} g'_{i'} g'_{k'} + 2 a'_{i'j'k'l'} \frac{\partial H_{jj'}}{\partial x_n} p_j p'_{l'} g'_{i'} g'_{k'} + 2G \frac{\partial g'_{i'}}{\partial x_n} g'_{i'} \ , \tag{18}$$

where the last term is zero from the same reason as in equation (14). Using equations (11) and (16), we obtain

$$\frac{\partial G}{\partial x_n} = -2\eta_n^0 + 2 \frac{\partial H_{jj'}}{\partial x_n} H_{jk'} v'_{j'} p'_{k'} \ , \tag{19}$$

where

$$\eta_n^0 = -\frac{1}{2} \frac{\partial a'_{i'j'k'l'}}{\partial x_n} p'_{j'} p'_{l'} g'_{i'} g'_{k'} \ . \tag{20}$$

Considering equations (6) and (19), the eta-vector component η_n can therefore be expressed in terms of quantities belonging to the LA coordinate system as

$$\eta_n = \eta_n^0 - \frac{\partial H_{jj'}}{\partial x_n} H_{jk'} v'_{j'} p'_{k'} \ . \tag{21}$$

A minor rewrite of the second term in equation (19) yields

$$\frac{\partial G}{\partial x_n} = -2\eta_n^0 + \frac{\partial H_{jj'}}{\partial x_n} H_{jk'} (v'_{j'} p'_{k'} - v'_{k'} p'_{j'}) \ , \tag{22}$$

or equivalently,

$$\eta_n = \eta_n^0 - \frac{1}{2} \frac{\partial H_{jj'}}{\partial x_n} H_{jk'} (v'_{j'} p'_{k'} - v'_{k'} p'_{j'}) \ . \tag{23}$$

We see from equations (21) and (23) that the vector $\boldsymbol{\eta}$ reduces to the vector $\boldsymbol{\eta}^0$ in the situation where the LA coordinate system has fixed orientation ($\partial H_{ii'}/\partial x_n = 0$ for every point of the medium). The second term in equation (23) depends on factors of the type $(v'_{j'} p'_{k'} - v'_{k'} p'_{j'})$, which constitute the components of the cross product

$$\mathbf{w}' \equiv \mathbf{v}' \times \mathbf{p}' = (v'_2 p'_3 - v'_3 p'_2, v'_3 p'_1 - v'_1 p'_3, v'_1 p'_2 - v'_2 p'_1)^T \ . \tag{24}$$

Ray-tracing equations

Collecting the results in equations (15) and (23), ray tracing in the presence of rotated anisotropy can be formulated by the system of ordinary differential equations

$$\frac{dx_n}{dT} = H_{nn'} v'_{n'} \ ,$$

$$\frac{dp_n}{dT} = \eta^0_n - \frac{1}{2}\frac{\partial H_{jj'}}{\partial x_n} H_{jk'}(v'_{j'} p'_{k'} - v'_{k'} p'_{j'}) \ . \tag{25}$$

The evaluation of the ray-tracing equations (25) relies on the knowledge of the slowness vector in LA coordinates, $\mathbf{p}'$. Knowing the slowness vector in global coordinates, $\mathbf{p}$, and the rotation matrix, $\mathbf{H}$, in the current ray point, $\mathbf{x}$, slowness vector components in LA coordinates are obtained by

$$p'_{n'} = H_{nn'} p_n \ . \tag{26}$$

The ray-tracing system (25) is valid for any type of anisotropic symmetry specified with respect to the LA coordinate system. For the most general anisotropic medium, expressions for the ray-velocity and eta vector components $v'_{n'}$ and η^0_n are given, respectively, by equations (16) and (20). If the anisotropy is of, say, TI or OR type locally, it is useful to write special versions of equations (16) and (20) that only include the elastic moduli $a'_{i'j'k'l'}$ actually influencing the calculations. Since the derivation of ray-tracing equations for non-rotated TI and OR media is straightforward and has been described several times in the literature, we do not include such equations in this paper. We remark, though, that, due to the common use of parameters of the Thomsen type (Thomsen, 1986; Tsvankin, 1997), the ray-tracing equations for non-rotated TI and OR media may preferably be written explicitly in terms of such parameters.

Initialization of the numerical integration of the ray-tracing equations (25) requires that the slowness vector $\mathbf{p}^0$ is specified in the starting point $\mathbf{x}^0$, see equation (7). The initialization consists of three steps: (1) transform the unit vector $\mathbf{n}^0$ to the LA coordinate system,

$$n'^0_{i'} = H_{ii'} n^0_i \ ; \tag{27}$$

(2) obtain the squared phase velocity $(c^0)^2$ as a selected root of a cubic equation, given in a general form as

$$\det\left[a'_{i'j'k'l'}(\mathbf{x}^0) n'^0_{j'} n'^0_{l'} - (c^0)^2 \delta_{i'k'}\right] \ ; \tag{28}$$

(3) when c^0 is known, use equation (7) to compute the slowness vector $\mathbf{p}^0$.

PARAMETERIZING THE MATRIX H BY EULER ANGLES

The 3×3 matrix $\mathbf{H}$ contains nine elements that are all required to vary smoothly within the model. Various possibilities exist for parameterization of the 3×3 matrix $\mathbf{H}$ in terms of only a few parameters. In this section we parameterize the matrix $\mathbf{H}$ by three *Euler angles* (Goldstein, 1980; Weisstein, 1999) denoted as λ, μ, and ν. Several conventions exist for the Euler angles, among these the so-called x, y,

and xyz conventions. Here we use a variant of the xyz convention, which means that the rotation of any vector with respect to a fixed Cartesian coordinate system can be decomposed in terms of three subsequent rotations around local coordinate axes generically referred to as x, y, and z. It also means that the unit basis vectors for a local coordinate system, specified relative to the global coordinate system and collected as columns of a 3×3 matrix $\mathbf{H}$, can be expressed in terms of the angles λ, μ, and ν by the matrix product

$$\mathbf{H} = \mathbf{H}^{\lambda}\,\mathbf{H}^{\mu}\,\mathbf{H}^{\nu}\ . \tag{29}$$

Here, the matrices on the right-hand side have the definitions

$$\mathbf{H}^{\lambda} = \begin{pmatrix} \cos\lambda & 0 & \sin\lambda \\ 0 & 1 & 0 \\ -\sin\lambda & 0 & \cos\lambda \end{pmatrix}\ , \tag{30}$$

$$\mathbf{H}^{\mu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\mu & -\sin\mu \\ 0 & \sin\mu & \cos\mu \end{pmatrix}\ , \tag{31}$$

$$\mathbf{H}^{\nu} = \begin{pmatrix} \cos\nu & -\sin\nu & 0 \\ \sin\nu & \cos\nu & 0 \\ 0 & 0 & 1 \end{pmatrix}\ . \tag{32}$$

Thus, in the current context, the generic axes of rotation x, y, and z correspond, respectively, to the axes of rotation belonging to the angles μ, λ, and ν. One could think of introducing a specific notation for the local axes of rotation, but we do not find it necessary.

The angles λ and μ (Figure 1) have an analogy to the longitude and latitude angles that specify a point on the Earth's surface. In this context, the polar axis is parallel to the vector $(0,1,0)^T$, which points to the "north" pole. The "longitude" λ and "latitude" μ are positive toward the "east" and "south", respectively. The angle ν defines rotation around the axis specified by the vector $\mathbf{u}$ in Figure 1. At the poles, i.e., for $\mu = \pm\pi/2$, the angle λ may take any value; it is indeterminate. For model building and ray tracing it is useful to have a one-to-one relationship between the rotation matrix $\mathbf{H}$ and the parameters λ, μ, ν. In particular, it is essential for ray tracing that λ, μ, ν are C_2 continuous. By choosing the order of rotations as done in equation (29), the x_2 axis of the global coordinate system represents a singular direction, the x_3 axis being non-singular. This is beneficial with respect to simulation and processing of seismic waves in reflection seismology, where the x_3 direction (depth) is of particular importance.

Using the chain rule, differentiation of equation (29) yields

$$\frac{\partial \mathbf{H}}{\partial x_n} = \frac{\partial \mathbf{H}}{\partial \lambda}\frac{\partial \lambda}{\partial x_n} + \frac{\partial \mathbf{H}}{\partial \mu}\frac{\partial \mu}{\partial x_n} + \frac{\partial \mathbf{H}}{\partial \nu}\frac{\partial \nu}{\partial x_n} \ . \tag{33}$$

Knowing the matrix $\mathbf{H}$ and its derivative $\partial\mathbf{H}/\partial x_n$, we have all the information required for numerical integration of ray-tracing equations (25).

ETA VECTOR FOR SPECIAL CASES

In this section we elaborate further on the expression for the eta-vector component η_n in equation (21). The objective is to provide better insight and more efficient calculations in a number of important special cases for the rotation of the local anisotropy. The derivations require a differentiation of the matrix $\mathbf{H}$ as stated in equation (33). Furthermore, we utilize the fact that a matrix product of the type $\partial\mathbf{H}^T/\partial x_n\mathbf{H}$, which corresponds to the product $\partial H_{jj'}/\partial x_n\, H_{jk'}$ in equation (21), is antisymmetric.

Generally rotated anisotropy

Inserting equations (30)-(32) into equation (29), we get

$$\mathbf{H} = \begin{pmatrix} \cos\lambda\cos\nu + \sin\lambda\sin\mu\sin\nu & -\cos\lambda\sin\nu + \sin\lambda\sin\mu\cos\nu & \sin\lambda\cos\mu \\ \cos\mu\sin\nu & \cos\mu\cos\nu & -\sin\mu \\ -\sin\lambda\cos\nu + \cos\lambda\sin\mu\sin\nu & \sin\lambda\sin\nu + \cos\lambda\sin\mu\cos\nu & \cos\lambda\cos\mu \end{pmatrix} . \tag{34}$$

After a sequence of algebraic manipulations, using the antisymmetric property of the matrix product $\partial\mathbf{H}^T/\partial x_n\mathbf{H}$, we obtain from equation (21) the eta-vector component in terms of the angles λ, μ, and ν as

$$\begin{aligned} \eta_n = \eta_n^0 \\ -(w_1'\cos\nu - w_2'\sin\nu)\frac{\partial\mu}{\partial x_n} \\ -[(w_1'\sin\nu + w_2'\cos\nu)\cos\mu - w_3'\sin\mu]\frac{\partial\lambda}{\partial x_n} \\ -w_3'\frac{\partial\nu}{\partial x_n} \ . \end{aligned} \tag{35}$$

Here, the quantities w_1', w_2', and w_3' are the components of the vector $\mathbf{w}'$, see equation (24).

Rotated anisotropy in the horizontal plane

Consider an anisotropic medium, where the rotation of the LA coordinate system is only performed with respect to the x_3 axis of the global coordinate system. This situation is accomplished by setting $\lambda = \mu = 0$ in equation (34) for the matrix $\mathbf{H}$, which yields

$$\mathbf{H} = \begin{pmatrix} \cos\nu & -\sin\nu & 0 \\ \sin\nu & \cos\nu & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{36}$$

The eta-vector component becomes

$$\eta_n = \eta_n^0 - w_3' \frac{\partial \nu}{\partial x_n} . \tag{37}$$

Equations (36) and (37) can be used for an OR medium with one horizontal and two vertical symmetry planes. Each column vector of matrix $\mathbf{H}$ is then orthogonal to one symmetry plane, and the angle ν defines the azimuths of the two vertical symmetry planes. The axis for specification of P- and S-wave velocities V_P and V_S in Thomsen/Tsvankin's parameterizations is in this case the x_3 axis of the global coordinate system.

For a TI medium with horizontal symmetry axis (HTI medium), it is commonly preferred to let the velocities V_P and V_S correspond to the vertical direction. This can be accomplished by considering the HTI medium as a special case of an OR medium, for which the rotation matrix $\mathbf{H}$ and the eta-vector component η_n are given by equations (36)-(37). The angle ν then defines the azimuth of the axis of symmetry.

Zero rotation angle ν

In a medium where $\nu = 0$, the matrix $\mathbf{H}$ is given by

$$\mathbf{H} = \begin{pmatrix} \cos\lambda & \sin\lambda\sin\mu & \sin\lambda\cos\mu \\ 0 & \cos\mu & -\sin\mu \\ -\sin\lambda & \cos\lambda\sin\mu & \cos\lambda\cos\mu \end{pmatrix} . \tag{38}$$

Since $\nu = 0$ everywhere, equation (35) simplifies to the following:

$$\eta_n = \eta_n^0 - w_1' \frac{\partial \mu}{\partial x_n} - (w_2' \cos\mu - w_3' \sin\mu) \frac{\partial \lambda}{\partial x_n} . \tag{39}$$

Equation (39) is especially useful for the situation where the anisotropy is TI. It may also be used for lower anisotropic symmetries, with the underlying understanding that the orientations of the x_1' and x_2' axes of the LA coordinate system are determined only by the angles λ and μ.

A special rotation for a HTI medium

Consider a rotated anisotropic medium where $\lambda = \pi/2$, $\nu = 0$, while μ may vary within the range $[-\pi/2, \pi/2]$. The matrix $\mathbf{H}$ becomes

$$\mathbf{H} = \begin{pmatrix} 0 & \sin\mu & \cos\mu \\ 0 & \cos\mu & -\sin\mu \\ -1 & 0 & 0 \end{pmatrix} . \tag{40}$$

The rotation matrix in equation (40) applies to HTI and OR media, for which the velocities V_P and V_S in Thomsen/Tsvankin's parameterizations are specified along a horizontal (symmetry) axis. The direction of this axis is given by the third column vector of the matrix $\mathbf{H}$. The OR medium has one horizontal and two vertical symmetry planes; in particular, the angle μ defines the azimuths of the two vertical symmetry planes. Equation (40) yields the component of the eta vector as

$$\eta_n = \eta_n^0 - w_1' \frac{\partial \mu}{\partial x_n} . \tag{41}$$

Rotated anisotropy in a vertical plane

For $\mu = 0$ and $\nu = 0$, the rotation of the LA coordinate system is solely with respect to the x_2 axis of the global coordinate system. Thus, the matrix $\mathbf{H}$ becomes

$$\mathbf{H} = \begin{pmatrix} \cos\lambda & 0 & \sin\lambda \\ 0 & 1 & 0 \\ -\sin\lambda & 0 & \cos\lambda \end{pmatrix} . \tag{42}$$

The rotation matrix in equation (42) is particularly useful for the case where the plane $x_2 = 0$ is a symmetry plane and all rays are confined to this plane. Considering equation (35), the eta-vector component can be obtained by

$$\eta_n = \eta_n^0 - w_2' \frac{\partial \lambda}{\partial x_n} . \tag{43}$$

NUMERICAL EXAMPLES

Models and configuration

In this section we demonstrate the application of the theory on simple slightly heterogeneous models possessing rotated anisotropy. These models consist of a single,

continuous, layer. The heterogeneity of the models is formed by use of two isosurfaces. The upper surface is a horizontal plane ($x_3 = 0$); the lower surface is only slightly curved with maximum dip approximately along the line $x_2 = x_1$. The material parameters constituting the models are required to be constant along each isosurface, and in between they vary linearly in the vertical direction. Since the lower isosurface varies with the lateral coordinates, this is also the case for the material properties.

Consider two models referred to as HTI_{FIX} and HTI_{ROT} where the anisotropy is of the type HTI. For both models, the elastic moduli belonging to LA coordinate systems at the upper and lower isosurfaces are identical (Table 1). However, the models HTI_{FIX} and HTI_{ROT} differ by the way the LA coordinate system is rotated with respect to the global coordinate system. In the model HTI_{FIX}, the symmetry axis is parallel to the global x_1 axis on both isosurfaces; hence, the direction of the symmetry axis is fixed throughout the model. In the model HTI_{ROT}, the symmetry axis is parallel to the global x_1 axis along the lower isosurface, as in the model HTI_{FIX}, but on the upper isosurface the symmetry axis makes an angle of -45° with the x_1 axis. We therefore have a medium where the angle between the x_1 axis and the symmetry axis changes linearly from -45° to 0° along any vertical line between the upper and lower isosurface.

In addition to the two HTI models, consider a third model, referred to as OR_{ROT}, where the anisotropy is of the type OR. The geometry of this model is exactly the same as for the two HTI models, only now, the local elastic moduli at the two isosurfaces correspond to orthorhombic anisotropy (Table 2). The values for the latter moduli are almost identical to those used by Schoenberg and Helbig (1997). As in the case of HTI models, the vertical axis of the local coordinate system is first rotated to the global x_1 axis. Conversely to the HTI_{ROT} model, elastic moduli are then rotated by 0° and -45° at the upper and lower isosurfaces, respectively.

In the first model, HTI_{FIX}, the three angles parameterizing the matrix $\mathbf{H}$ have the values $\lambda = 90°$, $\mu = 0$, and $\nu = 0$ on both isosurfaces. For the second model, HTI_{ROT}, we have $\lambda = 90°$, $\mu = -45°$, and $\nu = 0$ on the upper isosurface and $\lambda = 90°$, $\mu = 0$, and $\nu = 0$ on the lower isosurface. Finally, in the third model, OR_{ROT}, the specifiation of angles λ and ν for model HTI_{ROT} is retained, but now $\mu = 0°$ at the upper isosurface and $\mu = -45°$ at the lower isosurface. In all three models, the three angles λ, μ, ν and the elastic moduli in Tables 1-2 are interpolated linearly between the two isosurfaces.

Consider a VSP survey configuration with a source point located in the origin of the global coordinate system and a vertical receiver line, for which $x_1 = 1$ km and $x_2 = 0$ km. The receivers are distributed between depths $x_3 = 0.04$ km and $x_3 = 0.96$ km, with equidistant spacing 0.04 km. Given this survey configuration, we did two ray-tracing simulations of the direct P wave for each model. The first ray-tracing approach is formulated in LA coordinates and makes use of the elastic moduli in Tables 1-2, the angles λ,μ,ν, and the first-order derivatives of these parameters with respect to the global coordinates x_1,x_2, and x_3. The second ray-tracing approach (program package ANRAY) is formulated in global coordinates (Červený, 1972; Gajewski and Pšenčík, 1987). Before ray tracing with the second approach a model preparation step is needed, consisting of transforming the two sets of elastic moduli in Tables 1-2, one set for each isosurface, into two full sets of 21 elastic moduli belonging to the global coordinate system.

Simulations using model HTI_{FIX}

In the model HTI_{FIX}, the matrix $\mathbf{H}$ corresponds to equation (40) with the angle μ zero,

$$\mathbf{H} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} . \tag{44}$$

Moreover, the second term on the right-hand side of equation (41) vanishes, which means that

$$\eta_n = \eta_n^0 . \tag{45}$$

Ray diagrams and traveltimes obtained using model HTI_{FIX} are shown in Figure 2. Considering equation (12), the constant matrix $\mathbf{H}$ means that linear interpolation of elastic moduli of the form a_{ijkl} can be expected to be perfectly consistent with linear interpolation of elastic moduli of the form $a'_{i'j'k'l'}$. In other words, equation (12) will be satisfied everywhere, even if the elastic moduli a_{ijkl} and $a'_{i'j'k'l'}$ have been interpolated independently. As a consequence, the traveltimes computed by both ray-tracing approaches should, from theory, be the same, which is confirmed by Figure 3. The largest relative traveltime difference for the two approaches is 0.03%, in magnitude.

Simulations using model HTI_{ROT}

When the symmetry axes at the upper and lower isosurfaces are non-parallel, linear interpolation of the elastic moduli a_{ijkl} is generally not consistent with linear interpolation of the elastic moduli $a'_{i'j'k'l'}$. This is because some of the elements of the matrix $\mathbf{H}$ become strongly nonlinear with respect to the out-of-plane angle μ. Thus, the two ray-tracing approaches will yield slightly inconsistent values of the elastic moduli at a given location, which in the end leads to different ray diagrams and traveltime curves (Figures 4). The largest relative difference in traveltime (Figure 5) is 0.37%, in magnitude. The results of ray tracing formulated in LA coordinates should be considered superior, because linear interpolation of the elastic moduli $a'_{i'j'k'l'}$ and the angle μ conserves the TI character in any point of the model. For the ray-tracing approach formulated in global coordinates, strict TI is only obtained along the isosurfaces on which we specified the elastic moduli. In between these surfaces the medium deviates slightly from TI.

Simulations using model OR_{ROT}

Ray diagrams and traveltimes for the model OR_{ROT} are shown in Figure 6. Comparing the traveltimes to those obtained by ray tracing formulated in global coordi-

nates, we find a maximum deviation of approximately 2.5%. The traveltime deviations seem to be quite sensitive to the strength of the angular variation of the phase velocity. On the other hand, the traveltime deviations are quite insensitive to the inhomogeneity of a medium, i.e., to the spatial variation of elastic moduli. Again, the results from the ray tracing formulated in LA coordinates should be considered superior, because linear interpolation of the elastic moduli $a'_{i'j'k'l'}$ and the angle ν conserves the anisotropic symmetry in any point of the model.

DISCUSSION AND CONCLUSIONS

In this paper we have introduced a ray-tracing approach based on local anisotropy (LA) coordinates valid for a heterogeneous triclinic medium. Ray tracing using LA coordinates has a number of advantages compared to conventional ray tracing for anisotropic media, which is formulated with respect to 21 elastic moduli given in the global coordinate system. Although the new approach is generally applicable to arbitrary anisotropy, we foresee that the main use will be for cases where the anisotropic medium is locally of higher symmetry, e.g., of the symmetries TI or OR.

For the numerical experiments, we used a model representation where elastic moduli $a'_{i'j'k'l'}$ in LA coordinates where specified on two isosurfaces. In addition, we provided at these surfaces three angles λ, μ, and ν determining the orientation of the LA coordinate system. It must be strongly emphasized, however, that the method has a much wider potential than we managed to illustrate in the above tests. The wider potential concerns the generality of the model representation as well as the functionality of the ray tracer and is to be discussed in the following.

Ray tracing based on LA coordinates is more user-friendly than ray tracing formulated in global coordinates. One reason is that the number of material parameters specified by the user can be strongly reduced, to maximum five and nine parameters for anisotropy of type TI and OR, respectively. Another reason is the possibility of applying the parameterizations of Thomsen (1986) for TI media and Tsvankin (1997) for OR media, which are now widely used and commonly considered much more user-friendly than specification in terms of elastic moduli. In addition to the parameters specifying the elasticity, the user has to provide the angles λ, μ, and ν specifying the orientation of the LA coordinate system. For a TI medium, two of these angles specify the direction of the symmetry axis, commonly the angles λ and μ, while for an OR medium the three angles collectively yield the orientation of the three mutually orthogonal symmetry planes. Since it may be problematic to estimate λ and μ based on seismic data, an approach commonly considered adequate is to determine the two angles automatically from the dip of smoothly varying layered structures, assuming that the axis is normal to the bedding everywhere. This means that a generally smoothly varying symmetry axis of a TI medium can be specified without complicated user intervention, using interpolation or function fit based on interfaces provided in the anisotropic velocity model.

Ray tracing using LA coordinates is generally more efficient than ray tracing in global coordinates. The efficiency of any ray-tracing procedure is strongly dependent on the number of volumetric model parameters to be evaluated, e.g., the functions

$a_{ijkl}(\mathbf{x})$, $V_P(\mathbf{x})$, $\epsilon(\mathbf{x})$, etc., as well as on the representation of such functions, e.g., constant and linear representations, mono-/bi-/tri-cubic spline representations, etc.. Ray tracing formulated with respect to LA coordinates honors efficiency because the number of model parameter evaluations is reduced to a minimum, and the approach takes advantage of the fact that some model parameters are typically of a much simpler nature than others, e.g., when the function V_P varies strongly while the anisotropy parameters ϵ and δ are defined as constants or slowly varying functions. Note especially that ray tracing based on LA coordinates is performed without applying the tensor transformation in equation (12) explicitly, which would have resulted in significantly slower computations.

Ray tracing formulated with respect to LA coordinates will, by definition, conserve the local symmetry properties of the medium under consideration. As we have seen in the numerical tests, this is not necessarily the case for ray tracing using global coordinates.

Ray tracing using LA coordinates is generally less demanding with respect to computer memory than ray tracing in global coordinates. The reason is obvious: ray tracing in global coordinates will need to store data in memory for all 21 elastic moduli on three-dimensional grids (or their equivalents). Moreover, such grids may have to be densely sampled in order to represent accurately the local anisotropic properties of the medium. The numerical simulations demonstrated that ray tracing in global coordinates failed to retain the true local anisotropic symmetry, since the elastic moduli were only specified at two locations along the depth axis, with linear interpolation in between. Enhancement of the sampling and interpolation in the depth direction would have given better results for the ray tracing using global coordinates; but, on the other hand, this would have lead to a substantial increase in the usage of computer memory.

Ray tracing using LA coordinates is quite simple to implement if one already has available a ray tracer for a VTI medium or an OR medium where the symmetry planes are aligned with the main planes of the global coordinate system. One may actually re-use the software code for estimation of the ray-velocity and eta-vectors in these non-rotated cases and implement in addition the effects of rotating the anisotropy.

The rotation matrix $\mathbf{H}$ introduced in this paper avoids problems with indeterminate angles for rays whose trajectories are not parallel or close to parallel to the x_2 axis of the global coordinate system. This is an important aspect, considering, in particular, the basic role of vertical or nearly vertical rays in reflection seismology. If rays with trajectories parallel or nearly parallel to the x_2 axis are to be calculated, the order of the three matrix rotations (30)-(32) must be changed appropriately.

The scalar product of the ray-velocity vector and the slowness vector has a fundamental role in the ray theory for anisotropic media, since it has to be equal one everywhere along a ray. We have showed that the cross product of the ray-velocity vector and the slowness vector plays an important role in the presence of rotated anisotropy. The latter cross product contributes to the equations for the eta vector, and it vanishes when the two vectors are parallel, which is the case in an isotropic medium and in special directions of an anisotropic medium.

In this paper, we concentrated on expressing the ray-tracing equations in LA coordinates and presented examples for P waves in a smooth medium. Generalization for elementary P-, S-, and converted waves in layered media is straightforward. A

natural future extension is to apply the approach of this paper to dynamic ray-tracing equations and equations of ray perturbation theory. Introduction of the LA coordinates described here may also be considered for other modelling methods than ray tracing, e.g., finite-difference modelling, eikonal solvers, etc.

ACKNOWLEDGMENTS

The authors are grateful to two anonymous referees for their stimulating comments. Einar Iversen acknowledges the support from the Research Council of Norway, project 163390/S30. Ivan Pšenčík has been supported by the Consortium Project "Seismic Waves in Complex 3D Structures" and by research project A3012309 of the Grant Agency of the Acad. Sci. of the Czech Republic.

REFERENCES

Červený, V., 1972, Seismic rays and ray intensities in inhomogeneous anisotropic media: Geophys. J. Roy. Astr. Soc., **29**, 1–13.

Dickens, T., 2004, Ray tracing in tilted transversely isotropic media: A group velocity approach: 74th Ann. Internat. Mtg., Soc. of Expl. Geophys., Expanded Abstracts, 973–976.

Gajewski, D., and Pšenčík, I., 1987, Computation of high-frequency seismic wavefields in 3-D laterally inhomogeneous anisotropic media: Geophys. J. Roy. Astr. Soc., **91**, 383–411.

Goldstein, H., 1980, Classical mechanics, 2nd ed.: Addison-Wesley Publ. Co.

Hokstad, K., Engell-Sørensen, L., and Maaø, F., 2002, 3D elastic finite-difference modeling in tilted transversely isotropic media: 72nd Ann. Internat. Mtg., Soc. of Expl. Geophys., Expanded Abstracts, 1951–1954.

Isaac, J. H., and Lawton, D. C., 1999, Image mispositioning due to dipping TI media: A physical seismic modeling study: Geophysics, **64**, 1230–1238.

Jech, J., 1983, Computation of rays in an inhomogeneous transversely isotropic medium with a non-vertical axis of symmetry: Studia Geophysica et Geodaetica, **27**, 114–121.

Kumar, D., Sen, M. K., and Ferguson, R. J., 2004, Traveltime calculation and prestack depth migration in tilted transversely isotropic media: Geophysics, **69**, 37–44.

Santos, M. A. C., Filho, D. M. S., and Osório, P. L. M., 2004, A finite difference scheme for locally transverse isotropic media applied to a highly tectonic deformed model: 74th Ann. Internat. Mtg., Soc. of Expl. Geophys., Expanded Abstracts, 1913–1916.

Schoenberg, M., and Helbig, K., 1997, Orthorhombic media: Modeling elastic wave behavior in a vertically fractured earth: Geophysics, **62**, 1954–1974.

Thomsen, L., 1986, Weak elastic anisotropy: Geophysics, **51**, 1954–1966.

Tsvankin, I., 1997, Anisotropic parameters and P-wave velocity for orthorhombic media: Geophysics, **62**, 1292–1309.

Vestrum, R. W., Lawton, D. C., and Schmid, R., 1999, Imaging structures below dipping TI media: Geophysics, **64**, 1239–1246.

Weisstein, E. W., 1999, Euler angles: From MathWorld–A Wolfram Web Resource, http://mathworld.wolfram.com/EulerAngles.html.

APPENDIX A–

THOMSEN'S PARAMETERS FOR A TRANSVERSELY ISOTROPIC MEDIUM

Consider a transversely isotropic medium specified with respect to the LA coordinate system (x_1', x_2', x_3'), such that the axis of rotational symmetry is the x_3' axis. Using Voigt notation (Appendix C) for the elastic moduli, Thomsen's (1986) parameters have the following definitions,

$$V_P = A'^{1/2}_{33} \ , \tag{A-1}$$

$$V_S = A'^{1/2}_{44} \ , \tag{A-2}$$

$$\epsilon = \frac{A'_{11} - A'_{33}}{2A'_{33}} \ , \tag{A-3}$$

$$\delta = \frac{(A'_{13} + A'_{44})^2 - (A'_{33} - A'_{44})^2}{2A'_{33}(A'_{33} - A'_{44})} \ , \tag{A-4}$$

$$\gamma = \frac{A'_{66} - A'_{44}}{2A'_{44}} \ . \tag{A-5}$$

The quantities V_P and V_S in equations (A-1)-(A-2) are the P- and S-wave phase velocities along the x_3' axis. The three dimensionless parameters ϵ, δ, and γ in equations (A-3)-(A-5) are all zero for an isotropic medium.

APPENDIX B–

TSVANKIN'S PARAMETERS FOR AN ORTHORHOMBIC MEDIUM

Tsvankin (1997) used parameters of the Thomsen type, equations (A-1)-(A-5), for parameterization of an orthorhombic medium. The transformations from elastic moduli in the LA coordinate system to Tsvankin's parameters read

$$V_P = A'^{1/2}_{33} \ , \tag{B-1}$$

$$V_S = A'^{1/2}_{55} \ , \tag{B-2}$$

$$\epsilon^{(2)} = \frac{A'_{11} - A'_{33}}{2A'_{33}} \ , \tag{B-3}$$

$$\delta^{(2)} = \frac{(A'_{13} + A'_{55})^2 - (A'_{33} - A'_{55})^2}{2A'_{33}(A'_{33} - A'_{55})} \ , \tag{B-4}$$

$$\gamma^{(1)} = \frac{A'_{66} - A'_{55}}{2A'_{55}} \ , \tag{B-5}$$

$$\epsilon^{(1)} = \frac{A'_{22} - A'_{33}}{2A'_{33}} \ , \tag{B-6}$$

$$\delta^{(1)} = \frac{(A'_{23} + A'_{44})^2 - (A'_{33} - A'_{44})^2}{2A'_{33}(A'_{33} - A'_{44})} \ , \tag{B-7}$$

$$\gamma^{(2)} = \frac{A'_{66} - A'_{44}}{2A'_{44}} \ , \tag{B-8}$$

$$\delta^{(3)} = \frac{(A'_{12} + A'_{66})^2 - (A'_{11} - A'_{66})^2}{2A'_{11}(A'_{11} - A'_{66})} \ . \tag{B-9}$$

As with Thomsen's parameters, the symbols V_P and V_S are used for the P- and S-wave phase velocities along the x'_3 axis. The medium is isotropic if all the dimensionless parameters $\epsilon^{(i)}$, $\delta^{(i)}$, and $\gamma^{(i)}$ are zero. The orthorhombic medium becomes a transversely isotropic medium with symmetry axis along the x'_3 axis if $\epsilon^{(1)} = \epsilon^{(2)} = \epsilon$; $\delta^{(1)} = \delta^{(2)} = \delta$; $\gamma^{(1)} = \gamma^{(2)} = \gamma$; $\delta^{(3)} = 0$.

APPENDIX C–

VOIGT NOTATION

Elastic moduli a'_{ijkl} have an alternative representation as matrix elements $A'_{\alpha\beta}$ (Voigt notation). Mapping $\alpha \leftrightarrow ij$ follows the convention

$$1 \leftrightarrow 11\ , \quad 2 \leftrightarrow 22\ , \quad 3 \leftrightarrow 33\ , \quad 4 \leftrightarrow 23\ , \quad 5 \leftrightarrow 13\ , \quad 6 \leftrightarrow 12\ . \tag{C-1}$$

TABLES

Material parameter	Unit	Value on upper surface	Value on lower surface
A'_{11}	km^2/s^2	15.7100	35.3475
A'_{22}	km^2/s^2	15.7100	35.3475
A'_{33}	km^2/s^2	13.3900	30.1275
A'_{44}	km^2/s^2	4.9800	11.2050
A'_{55}	km^2/s^2	4.9800	11.2050
A'_{66}	km^2/s^2	5.3300	11.9925
A'_{23}	km^2/s^2	4.4600	10.0350
A'_{13}	km^2/s^2	4.4600	10.0350
A'_{12}	km^2/s^2	5.0500	11.3625
V_P	km/s	3.659	5.489
V_S	km/s	2.232	3.347
ϵ		0.087	0.087
δ		0.082	0.082
γ		0.035	0.035

TABLE 1. Upper part: Values of the elastic moduli used for ray tracing in the models HTI_{FIX} and HTI_{ROT}. The elastic moduli are written in Voigt notation and belong to LA coordinate systems at the upper and lower isosurfaces. Lower part: Values of Thomsen's parameters, see Appendix A, corresponding to the given elastic moduli.

Material parameter	Unit	Value on upper surface	Value on lower surface
A'_{11}	km^2/s^2	9.000	19.800
A'_{22}	km^2/s^2	9.840	21.650
A'_{33}	km^2/s^2	5.940	13.070
A'_{44}	km^2/s^2	2.000	4.400
A'_{55}	km^2/s^2	1.600	3.520
A'_{66}	km^2/s^2	2.180	4.800
A'_{23}	km^2/s^2	2.400	5.280
A'_{13}	km^2/s^2	2.250	4.950
A'_{12}	km^2/s^2	3.600	7.920
V_P	km/s	2.437	3.615
V_S	km/s	1.265	1.876
$\epsilon^{(2)}$		0.258	0.257
$\delta^{(2)}$		-0.078	-0.078
$\gamma^{(2)}$		0.181	0.182
$\epsilon^{(1)}$		0.328	0.328
$\delta^{(1)}$		0.082	0.082
$\gamma^{(1)}$		0.045	0.045
$\delta^{(3)}$		-0.107	-0.106

TABLE 2. Upper part: Values of the elastic moduli used for ray tracing in the model OR_{ROT}. The elastic moduli are written in Voigt notation and belong to LA coordinate systems at the upper and lower isosurfaces. Lower part: Values of Tsvankin's parameters, see Appendix B, corresponding to the given elastic moduli.

FIGURES

FIG. 1. The direction of a vector $\mathbf{u}$ is specified by the in-plane angle, λ, which is in the plane $x_2 = 0$ of the global coordinate system, and the out-of-plane angle, μ. The angle λ is indeterminate when the vector $\mathbf{u}$ is parallel to the x_2 axis.

FIG. 2. Results of ray tracing formulated in LA coordinates, using the model HTI_{FIX}: Traveltimes (top) and corresponding rays projected into the $x_1 - x_2$ plane (middle) and into the $x_1 - x_3$ plane (bottom) of the global coordinate system.

FIG. 3. Relative differences in traveltime (in %) for the ray tracing approaches formulated in LA coordinates and global coordinates, using the model HTI_{FIX}.

FIG. 4. Results of ray tracing formulated in LA coordinates, using the model HTI_{ROT}: Traveltimes (top) and corresponding rays projected into the $x_1 - x_2$ plane (middle) and into the $x_1 - x_3$ plane (bottom) of the global coordinate system.

FIG. 5. Relative differences in traveltime (in %) for the ray tracing approaches formulated in LA coordinates and global coordinates, using the model HTI_{ROT}.

FIG. 6. Results of ray tracing formulated in LA coordinates, using the model OR_{ROT}: Traveltimes (top) and corresponding rays projected into the $x_1 - x_2$ plane (middle) and into the $x_1 - x_3$ plane (bottom) of the global coordinate system.

FIG. 7. Relative differences in traveltime (in %) for the ray tracing approaches formulated in LA coordinates and global coordinates, using the model OR_{ROT}.

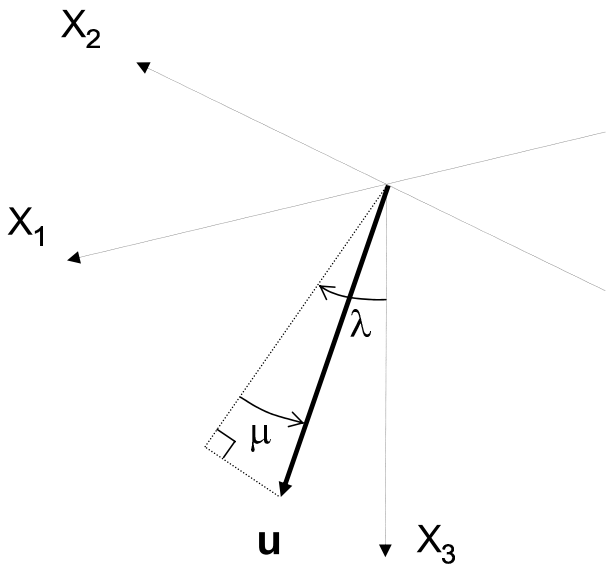

FIG. 1. The direction of a vector **u** is specified by the in-plane angle, λ, which is in the plane $x_2 = 0$ of the global coordinate system, and the out-of-plane angle, μ. The angle λ is indeterminate when the vector **u** is parallel to the x_2 axis.

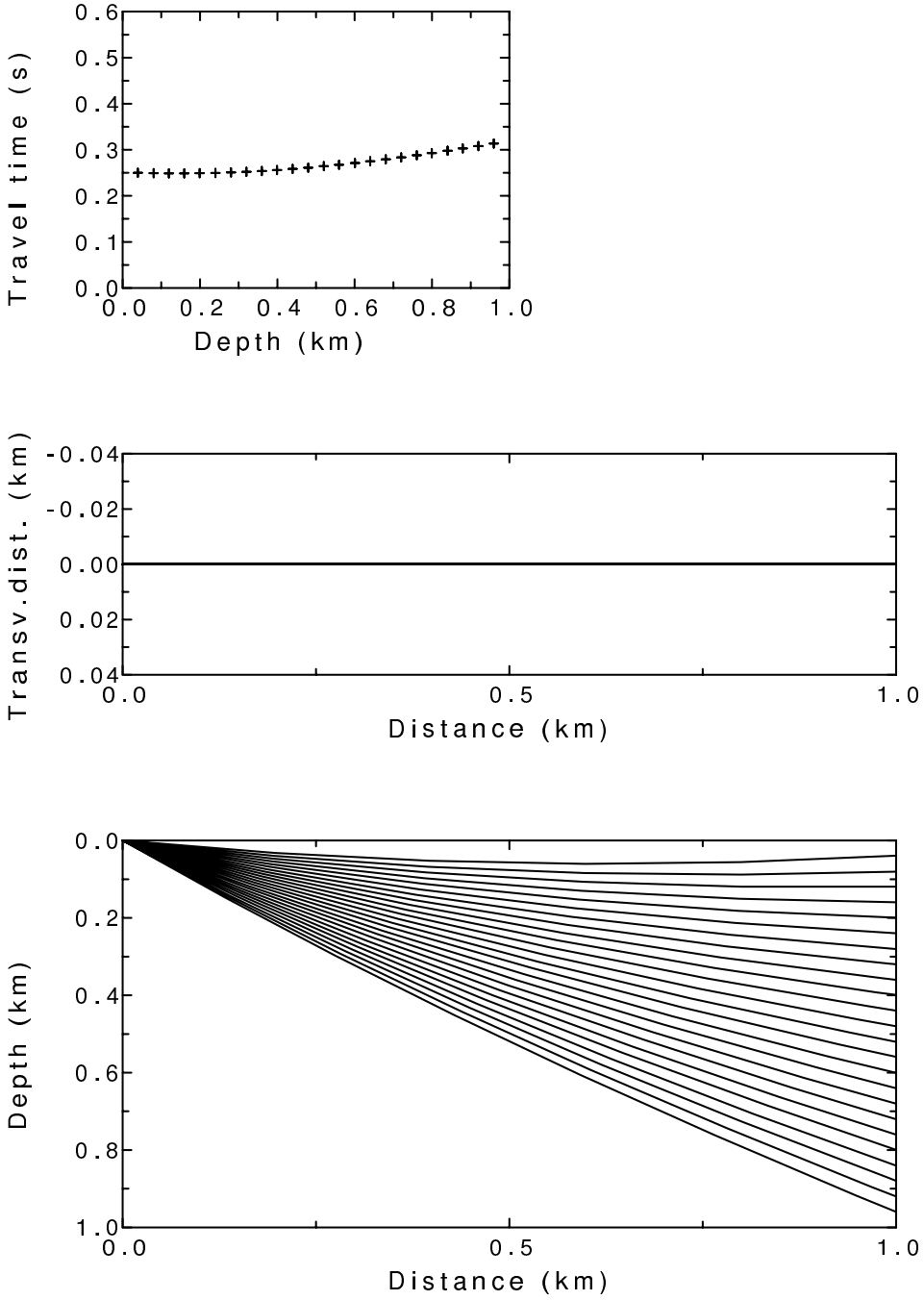

FIG. 2. Results of ray tracing formulated in LA coordinates, using the model HTI_{FIX}: Traveltimes (top) and corresponding rays projected into the $x_1 - x_2$ plane (middle) and into the $x_1 - x_3$ plane (bottom) of the global coordinate system.

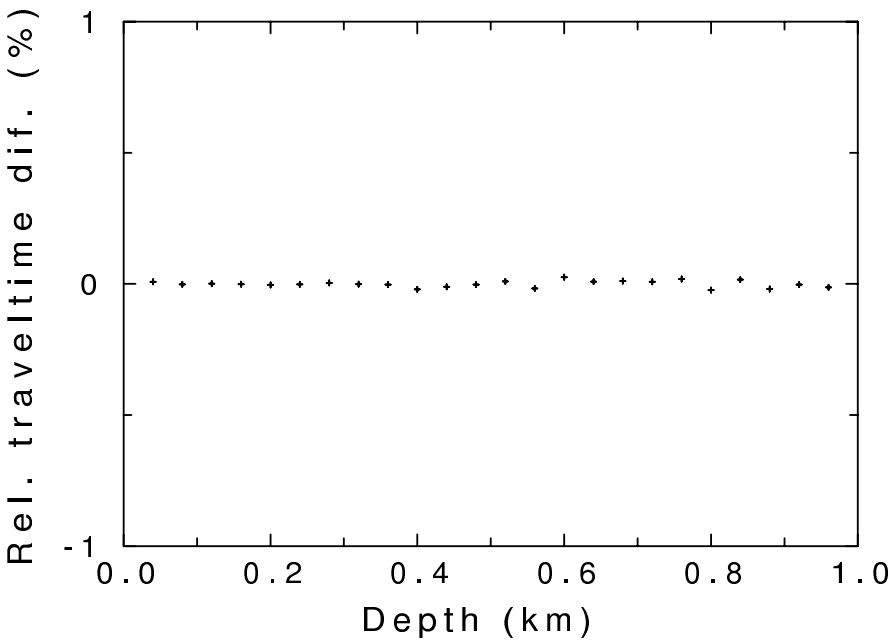

FIG. 3. Relative differences in traveltime (in %) for the ray tracing approaches formulated in LA coordinates and global coordinates, using the model HTI_{FIX}.

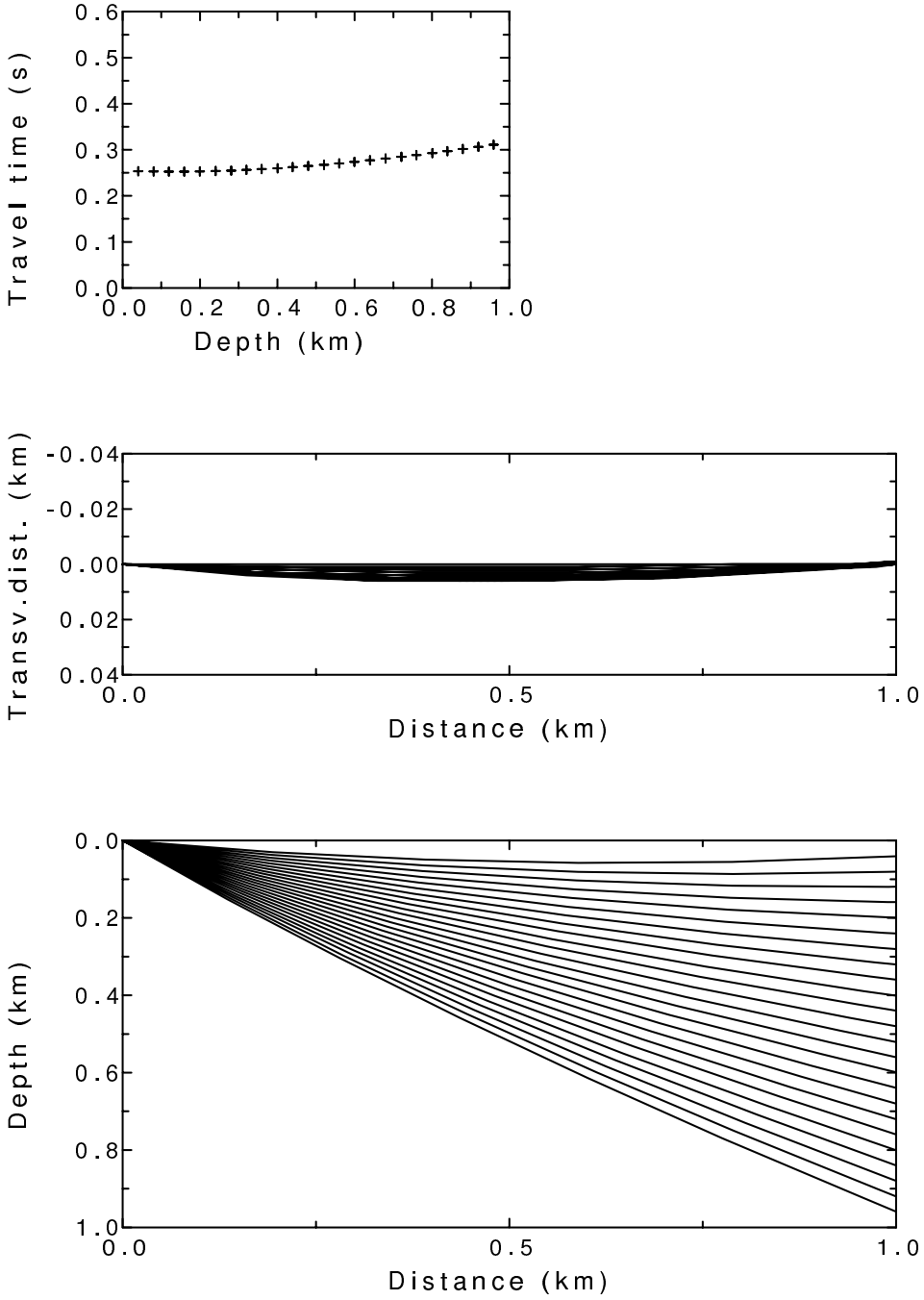

FIG. 4. Results of ray tracing formulated in LA coordinates, using the model HTI_{ROT}: Traveltimes (top) and corresponding rays projected into the $x_1 - x_2$ plane (middle) and into the $x_1 - x_3$ plane (bottom) of the global coordinate system.

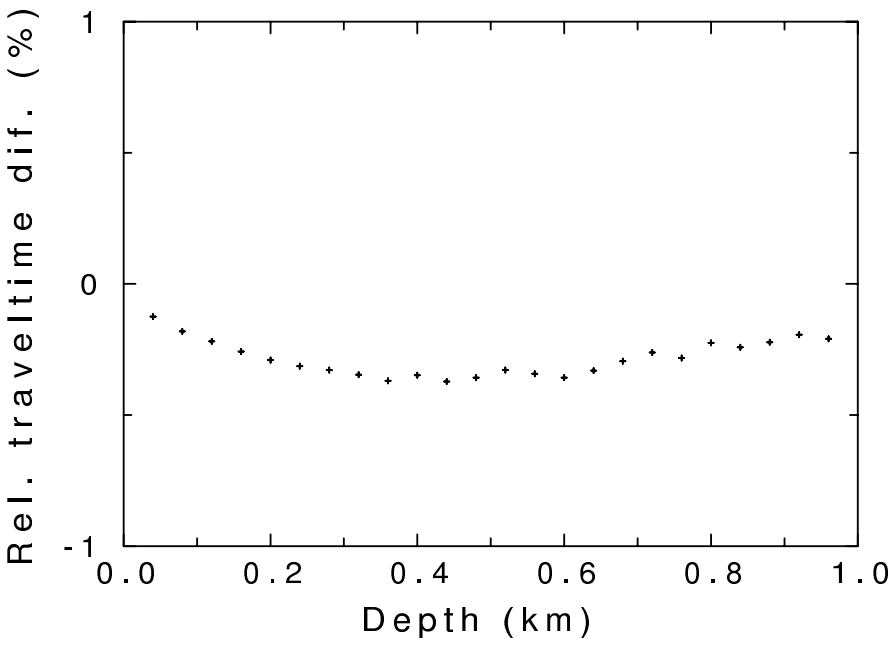

FIG. 5. Relative differences in traveltime (in %) for the ray tracing approaches formulated in LA coordinates and global coordinates, using the model HTI_{ROT}.

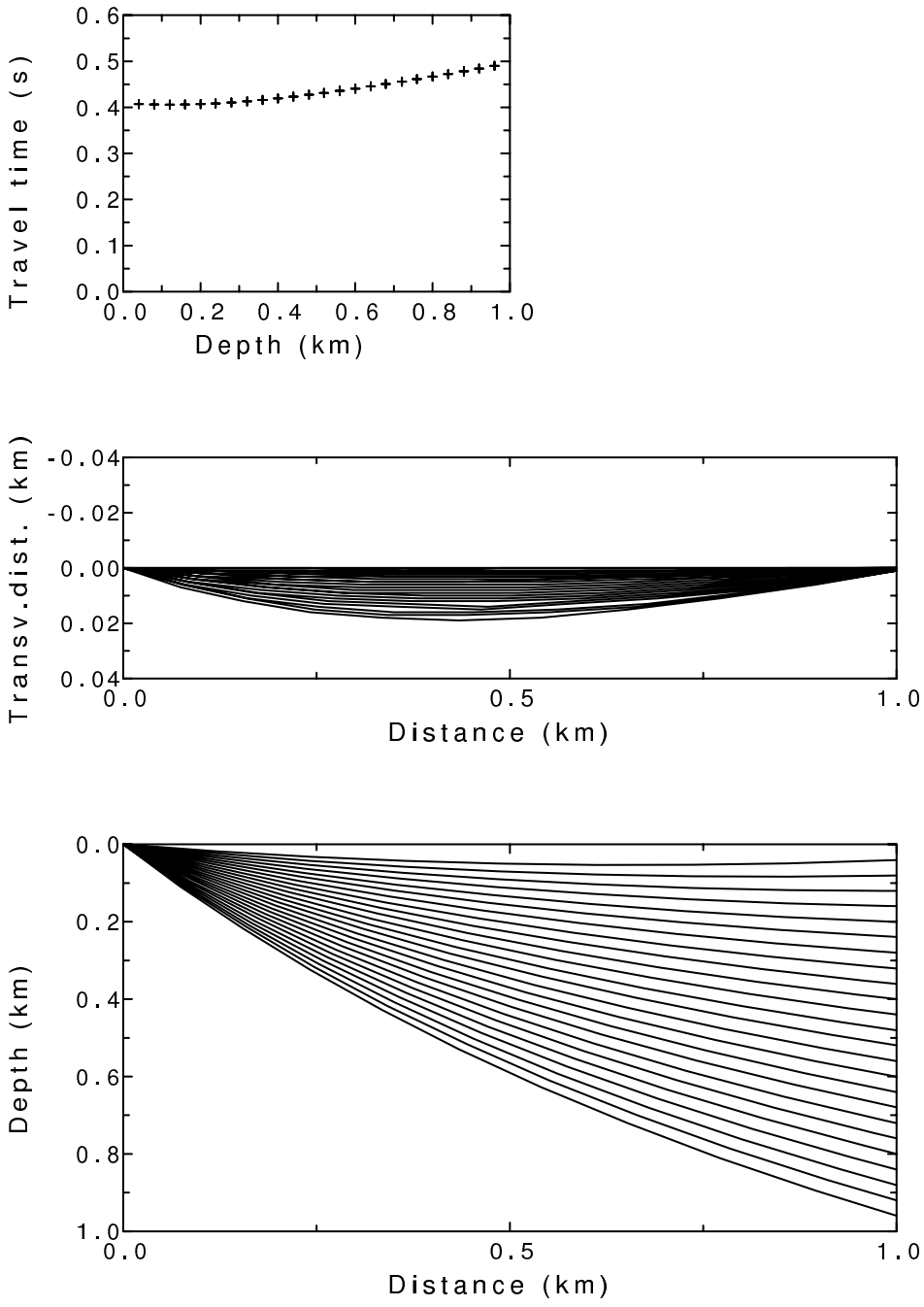

FIG. 6. Results of ray tracing formulated in LA coordinates, using the model OR_{ROT}: Traveltimes (top) and corresponding rays projected into the $x_1 - x_2$ plane (middle) and into the $x_1 - x_3$ plane (bottom) of the global coordinate system.

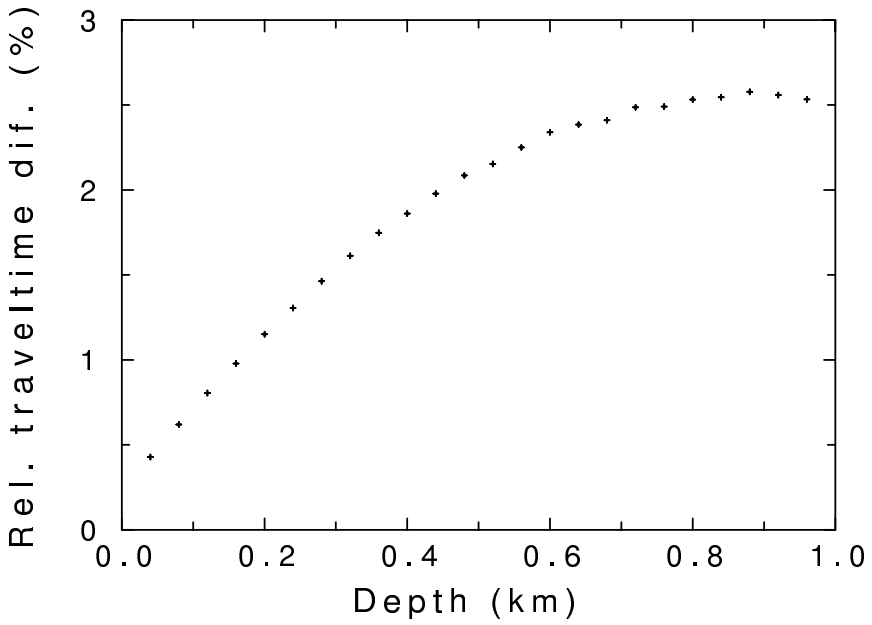

FIG. 7. Relative differences in traveltime (in %) for the ray tracing approaches formulated in LA coordinates and global coordinates, using the model OR_{ROT}.