Errors due to the anisotropic-common-ray approximation of the coupling ray theory

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Summary

The common–ray approximation considerably simplifies the numerical algorithm of the coupling ray theory for S waves, but may introduce errors in travel times due to the perturbation from the common reference ray. These travel–time errors can deteriorate the coupling–ray–theory solution at high frequencies. It is thus of principal importance for numerical applications to estimate the errors due to the common–ray approximation.

The anisotropic–common–ray approximation of the coupling ray theory is more accurate than the isotropic–common–ray approximation. We derive the equations for estimating the travel–time errors due to the anisotropic–common–ray (and also isotropic–common–ray) approximation of the coupling ray theory. The errors of the common–ray approximations are calculated along the *anisotropic common rays* in smooth velocity models without interfaces. The derivation is based on the general equations for the second–order perturbations of travel time.

The derived equations are numerically tested in three smooth 1–D velocity models of differing degree of anisotropy. The first–order and second–order perturbation expansions of travel time from the anisotropic common rays to anisotropic–ray–theory rays are compared with the anisotropic–ray–theory travel times. In the numerical example, the errors of the anisotropic–common–ray approximation are considerably smaller than the errors of the isotropic–common–ray approximation.

Keywords

Coupling ray theory, common–ray approximation, travel time, perturbation theory, seismic anisotropy, inhomogeneous media.

1. Introduction

In the isotropic ray theory, the S–wave polarization vectors do not rotate about the ray, whereas in the anisotropic ray theory they coincide with the eigenvectors of the Christoffel matrix which may rotate rapidly about the ray. In "weakly anisotropic" models, at moderate frequencies, the S–wave polarization tends to remain unrotated round the ray but is partly attracted by the rotation of the eigenvectors of the Christoffel matrix. The intensity of the attraction increases with frequency. This behaviour of the S–wave polarization is described by the *coupling ray theory* proposed by Coates & Chapman (1990). The coupling ray theory is applicable to S waves at all degrees of anisotropy, from isotropic models to considerably anisotropic ones. The numerical algorithm for calculating the frequency–dependent complex–valued S–wave polarization vectors of the coupling ray theory has been designed by Bulant & Klimeš (2002).

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There are many commonly used *quasi–isotropic approximations* of the coupling ray theory (Pšenčík, 1998a), which diminish the accuracy of the coupling ray theory both with increasing frequency and increasing degree of anisotropy. Most of these quasi–isotropic approximations can be avoided with minimal effort (Bulant & Klimeš, 2002; 2004a), except for the *common–ray approximation* for S waves. In the common–ray approximation, only one reference ray is traced for both anisotropic–ray–theory S waves, and both S–wave anisotropic–ray–theory travel times are approximated by the perturbation expansion from the common reference ray. The common–ray approximation thus considerably simplifies the coding of the coupling ray theory and numerical calculations, but may introduce errors in travel times due to the perturbation. These travel–time errors can deteriorate the coupling–ray–theory solution at high frequencies. It is thus of principal importance for numerical applications to estimate the travel–time errors due to the common–ray approximation, and then the related error of the wavefield.

The *anisotropic–common–ray approximation* (Bulant & Klimeš, 2004b) of the coupling ray theory is more accurate than the *isotropic–common–ray approximation*. The anisotropic common rays are traced in the anisotropic medium using the averaged Hamiltonian of both anisotropic–ray–theory S waves (Bakker, 2002; Klimeš, 2003).

In the common–ray approximation, the S–wave travel times are usually approximated by the first–order perturbation expansion from the common reference ray. The errors of S–wave travel times may then be approximated by second–order terms in the perturbation expansion. Klimeš & Bulant (2004) derived the equations for estimating the travel–time errors due to the isotropic–common–ray and anisotropic–common–ray approximations by numerical quadrature along the *isotropic common rays* in smooth velocity models without interfaces.

In this paper, we propose and numerically test the equations for estimating the travel–time errors due to the anisotropic–common–ray (and also isotropic–common–ray) approximation by numerical quadrature along the *anisotropic common rays* in smooth velocity models without interfaces. The derivation for estimating the errors along the anisotropic common rays is based on the general equations for the second–order perturbations of travel time by Klimeš (2002), and is analogous to the derivation by Klimeš & Bulant (2004) for estimating the errors along the isotropic common rays. The equations for estimating the travel–time errors due to the common–ray approximations of the coupling ray theory for S waves are derived in a form suitable for dynamic ray tracing in ray–centred coordinates attached to the anisotropic common rays.

Note that the perturbation expansion is the Taylor expansion with respect to the perturbation parameters, see (19), which parametrize the Hamiltonian, see (18). We refer here to the partial derivatives with respect to the perturbation parameters as perturbation derivatives, in order to distinguish them from the partial derivatives with respect to the spatial coordinates.

2. Common–ray approximations

For the derivation of the coupling ray theory refer to Coates & Chapman (1990) and Červený (2001). For the description of the numerical algorithm refer to Červený (2001) and Bulant & Klimeš (2002). For the algorithms of anisotropic common S–wave ray tracing and of corresponding dynamic ray tracing in a smooth anisotropic medium refer to Klimeš (2003). The algorithm of the coupling ray theory by Bulant & Klimeš (2002) is independent of the reference ray and was applied by Bulant & Klimeš (2004b) to the anisotropic common rays.

Here we shall concentrate only on estimating the travel–time errors due to the anisotropic–common–ray (and also isotropic–common–ray) approximation by numerical quadrature along the anisotropic common rays. The estimates are based on the equations for the second–order perturbation derivatives of travel time derived by Klimeš (2002).

2.1. Selection of the reference ray

The isotropic ray theory is always the limiting case of the coupling ray theory for decreasing anisotropy at a fixed frequency. On the other hand, the high–frequency limit of the coupling ray theory at a fixed anisotropy depends on the choice of the reference ray, and even on the choice of the *system* of reference rays, because the amplitudes are determined by the paraxial reference rays. The reference ray should be "close to the ray of the coupled S wave under study". Unfortunately, this is nothing but a rough statement, because the ray of the coupled S wave has not been defined yet.

From the point of view of the high–frequency asymptotic validity, the frequency–independent reference ray is best represented by the *anisotropic–ray–theory reference ray*, provided we choose the initial condition for the polarization vector in the coupling equation given by the eigenvector of the Christoffel matrix corresponding to the reference ray. The anisotropic–ray–theory travel time corresponding to the selected polarization is then exact, and only the difference between the two anisotropic–ray–theory S–wave travel times is approximate. The coupling ray theory may then also be used at high frequencies because the approximate travel–time difference influences only the coupling due to low–frequency scattering. The coupling ray theory then correctly converges to the anisotropic ray theory for high frequencies. For other choices of reference rays, the high–frequency limit of the coupling ray theory at a fixed anisotropy is incorrect, although the differences may be small at the finite frequencies under consideration. Note that the anisotropic–ray–theory reference ray can be traced only if the eigenvectors of the Christoffel matrix vary continuously along the whole ray (Vavryčuk, 2001).

In the *anisotropic–common–ray approximation*, the common reference ray is traced using the averaged Hamiltonian of both anisotropic–ray–theory S waves (Bakker, 2002; Klimeš, 2003).

In the less accurate *isotropic–common–ray approximation*, the reference ray is traced in the reference isotropic model. Moreover, the reference isotropic model may be selected in different ways, yielding quasi–isotropic approximations of differing accuracies.

2.2. Travel times in the coupling ray theory

Let $a_{ijkl} = a_{ijkl}(x_m)$ be the density–normalized elastic moduli describing a *smooth anisotropic model*, in which the *anisotropic common rays* are traced using the averaged Hamiltonian of both anisotropic–ray–theory S waves. For completeness, we also introduce S–wave velocity $v_0 = v_0(x_m)$ in the smooth reference isotropic model in which the isotropic common rays could be traced.

Assume a phase–space *reference ray*, parametrized by reference travel time τ, with reference slowness vectors $p_i(\tau)$ known at all its points $x_j(\tau)$. In this Section 2.2, the reference ray is general, including the anisotropic common ray and isotropic common ray as special cases. Using the reference slowness vectors, we can calculate reference Christoffel matrix

$$\Gamma_{jk}(\tau) = p_i(\tau)\, a_{ijkl}(x_m(\tau))\, p_l(\tau) \tag{1}$$

and its eigenvectors $g_{i\alpha}(\tau)$, $\alpha = 1,2,3$ along the reference ray. Whereas the Einstein summation over the pairs of identical Roman indices (both subscripts and superscripts) $i,j,k,... = 1,2,3$ or $I,J,K,... = 1,2$ is used throughout this paper, no implicit summation applies to Greek subscripts $\alpha,\beta,...$ indexing the eigenvectors of the Christoffel matrix or the perturbation parameters. Assume that eigenvectors $g_{i1}(\tau)$ and $g_{i2}(\tau)$ correspond to the S waves. For application of the coupling ray theory, the eigenvectors should vary continuously along the reference ray (Bulant & Klimeš, 2002). This condition is not required in regions where the two S–wave eigenvalues of the Christoffel matrix are approximately equal.

Let us denote by $\tau_\alpha(\tau)$ the anisotropic–ray–theory travel time corresponding to the selected eigenvector $g_{i\alpha}(\tau)$ of the Christoffel matrix. It may be approximated by a quadrature along the unperturbed reference ray,

$$\frac{\mathrm{d}\tau_\alpha}{\mathrm{d}\tau} = [\Gamma_{jk} g_{j\alpha} g_{k\alpha}]^{-\frac{1}{2}} \quad . \tag{2}$$

Travel–time approximation (2), suggested for the coupling ray theory by Bulant & Klimeš (2002), would become exact for a reference ray following the path of the respective anisotropic–ray-theory ray. Travel–time approximation (2) can be derived as the first–order part of perturbation expansion (19), corresponding to Hamiltonian

$$H_\alpha(x_m, p_n) = -[G_\alpha(x_m, p_n)]^{-\frac{1}{2}} \quad , \tag{3}$$

where $G_\alpha(x_m, p_n)$ is the eigenvalue of Christoffel matrix (1), corresponding to eigenvector $g_{i\alpha}$, see Section 2.5. Hamiltonian (3) is a homogeneous function of the minus first degree with respect to the slowness vector.

2.3. Reference Hamiltonian, anisotropic–ray–theory Hamiltonians, and isotropic–ray–theory Hamiltonian

We denote by $H_0(x_m, p_n)$ the *reference Hamiltonian* used to trace the reference ray.

For the anisotropic common ray, Klimeš (2003) assumed the reference Hamiltonian to be equal to the averaged Hamiltonian

$$\overline{H}_0(x_m, p_n) = \tfrac{1}{2}\left\{\tfrac{1}{2}\{[G_1(x_m, p_n)]^{\frac{N}{2}} + [G_2(x_m, p_n)]^{\frac{N}{2}}\}\right\}^{\frac{2}{N}} \tag{4}$$

of both S–wave polarizations (Klimeš, 2003, eqs. 28 and 36), where constant N specifies the way of averaging. Note that Bakker (2002) considered $N = 2$. Hamiltonian (4) is a homogeneous function of the second degree with respect to the slowness vector. The value of Hamiltonian (4) along the reference ray is $\overline{H}_0(x_m, p_n) = \frac{1}{2}$.

The first–order phase–space derivatives of Hamiltonian (4) are determined by equations (29) and (37) of Klimeš (2003). The second–order phase–space derivatives of Hamiltonian (4) are determined by equations (31) and (38) of Klimeš (2003). These equations for the first–order and second–order phase–space derivatives of Hamiltonian (4) have been coded in Fortran 77 (Bucha & Bulant, 2003, package MODEL, file `hder.for`, subroutine `HDER`).

Since the equations for the perturbation derivatives of travel time were derived by Klimeš (2002) on the condition that the Hamiltonian has equal value for all orthonomic systems of rays under consideration, reference Hamiltonian $H_0(x_m, p_n)$ must be a homogeneous function of the same degree with respect to the slowness vector as Hamiltonian (3). We thus select reference Hamiltonian

$$H_0(x_m, p_n) = -[2\overline{H}_0(x_m, p_n)]^{-\frac{1}{2}} \quad , \tag{5}$$

which is a homogeneous function of the minus first degree with respect to the slowness vector, and which yields equal orthonomic systems of rays as Hamiltonian (4). Along the reference ray, the value of Hamiltonian (5) is

$$H_0(x_m, p_n) = -1 \quad , \tag{6}$$

the values of the first–order partial derivatives with respect to the spatial coordinates (denoted by Roman subscripts following a comma) of Hamiltonian (5) are

$$\boxed{H_{0,i} = \overline{H}_{0,i} \quad ,} \tag{7}$$

the values of the first–order partial derivatives with respect to the components of the slowness vector (denoted by Roman superscripts following a comma) of Hamiltonian (5) are

$$\boxed{H_0^{,i} = \overline{H}_0^{,i} \quad ,} \tag{8}$$

the values of the second–order partial derivatives of Hamiltonian (5) with respect to the spatial coordinates are

$$\boxed{H_{0,ij} = \overline{H}_{0,ij} - 3\overline{H}_{0,i}\overline{H}_{0,j} \quad ,} \tag{9}$$

and the values of the second–order partial derivatives of Hamiltonian (5) with respect to the components of the slowness vector are

$$\boxed{H_0^{,ij} = \overline{H}_0^{,ij} - 3\overline{H}_0^{,i}\overline{H}_0^{,j} \quad .} \tag{10}$$

The value of Hamiltonian (3), corresponding to unit S–wave eigenvector $g_{i\alpha}$, $\alpha = 1, 2$ of the Christoffel matrix, at the reference ray, is

$$\boxed{H_\alpha = -(a_{ijkl}g_{i\alpha}p_j g_{k\alpha}p_l)^{-\frac{1}{2}} \quad .} \tag{11}$$

The partial derivatives of Hamiltonian (11) with respect to the spatial coordinates are

$$\boxed{H_{\alpha,i} = -\tfrac{1}{2}(H_\alpha)^3 a_{mjkl,i}g_{m\alpha}p_j g_{k\alpha}p_l \quad .} \tag{12}$$

The partial derivatives of Hamiltonian (11) with respect to the components of the slowness vector are

$$\boxed{H_\alpha^{;i} = -(H_\alpha)^3 a_{mikl} g_{m\alpha} g_{k\alpha} p_l \quad . \qquad (13)}$$

The values of phase–space derivatives (12) and (13) are calculated at the reference ray in this paper. Partial derivatives (13) satisfy Euler's relation

$$H_\alpha^{;i} p_i = -H_\alpha \tag{14}$$

for a homogeneous function of degree -1 with respect to p_i.

The quantities enclosed above in boxes are necessary for estimating the errors due to the anisotropic–common–ray approximation of the coupling ray theory.

To be able to also calculate the travel–time perturbations from the anisotropic common ray towards the isotropic–ray–theory ray, we also introduce homogeneous Hamiltonian

$$H_3(x_m, p_n) = -[v_0(x_m)]^{-1}(p_i p_i)^{-\frac{1}{2}} \tag{15}$$

for the reference isotropic medium. Hamiltonian (15) is a homogeneous function of the minus first degree with respect to the slowness vector. The partial derivatives of Hamiltonian (15) with respect to spatial coordinates x_i are

$$H_{3,i} = v_{0,i}\,(v_0)^{-2}(p_k p_k)^{-\frac{1}{2}} \quad . \tag{16}$$

The partial derivatives of Hamiltonian (15) with respect to the components of the slowness vector are

$$H_3^{;i} = (v_0)^{-1}(p_k p_k)^{-\frac{3}{2}} p_i \quad . \tag{17}$$

2.4. Parametric system of the Hamiltonians and errors due to the common–ray approximations

We consider the parametric set $H = H(x_m, p_n, f_\alpha)$ of Hamiltonians parametrized by f_α,

$$H(x_m, p_n, f_\alpha) = H_0(x_m, p_n) + \sum_\alpha [H_\alpha(x_m, p_n) - H_0(x_m, p_n)]\, f_\alpha \quad . \tag{18}$$

where $\alpha = 1, 2$ for perturbations towards the anisotropic–ray–theory rays. If we are also interested in the perturbations towards a reference isotropic model, $\alpha = 1, 2, 3$. Parameters f_α are called *perturbation parameters* (Baumgärtel, 1985) or model parameters (Tarantola, 1987; Klimeš, 2002). Parametric system (18) of the Hamiltonians then generates parametric system $\tau(x_m, f_\alpha)$ of the travel–time fields corresponding to the individual Hamiltonians $H(x_m, p_n, f_\alpha)$.

For an S wave, we obtain the reference Hamiltonian for $(f_1, f_2, f_3) = (0, 0, 0)$, the Hamiltonian corresponding to the first anisotropic–ray–theory S wave for $(f_1, f_2, f_3) = (1, 0, 0)$, the Hamiltonian corresponding to the second anisotropic–ray–theory S wave for $(f_1, f_2, f_3) = (0, 1, 0)$, and the Hamiltonian corresponding to the isotropic–ray–theory ray traced in the reference isotropic model for $(f_1, f_2, f_3) = (0, 0, 1)$.

The Taylor expansion with respect to perturbation parameters f_α is called *perturbation expansion.* The second–order perturbation expansion of travel time is

$$\tau(x_m, f_\alpha) \approx \tau(x_m) + \sum_\alpha \tau_{,\alpha}(x_m) f_\alpha + \tfrac{1}{2} \sum_\alpha \sum_\beta \tau_{,\alpha\beta}(x_m) f_\alpha f_\beta \quad , \tag{19}$$

where the Greek subscripts following a comma denote the partial derivatives with respect to perturbation parameters f_α, called hereinafter the *perturbation derivatives.* Note that Klimeš (2002) refers to the perturbation derivatives briefly as the "perturbations". Here, travel time $\tau(x_m)$ and its perturbation derivatives $\tau_{,\alpha}(x_m)$, $\tau_{,\alpha\beta}(x_m)$, with arguments f_α omitted, correspond to the reference system of rays $(f_1, f_2, f_3) = (0, 0, 0)$.

Travel–time approximation (2) corresponds to the first–order part of perturbation expansion (19), see Section 2.5. The error of travel–time approximation (2) may thus be approximated by the quadratic term in perturbation expansion (19) with Hamiltonians (3).

We may now express the estimates of the travel–time errors due to the *anisotropic–common–ray* (ACR) *approximation* (Bakker, 2002; Klimeš, 2003) of anisotropic travel times in terms of the second–order perturbation derivatives of travel time. The errors of travel–time approximation (2) from anisotropic common ray $(f_1, f_2, f_3) = (0, 0, 0)$ to anisotropic–ray–theory ray $(f_1, f_2, f_3) = (1, 0, 0)$ and to anisotropic–ray–theory ray $(f_1, f_2, f_3) = (0, 1, 0)$ are approximately

$$\boxed{\delta\tau_\alpha^{\mathrm{ACR}} = \frac{\tau_{,\alpha\alpha}}{2} \quad ,} \tag{20}$$

where $\alpha = 1, 2$. Equation (20) with $\alpha = 3$ analogously yields the second–order term in the perturbation expansion from the anisotropic common ray to the isotropic–ray–theory ray traced in the reference isotropic model.

We may also use the second–order perturbation derivatives of travel time, calculated along the anisotropic common ray, to estimate the accuracy of the *isotropic–common–ray* (ICR) *approximation.* The estimates of errors of travel–time approximation (2) from isotropic common ray $(f_1, f_2, f_3) = (0, 0, 1)$ to anisotropic–ray–theory ray $(f_1, f_2, f_3) = (1, 0, 0)$ and to anisotropic–ray–theory ray $(f_1, f_2, f_3) = (0, 1, 0)$ are

$$\delta\tau_\alpha^{\mathrm{ICR}} = \tau_{,\alpha\alpha} - 2\tau_{,\alpha 3} + \tau_{,33} \quad , \tag{21}$$

where $\alpha = 1, 2$. Error estimates (21) represent the contributions of the quadratic terms in perturbation expansion (19) from the isotropic common ray to the anisotropic–ray–theory S–wave rays.

To calculate the first–order and second–order perturbation derivatives $\tau_{,\alpha}$ and $\tau_{,\alpha\beta}$ of travel time, we need the first–order and second–order phase–space and perturbation derivatives of the Hamiltonian. The phase–space and perturbation derivatives of Hamiltonian (18) in the reference model $f_\alpha = 0$ are

$$H_{,i} = H_{0,i} \quad , \tag{22}$$

$$H^{,i} = H_0^{;i} \quad , \tag{23}$$

$$H_{,ij} = H_{0,ij} \quad , \tag{24}$$

$$H^{,ij} = H_0^{;ij} \quad , \tag{25}$$

$$H_{,\alpha} = H_\alpha - H_0 \quad , \tag{26}$$

$$H_{,i\alpha} = H_{\alpha,i} - H_{0,i} \quad , \tag{27}$$

$$H^{;i}_{,\alpha} = H^{;i}_\alpha - H^{;i}_0 \quad , \tag{28}$$

$$H_{,\alpha\beta} = 0 \quad , \tag{29}$$

where the Greek subscripts following a comma denote the partial derivatives with respect to perturbation parameters f_α, analogously as for travel time.

2.5. First–order perturbation expansion of travel time

The first–order perturbation derivatives of travel time can be determined by numerical quadrature (Klimeš, 2002, eqs. 19 and 25)

$$\tau_{,\alpha}(\tau) = \tau_{,\alpha}(\tau^0) - \int_{\tau^0}^{\tau} \mathrm{d}\tau \; H_{,\alpha} \tag{30}$$

along the reference ray, with zero initial conditions $\tau_{,\alpha}(\tau^0) = 0$ at the point source.

Inserting (26) for $H_{,\alpha}$ into equation (30) and considering (6), we obtain

$$\tau(\tau) + \tau_{,\alpha}(\tau) = \tau(\tau^0) + \tau_{,\alpha}(\tau^0) - \int_{\tau^0}^{\tau} \mathrm{d}\tau \; H_{\alpha} \quad . \tag{31}$$

This is the first–order part of perturbation expansion (19) of travel time from anisotropic common ray $(f_1, f_2, f_3) = (0, 0, 0)$ to anisotropic–ray–theory rays $(f_1, f_2, f_3) = (1, 0, 0)$ and $(f_1, f_2, f_3) = (0, 1, 0)$, or to isotropic common ray $(f_1, f_2, f_3) = (0, 0, 1)$.

Equation (31) with (11) for $\alpha = 1, 2$ yields travel–time approximation (2) used in the coupling ray theory. Equation (31) with (15) for $\alpha = 3$ represents the first–order perturbation expansion from the anisotropic common ray to the isotropic common ray.

2.6. Dynamic ray tracing and ray–centred coordinates

We define the matrices of the partial derivatives of spatial coordinates x_i and of the slowness–vector components $p_i = \tau_{,i}$ with respect to 3 *ray coordinates*: ray take–off parameters γ_1, γ_2, and independent parameter $\gamma_3 = \tau$ along rays,

$$Q_{ia} = \frac{\partial x_i}{\partial \gamma_a} \quad , \qquad P_{ia} = \frac{\partial \tau_{,i}}{\partial \gamma_a} \quad . \tag{32}$$

Matrices Q_{ia} and P_{ia} can be calculated by numerically solving a set of linear ordinary differential equations called the *dynamic ray tracing equations* (Babich & Buldyrev, 1972; Červený, 1972; Arnaud, 1973; Červený & Hron, 1980). Matrices Q_{ia} and P_{ia} describe, by definition, the properties of the orthonomic system of rays corresponding to the travel time under consideration. They may be expressed in terms of their initial values and the paraxial ray propagator matrix (Červený, 2001).

The second–order spatial derivatives $\tau_{,ij}$ of travel time can be expressed in terms of matrices Q_{ia} and P_{ia}. Equation

$$P_{ia} = \tau_{,ij} \; Q_{ja} \tag{33}$$

is a direct consequence of the above definitions (32).

Let us denote by h^i_1, h^i_2, h^i_3 the contravariant basis vectors, and by $\hat{h}_{i1}, \hat{h}_{i2}, \hat{h}_{i3}$ the covariant basis vectors of the *ray–centred coordinate system*. The basis vectors of the ray–centred coordinate system are calculated along the reference ray. The first two contravariant basis vectors h^i_M of the ray–centred coordinate system are tangent to the wavefront, the third contravariant basis vector $h^i_3 = H_0^{;i}$ is tangent to the ray. The first two covariant basis vectors $\hat{h}_{iM}$ of the ray–centred coordinate system are perpendicular to the ray, the third covariant basis vector $\hat{h}_{i3} = p_i$ is perpendicular to the wavefront. The contravariant basis vectors and covariant basis vectors are mutually biorthonormal,

$$h^i_a \hat{h}_{ib} = \delta_{ab} \quad . \tag{34}$$

The equations for calculating the basis vectors of the ray–centred coordinate system can be found, e.g., in Klimeš (2003).

The quantities covariantly transformed from spatial coordinates x_i to the local Cartesian basis of the ray–centred coordinate system are marked by a tilde $\tilde{}$. In particular,

$$\widetilde{Q}_{ia} = \hat{h}_{mi} Q_{ma} \quad , \tag{35}$$

$$\widetilde{P}_{ia} = h_i^m P_{ma} \quad , \tag{36}$$

$$\widetilde{\tau}_{,ij} = h_i^m \tau_{,mn} h_j^n \quad , \tag{37}$$

$$\widetilde{\tau}_{,i\alpha} = h_i^m \tau_{,m\alpha} \quad , \tag{38}$$

$$\boxed{\begin{aligned} \widetilde{H}_{0,i} &= H_{0,m} h_i^m \quad , & (39) \\ \widetilde{H}_{0,ij} &= H_{0,mn} h_i^m h_j^n \quad , & (40) \\ \widetilde{H}_0^{;ij} &= H_0^{;mn} \hat{h}_{mi} \hat{h}_{nj} \quad , & (41) \\ \widetilde{H}_{\alpha,i} &= H_{\alpha,m} h_i^m \quad , & (42) \\ \widetilde{H}_\alpha^{;i} &= H_\alpha^{;m} \hat{h}_{mi} \quad . & (43) \end{aligned}}$$

Since we consider matrices Q_{ia} and P_{ia} defined for $\gamma_3 = \tau$,

$$\widetilde{Q}_{3J} = 0 \quad . \tag{44}$$

The 2×2 submatrices $\widetilde{Q}_{IJ}$ and $\widetilde{P}_{IJ}$ of matrices (35) and (36) are usually calculated by the *dynamic ray tracing equations in ray–centred coordinates* (Klimeš, 1994, eq. 60)

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\widetilde{Q}_{IJ} = \widetilde{\widetilde{H}}{}_{0,K}^{;I}\,\widetilde{Q}_{KJ} + \widetilde{H}_0^{;IK}\,\widetilde{P}_{KJ} \quad , \tag{45}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\widetilde{P}_{IJ} = -\widetilde{\widetilde{H}}_{0,IK}\,\widetilde{Q}_{KJ} - \widetilde{\widetilde{H}}{}_{0,I}^{;K}\,\widetilde{P}_{KJ} \quad . \tag{46}$$

For the second–order phase–space derivatives

$$\widetilde{\widetilde{H}}_{0,IK} = \widetilde{H}_{0,IK} + 2\widetilde{H}_{0,I}\widetilde{H}_{0,K} \quad , \tag{47}$$

$\widetilde{\widetilde{H}}{}_{0,K}^{;I}$ and $\widetilde{\widetilde{H}}{}_0^{;IK} = \widetilde{H}_0^{;IK}$ of the reference Hamiltonian in ray–centred coordinates refer to Klimeš (1994, eqs. 51a, 51b, 51c) or to Klimeš (2003, eqs. 62–64).

We shall thus express the equations for estimating the travel–time errors due to the common–ray approximations of the coupling ray theory in terms of the 2×2 matrices $\widetilde{Q}_{IJ}$ and $\widetilde{P}_{IJ}$ in ray–centred coordinates.

2.7. First–order perturbation derivatives of the travel–time gradient

To calculate the second–order perturbation derivatives of travel time, we need to know at least the ray–normal projection

$$\tau_{,i\alpha}^{\perp} = E_i^{\perp j}\tau_{,j\alpha} \tag{48}$$

of the first–order perturbation derivatives $\tau_{,i\alpha}$ of the travel–time gradient (Klimeš, 2002, eqs. 54 and 55). The ray–normal projection matrix $E_i^{\perp j}$ can be expressed in terms of the first two basis vectors of the ray–centred coordinate system,

$$E_i^{\perp j} = \hat{h}_{iM} h_M^j \quad . \tag{49}$$

The covariant transform (Klimeš, 2002, eq. 69)

$$T^{\perp}_{J\alpha} = \tau^{\perp}_{,i\alpha}\, Q_{iJ} \tag{50}$$

of ray–normal derivatives (48) into ray coordinates can be determined by numerical quadrature (Klimeš, 2002, eq. 75)

$$\boxed{T^{\perp}_{J\alpha}(\tau) = T^{\perp}_{J\alpha}(\tau^0) + \int_{\tau^0}^{\tau} \mathrm{d}\tau\, K^{\perp}_{i\alpha} Q_{iJ}} \tag{51}$$

along the reference ray, with zero initial conditions $T^{\perp}_{J\alpha}(\tau^0) = 0$ at the point source.

The ray–normal integration kernel for the perturbation derivatives of the travel–time gradient is (Klimeš, 2002, eq. 80)

$$K^{\perp}_{i\alpha} = E^{\perp a}_{i}\,(-H_{,a\alpha} + H_{,a}\, H^{;r}_{,\alpha}\, p_r) - H^{;r}_{,\alpha}\, \tau^{\perp}_{,ri} \quad , \tag{52}$$

where

$$\tau^{\perp}_{,ij} = E^{\perp a}_{i} E^{\perp b}_{j} \tau_{,ab} \tag{53}$$

is the ray–normal projection of the second–order spatial derivatives $\tau_{,ij}$ of travel time.

Inserting equations (22), (27), (28) with (14) and (34), and (53) into (52), we arrive at

$$K^{\perp}_{i\alpha} = -E^{\perp a}_{i}\,(H_{\alpha,a} + H_{\alpha} H_{0,a}) - H^{;r}_{\alpha} E^{\perp a}_{r} \tau_{,ab} E^{\perp b}_{i} \quad . \tag{54}$$

Multiplying equation (54) by the first two columns Q_{iJ} of the matrix of geometrical spreading, inserting equation (49), and considering transforms (35), (36), (37) and relations (33), (34) and (44), we arrive at

$$\boxed{K^{\perp}_{i\alpha} Q_{iJ} = -(\widetilde{H}_{\alpha,I} + H_{\alpha} \widetilde{H}_{0,I})\, \widetilde{Q}_{IJ} - \widetilde{H}^{;I}_{\alpha} \widetilde{P}_{IJ} \quad ,} \tag{55}$$

where $\widetilde{Q}_{IJ}$ and $\widetilde{P}_{IJ}$ are the 2×2 submatrices of matrices (35) and (36). The 2×2 matrices $\widetilde{Q}_{IJ}$ and $\widetilde{P}_{IJ}$ are usually calculated by the dynamic ray tracing equations (45) and (46) in ray–centred coordinates.

Inserting equations (48) and (49) into (50) while considering transforms (35) and (38), we arrive at

$$T^{\perp}_{J\alpha} = \widetilde{\tau}_{,I\alpha}\, \widetilde{Q}_{IJ} \quad . \tag{56}$$

The first–order perturbation derivatives of the travel–time gradient expressed in ray–centred coordinates are then

$$\boxed{\widetilde{\tau}_{,I\alpha} = T^{\perp}_{J\alpha} \widetilde{Q}^{-1}_{JI} \quad ,} \tag{57}$$

where $\widetilde{Q}^{-1}_{JI}$ are the components of the matrix inverse to 2×2 matrix $\widetilde{Q}_{IJ}$ calculated by dynamic ray tracing (45) and (46) in ray–centred coordinates.

The value of $\widetilde{\tau}_{,I\alpha}$ at the point source, where $T^{\perp}_{J\alpha}(\tau^0) = 0$ and $\widetilde{Q}_{IJ}(\tau^0) = 0$, can be obtained from equation (57) by the l'Hospital rule with respect to $\tau \to \tau^0+$. Equation (45) and derivative (55) of equation (51) with $\widetilde{Q}_{IJ} = 0$ then yield

$$\boxed{\widetilde{\tau}_{,I\alpha}(\tau^0) = -\widetilde{G}_{IJ} \widetilde{H}^{;J}_{\alpha} \quad ,} \tag{58}$$

where $\widetilde{G}_{IJ}$ is the inverse matrix to $\widetilde{G}^{IJ} = \widetilde{H}^{;IJ}_{0}$.

2.8. Second–order perturbation derivatives of travel time

The second–order perturbation derivatives of travel time can be determined by numerical quadrature (Klimeš, 2002, eqs. 19 and 20)

$$\boxed{\tau_{,\alpha\beta}(\tau) = \tau_{,\alpha\beta}(\tau^0) + \int_{\tau^0}^{\tau} \mathrm{d}\tau \; K_{\alpha\beta}} \tag{59}$$

along the reference ray, with zero initial conditions $\tau_{,\alpha\beta}(\tau^0) = 0$ at the point source.

For Hamiltonian (3), the integration kernel for the second–order perturbation derivatives of travel time can be expressed as (Klimeš, 2002, eq. 64)

$$K_{\alpha\beta} = -H_{,\alpha\beta} - H^{;i}_{,\alpha}\tau^{\perp}_{,i\beta} - H^{;i}_{,\beta}\tau^{\perp}_{,i\alpha} - H^{;ij}\tau^{\perp}_{,i\alpha}\tau^{\perp}_{,j\beta} \quad . \tag{60}$$

Inserting relations (25), (28) and (29), and considering biorthonormality (34), equation (60) becomes

$$K_{\alpha\beta} = -H^{;i}_{\alpha}\tau^{\perp}_{,i\beta} - H^{;i}_{\beta}\tau^{\perp}_{,i\alpha} - H^{;ij}_{0}\tau^{\perp}_{,i\alpha}\tau^{\perp}_{,j\beta} \quad . \tag{61}$$

Inserting relations (48) and (49), and considering transforms (38), (41) and (43), equation (61) can be expressed in ray–centred coordinates as

$$\boxed{K_{\alpha\beta} = -\widetilde{H}^{;I}_{\alpha}\widetilde{\tau}_{,I\beta} - \widetilde{H}^{;I}_{\beta}\widetilde{\tau}_{,I\alpha} - \widetilde{H}^{;IJ}_{0}\widetilde{\tau}_{,I\alpha}\widetilde{\tau}_{,J\beta} \quad .} \tag{62}$$

Equation (59) cannot be integrated numerically through caustics, where $\widetilde{Q}_{IJ}$ is singular and $\widetilde{\tau}_{,I\alpha}$ in integration kernel (62) thus approach infinity. Fortunately, we may convert equation (59) analogously to Klimeš (2002, sec. 4.2), but in ray–centred coordinates. Equation (59) with integration kernel (62) may be converted using dynamic ray tracing equations (45) and (46) and equation (51) with (55) into

$$\boxed{\tau_{,\alpha\beta}(\tau) = \tau_{,\alpha\beta}(\tau^0) + \left[T^{\perp}_{I\alpha}\widetilde{Q}^{-1}_{IK}\widetilde{P}^{-1}_{JK}T^{\perp}_{J\beta}\right]^{\tau}_{\tau^0} + \int_{\tau^0}^{\tau} \mathrm{d}\tau \; K^{\mathrm{caust}}_{\alpha\beta}} \tag{63}$$

with the modified integration kernel

$$\begin{aligned} K^{\mathrm{caust}}_{\alpha\beta} =& (\widetilde{H}_{\alpha,I} + H_{\alpha}\widetilde{H}_{0,I})\,T^{\perp}_{J\beta}\widetilde{P}^{-1}_{JI} \\ &+ (\widetilde{H}_{\beta,I} + H_{\beta}\widetilde{H}_{0,I})\,T^{\perp}_{J\alpha}\widetilde{P}^{-1}_{JI} - \widetilde{\widetilde{H}}_{0,KL}T^{\perp}_{I\alpha}\widetilde{P}^{-1}_{IK}T^{\perp}_{J\beta}\widetilde{P}^{-1}_{JL} \quad . \end{aligned} \tag{64}$$

Inserting (47) into equation (64), the modified integration kernel reads

$$\boxed{\begin{aligned} K^{\mathrm{caust}}_{\alpha\beta} =& (\widetilde{H}_{\alpha,I} + H_{\alpha}\widetilde{H}_{0,I})\,T^{\perp}_{J\beta}\widetilde{P}^{-1}_{JI} \\ &+ (\widetilde{H}_{\beta,I} + H_{\beta}\widetilde{H}_{0,I})\,T^{\perp}_{J\alpha}\widetilde{P}^{-1}_{JI} - (\widetilde{H}_{0,KL} + 2\widetilde{H}_{0,I}\widetilde{H}_{0,K})T^{\perp}_{I\alpha}\widetilde{P}^{-1}_{IK}T^{\perp}_{J\beta}\widetilde{P}^{-1}_{JL} \; . \end{aligned}} \tag{65}$$

Combining equations (59) and (63) enables the optimization of calculating the second–order perturbation derivatives of travel time by numerical quadrature. The second–order perturbation derivatives of travel time are then accurately numerically calculated through caustics, although they are infinite at the caustics.

The second–order perturbation derivatives of travel time, calculated by quadratures (59) and (63), can be inserted into equation (20) for the travel–time errors due to the anisotropic–common–ray approximation, and into equation (21) for the travel–time errors due to the isotropic–common–ray approximation.

3. Numerical example

The calculation of the second–order travel–time perturbation expansion from an anisotropic common ray to anisotropic–ray–theory rays in smooth velocity models without interfaces have been coded and added to the Fortran 77 package CRT (Bucha & Bulant, 2005).

3.1. Model QI

A vertically heterogeneous 1–D anisotropic model QI was provided by Pšenčík & Dellinger (2001, model WA rotated by 45°) who performed the coupling–ray–theory calculations using the programs of package ANRAY (Pšenčík, 1998b) and compared the results with the reflectivity method. The density–normalized elastic moduli a_{ijkl} in $\mathrm{km}^2\mathrm{s}^{-2}$ at the surface (zero depth) are

$$
\begin{array}{c}
\\ 11 \\ 22 \\ 33 \\ 23 \\ 13 \\ 12
\end{array}
\begin{array}{c}
\begin{array}{cccccc} \quad 11 \quad & \quad 22 \quad & \quad 33 \quad & \quad 23 \quad & \quad 13 \quad & \quad 12 \quad \end{array} \\
\begin{pmatrix}
14.48500 & 4.52500 & 4.75000 & 0.00000 & 0.00000 & -0.58000 \\
 & 14.48500 & 4.75000 & 0.00000 & 0.00000 & -0.58000 \\
 & & 15.71000 & 0.00000 & 0.00000 & -0.29000 \\
 & & & 5.15500 & -0.17500 & 0.00000 \\
 & & & & 5.15500 & 0.00000 \\
 & & & & & 5.04500
\end{pmatrix}
\end{array}
\quad , \tag{66}
$$

and at the depth of 1 km they are

$$
\begin{array}{c}
\\ 11 \\ 22 \\ 33 \\ 23 \\ 13 \\ 12
\end{array}
\begin{array}{c}
\begin{array}{cccccc} \quad 11 \quad & \quad 22 \quad & \quad 33 \quad & \quad 23 \quad & \quad 13 \quad & \quad 12 \quad \end{array} \\
\begin{pmatrix}
22.08963 & 6.90063 & 7.24375 & 0.00000 & 0.00000 & -0.88450 \\
 & 22.08963 & 7.24375 & 0.00000 & 0.00000 & -0.88450 \\
 & & 23.95775 & 0.00000 & 0.00000 & -0.44225 \\
 & & & 7.86138 & -0.26688 & 0.00000 \\
 & & & & 7.86138 & 0.00000 \\
 & & & & & 7.69363
\end{pmatrix}
\end{array}
\quad . \tag{67}
$$

Here the rows correspond to the first couple of indices of a_{ijkl}, the columns correspond to the second couple of indices. The reference isotropic model is given by

$$
v_{\mathrm{P}}^2 = 15.00\,\mathrm{km}^2\mathrm{s}^{-2} \quad , \qquad v_{\mathrm{S}}^2 = 5.10\,\mathrm{km}^2\mathrm{s}^{-2} \tag{68}
$$

at the surface, and

$$
v_{\mathrm{P}}^2 = 23.00\,\mathrm{km}^2\mathrm{s}^{-2} \quad , \qquad v_{\mathrm{S}}^2 = 7.79\,\mathrm{km}^2\mathrm{s}^{-2} \tag{69}
$$

at the depth of 1 km. All the above values are interpolated linearly with depth. The density is constant.

The point source is located at position $(50, 50, 0)^{\mathrm{T}}$, 29 receivers $(51, 50, 0.010)^{\mathrm{T}}$, $(51, 50, 0.030)^{\mathrm{T}}$, $(51, 50, 0.050)^{\mathrm{T}}$, ..., $(51, 50, 0.570)^{\mathrm{T}}$ are located in a vertical well (distances in km), see Figure 1.

For comparison with the isotropic–ray–theory and anisotropic–ray–theory seismograms in model QI and for a more detailed discussion and description of this model refer to Pšenčík & Dellinger (2001).

3.2. Models QI2, QI4 and QI8

To emphasize the effects of perturbations of travel time, new models with an increased degree of anisotropy have been derived from the QI model.

The differences of the elastic moduli of model QI2 from the elastic moduli of the reference isotropic model (68), (69) are exactly twice larger than the differences of model QI. The density–normalized elastic moduli a_{ijkl} of model QI2 in km^{2}s^{-2} at the surface (zero depth) are

$$\begin{array}{c} \\ 11 \\ 22 \\ 33 \\ 23 \\ 13 \\ 12 \end{array}\begin{array}{c} \begin{array}{cccccc} 11 & 22 & 33 & 23 & 13 & 12 \end{array} \\ \begin{pmatrix} 13.97000 & 4.25000 & 4.70000 & 0.00000 & 0.00000 & -1.16000 \\ & 13.97000 & 4.70000 & 0.00000 & 0.00000 & -1.16000 \\ & & 16.42000 & 0.00000 & 0.00000 & -0.58000 \\ & & & 5.21000 & -0.35000 & 0.00000 \\ & & & & 5.21000 & 0.00000 \\ & & & & & 4.99000 \end{pmatrix} \end{array} , \qquad (70)$$

and at the depth of 1 km they are

$$\begin{array}{c} \\ 11 \\ 22 \\ 33 \\ 23 \\ 13 \\ 12 \end{array}\begin{array}{c} \begin{array}{cccccc} 11 & 22 & 33 & 23 & 13 & 12 \end{array} \\ \begin{pmatrix} 21.17926 & 6.38126 & 7.06750 & 0.00000 & 0.00000 & -1.76900 \\ & 21.17926 & 7.06750 & 0.00000 & 0.00000 & -1.76900 \\ & & 24.91550 & 0.00000 & 0.00000 & -0.88450 \\ & & & 7.93276 & -0.53376 & 0.00000 \\ & & & & 7.93276 & 0.00000 \\ & & & & & 7.59726 \end{pmatrix} \end{array} . \qquad (71)$$

Analogously, the differences of the elastic moduli of model QI4 from the elastic moduli of the reference isotropic model (68), (69) are exactly 4 times larger than the differences of model QI. The density–normalized elastic moduli a_{ijkl} of model QI4 in km^{2}s^{-2} at the surface (zero depth) are

$$\begin{array}{c} \\ 11 \\ 22 \\ 33 \\ 23 \\ 13 \\ 12 \end{array}\begin{array}{c} \begin{array}{cccccc} 11 & 22 & 33 & 23 & 13 & 12 \end{array} \\ \begin{pmatrix} 12.94000 & 3.70000 & 4.60000 & 0.00000 & 0.00000 & -2.32000 \\ & 12.94000 & 4.60000 & 0.00000 & 0.00000 & -2.32000 \\ & & 17.84000 & 0.00000 & 0.00000 & -1.16000 \\ & & & 5.32000 & -0.70000 & 0.00000 \\ & & & & 5.32000 & 0.00000 \\ & & & & & 4.88000 \end{pmatrix} \end{array} , \qquad (72)$$

and at the depth of 1 km they are

$$\begin{array}{c} \\ 11 \\ 22 \\ 33 \\ 23 \\ 13 \\ 12 \end{array}\begin{array}{c} \begin{array}{cccccc} 11 & 22 & 33 & 23 & 13 & 12 \end{array} \\ \begin{pmatrix} 19.35852 & 5.34252 & 6.71500 & 0.00000 & 0.00000 & -3.53800 \\ & 19.35852 & 6.71500 & 0.00000 & 0.00000 & -3.53800 \\ & & 26.83100 & 0.00000 & 0.00000 & -1.76900 \\ & & & 8.07552 & -1.06752 & 0.00000 \\ & & & & 8.07552 & 0.00000 \\ & & & & & 7.40452 \end{pmatrix} \end{array} . \qquad (73)$$

Note that we also calculated the errors of the anisotropic–common–ray approximation in model QI8, in which the differences of the elastic moduli from the elastic moduli of the reference isotropic model (68), (69) are exactly 8 times larger than the differences of model QI.

The data for models QI, QI2, QI4 and QI8 have been released on the compact disk of Bucha & Bulant (2002). Numerical examples in this paper have been calculated using the software and data by Bucha & Bulant (2005).

3.3. Effects of common–ray approximations

Two–point anisotropic common rays have been traced from the source to the receivers using the program CRT (Červený, Klimeš & Pšenčík, 1988), and the first–order and second–order perturbation derivatives of travel time have been calculated along these rays. Equation (20) has then been used to estimate the travel–time errors $\delta\tau_1^{\mathrm{ACR}}$ and $\delta\tau_2^{\mathrm{ACR}}$ of the anisotropic–common–ray approximation. Equation (20) has also been used to estimate the second–order term $\delta\tau_3^{\mathrm{ACR}}$ in the perturbation expansion of travel time from the anisotropic common ray to the isotropic common ray. If required, the travel–time errors $\delta\tau_1^{\mathrm{ICR}}$ and $\delta\tau_2^{\mathrm{ICR}}$ of the isotropic–common–ray approximation can also be roughly estimated using equation (21). The projection of the travel–time errors on the relative error of the coupled S wavefield depends on the polarization. For the projection of the travel–time errors on the relative errors of the time–harmonic Green tensor refer to Klimeš & Bulant (2004, sec. 3.4.).

To check the convergency and accuracy of the second–order perturbation expansion (19), we first compare the perturbation expansion from the isotropic common ray to the anisotropic common ray with the perturbation expansion from the anisotropic common ray to the isotropic common ray. The results of the comparison for the anisotropic common rays traced using the reference Hamiltonian (4) averaged with degree $N = -1$ are displayed in Tables 1, 2 and 3. Only the results at the 1^{st}, 8^{th}, 15^{th}, 22^{nd} and 29^{th} receivers (see Figure 1) are shown, because the variation of the quantities along the vertical profile in models QI, QI2 and QI4 is very moderate. Note that the neglected third–order and higher odd–order perturbation corrections map onto the differences between the ACR and ICR remaining terms in Tables 1, 2 and 3, whereas the neglected fourth–order and higher even–order perturbation corrections map onto the average of the ACR and ICR remaining terms. We may approximate the third–order terms in the perturbation expansions using the differences between the the ACR and ICR quadratic terms. If we subtract these approximate third–order terms from the ACR and ICR remaining terms, the differences between the ACR and ICR remaining terms become decreased onto the level of numerical errors. This means that the fifth–order terms in the perturbation expansion are not detectable in Tables 1, 2 and 3. The average of the ACR and ICR remaining terms in Table 3 is mostly caused by the fourth–order terms in the perturbation expansions.

For convenient comparison of the isotropic–common–ray and anisotropic–common–ray approximations, the perturbation expansion of travel time from the isotropic common ray to the anisotropic–ray–theory rays is compared with the exact anisotropic–ray–theory travel times in Tables 4, 5 and 6. The exact anisotropic–ray–theory travel times have been calculated by the program ANRAY (Gajewski & Pšenčík, 1990; Pšenčík, 1998b). Note that the quadratic terms in analogous Tables 1, 2 and 3 by Klimeš & Bulant (2004) slightly differ from the corresponding terms in our Tables 4, 5 and 6 because of a bug which has been fixed by Bucha & Bulant (2005).

The perturbation expansion of travel time from the anisotropic common ray to the anisotropic–ray–theory rays is then compared with the exact anisotropic–ray–theory

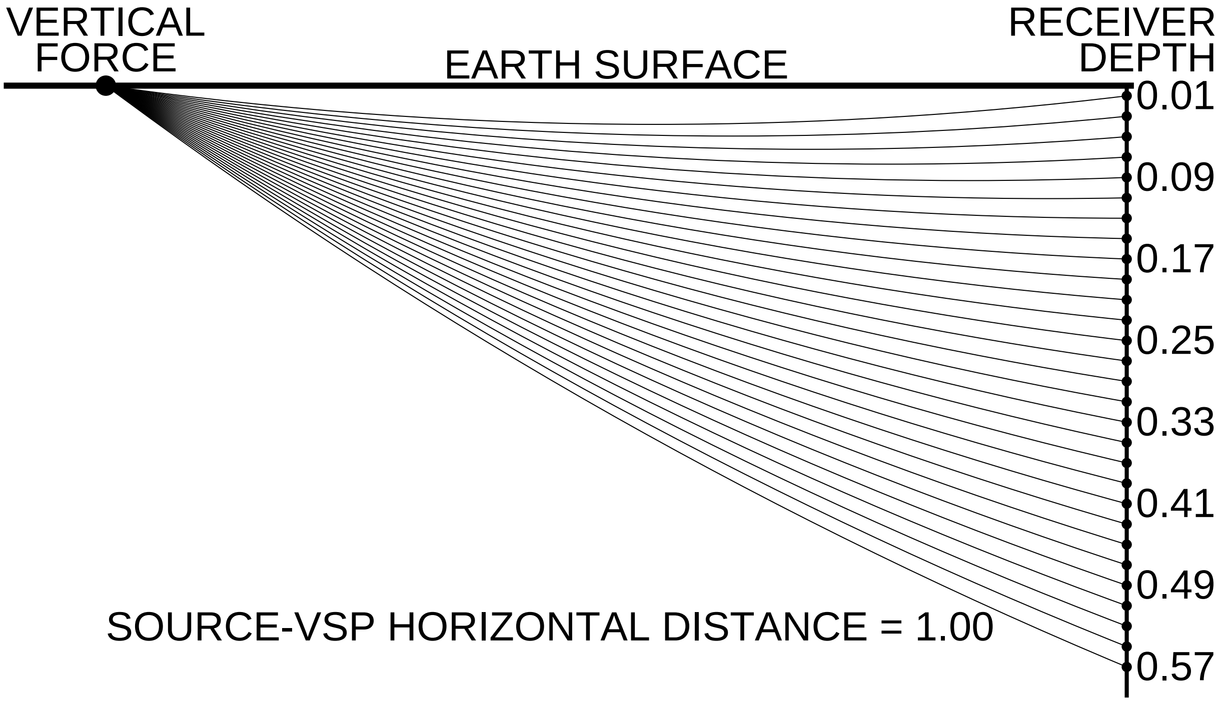

Figure 1. The source and receiver configuration in models QI, QI2, QI4 and QI8.

travel times. The results of the comparison for the anisotropic common rays traced using the reference Hamiltonian (4) averaged with degree $N = -1$ are displayed in Tables 7, 8 and 9. The results of the comparison for the anisotropic common rays traced using the reference Hamiltonian (4) averaged with degree $N = 2$ (Bakker, 2002) are displayed in Tables 10, 11 and 12.

The quadratic terms in the perturbation expansions displayed in Tables 4–12 represent the estimates of the errors due to the isotropic–common–ray (ICR) and anisotropic–common–ray (ACR) approximations of the anisotropic–ray–theory travel times.

In strongly anisotropic model QI8, the errors of the anisotropic–common–ray approximation of the anisotropic–ray–theory travel times are so large that the anisotropic–common–ray approximation of the coupling ray theory is considerably inaccurate and should not be applied. In model QI8, both the anisotropic common rays and the errors of the anisotropic–common–ray approximation considerably depend on degree N of homogeneous Hamiltonians averaged in equation (4).

Although the second–order perturbation expansion of travel time has successfully been applied by Klimeš & Bulant (2004) and by Bulant & Klimeš (2004b) to the estimation of the anisotropic–ray–theory travel times in simple 1–D models QI, QI2 and QI4 with constant gradients of the density–normalized elastic moduli, we cannot recommend approximation of travel time using the second–order perturbation expansion in more complex models, because the second–order perturbations may be infinitely large in the vicinity of caustics. The second–order perturbation expansion of travel time should be used especially for estimating and controlling the accuracy of the common–ray approximations outside caustics.

Table 1. Perturbation expansion of travel time from the isotropic common ray to the anisotropic common ray and from the anisotropic common ray to the isotropic common ray in model QI.

Rec. dep.	ICR time	ICR linear term	ICR quadratic term	ICR remaining term	ACR time	ACR linear term	ACR quadratic term	ACR remaining term
0.01	0.440993	0.000796	0.000060	0.000001	0.441850	-0.000917	0.000062	-0.000001
0.15	0.438077	0.000743	0.000058	0.000002	0.438880	-0.000862	0.000061	-0.000002
0.29	0.443550	0.000575	0.000056	0.000002	0.444183	-0.000692	0.000059	-0.000001
0.43	0.456339	0.000328	0.000054	0.000000	0.456721	-0.000440	0.000057	0.000000
0.57	0.475205	0.000037	0.000050	0.000003	0.475296	-0.000143	0.000054	-0.000003

Table 2. Perturbation expansion of travel time from the isotropic common ray to the anisotropic common ray and from the anisotropic common ray to the isotropic common ray in model QI2.

Rec. dep.	ICR time	ICR linear term	ICR quadratic term	ICR remaining term	ACR time	ACR linear term	ACR quadratic term	ACR remaining term
0.01	0.440993	0.003429	0.000250	-0.000005	0.444667	-0.003921	0.000240	0.000006
0.15	0.438077	0.003310	0.000248	-0.000002	0.441634	-0.003802	0.000241	0.000004
0.29	0.443550	0.002982	0.000252	0.000005	0.446788	-0.003485	0.000248	-0.000001
0.43	0.456339	0.002500	0.000255	-0.000002	0.459093	-0.003018	0.000258	0.000005
0.57	0.475205	0.001926	0.000257	0.000004	0.477392	-0.002453	0.000266	0.000000

Table 3. Perturbation expansion of travel time from the isotropic common ray to the anisotropic common ray and from the anisotropic common ray to the isotropic common ray in model QI4.

Rec. dep.	ICR time	ICR linear term	ICR quadratic term	ICR remaining term	ACR time	ACR linear term	ACR quadratic term	ACR remaining term
0.01	0.440993	0.016119	0.001701	-0.000241	0.458572	-0.018809	0.001016	0.000214
0.15	0.438077	0.015763	0.001735	-0.000231	0.455345	-0.018545	0.001059	0.000218
0.29	0.443550	0.014973	0.001861	-0.000212	0.460171	-0.018043	0.001196	0.000226
0.43	0.456339	0.013833	0.002024	-0.000175	0.472022	-0.017316	0.001407	0.000226
0.57	0.475205	0.012454	0.002173	-0.000117	0.489715	-0.016385	0.001661	0.000214

Tables 1, 2 and 3. *Rec. dep.* stands for the receiver depth along the vertical profile, see Figure 1. The *ICR time* is the travel time along the isotropic common ray traced in the reference isotropic model. The *ICR linear term* is the linear term in perturbation expansion (19) of travel time from the isotropic common ray to the anisotropic common ray. The *ICR quadratic term* stands for the quadratic term in the perturbation expansion from the isotropic common ray to the anisotropic common ray. The *ICR remaining term* stands for the difference between the travel time along the anisotropic common ray and the second–order perturbation expansion from the isotropic common ray (*ICR time* +*ICR linear term* +*ICR quadratic term* +*ICR remaining term* =*ACR time*). The *ACR time* is the travel time along the anisotropic common ray. The *ACR linear term* is the linear term in perturbation expansion (19) of travel time from the anisotropic common ray to the isotropic common ray. The *ACR quadratic term* stands for the quadratic term in the perturbation expansion from the anisotropic common ray to the isotropic common ray. The *ACR remaining term* stands for the difference between the travel time along the isotropic common ray and the second–order perturbation expansion from the anisotropic common ray (*ACR time* +*ACR linear term* +*ACR quadratic term* +*ACR remaining term* =*ICR time*). The ICR remaining term and ACR remaining term represent both the inaccuracy of numerical ray tracing and the third–order and higher–order terms in the perturbation expansion. The third–order and fourth–order terms are significant in model QI4 only.

Table 4. Perturbation expansion of travel time from the isotropic common ray to the anisotropic–ray–theory rays in model QI.

Rec. dep.	ICR time	ICR linear terms		ICR quadratic terms		ICR remaining terms		ACR q.terms
0.01	0.440993	-0.002392	0.003983	0.000251	0.000000	-0.000001	-0.000001	0.000065
0.15	0.438077	-0.002518	0.004003	0.000248	0.000000	0.000009	-0.000001	0.000066
0.29	0.443550	-0.002967	0.004117	0.000249	0.000001	0.000009	0.000001	0.000069
0.43	0.456339	-0.003661	0.004317	0.000251	0.000001	0.000010	-0.000002	0.000072
0.57	0.475205	-0.004520	0.004595	0.000250	0.000003	0.000022	0.000006	0.000076

Table 5. Perturbation expansion of travel time from the isotropic common ray to the anisotropic–ray–theory rays in model QI2.

Rec. dep.	ICR time	ICR linear terms		ICR quadratic terms		ICR remaining terms		ACR q.terms
0.01	0.440993	-0.004746	0.011604	0.000971	0.000000	0.000008	-0.000007	0.000235
0.15	0.438077	-0.004992	0.011612	0.000959	0.000000	0.000023	0.000001	0.000231
0.29	0.443550	-0.005874	0.011837	0.000960	0.000001	0.000036	0.000005	0.000229
0.43	0.456339	-0.007234	0.012235	0.000958	0.000001	0.000054	0.000001	0.000224
0.57	0.475205	-0.008912	0.012764	0.000945	0.000002	0.000069	0.000000	0.000217

Table 6. Perturbation expansion of travel time from the isotropic common ray to the anisotropic–ray–theory rays in model QI4.

Rec. dep.	ICR time	ICR linear terms		ICR quadratic terms		ICR remaining terms		ACR q.terms
0.01	0.440993	-0.009341	0.041580	0.003643	0.000488	0.000119	-0.000137	0.000365
0.15	0.438077	-0.009816	0.041343	0.003589	0.000548	0.000164	-0.000142	0.000333
0.29	0.443550	-0.011517	0.041462	0.003554	0.000711	0.000249	-0.000167	0.000272
0.43	0.456339	-0.014129	0.041796	0.003496	0.000953	0.000352	-0.000203	0.000200
0.57	0.475205	-0.017335	0.042243	0.003394	0.001224	0.000455	-0.000224	0.000136

Tables 4, 5 and 6. *Rec. dep.* stands for the receiver depth along the vertical profile, see Figure 1. The *ICR time* is the reference travel time along the isotropic common ray traced in the reference isotropic model. The *ICR linear terms* are the linear terms in perturbation expansion (19) of travel time from the isotropic common ray to the anisotropic–ray–theory rays and represent the travel–time corrections considered in the isotropic–common–ray approximation of the coupling ray theory. The *ICR quadratic terms* stand for the quadratic terms in the perturbation expansion, and represent the estimates of the errors due to the isotropic–common–ray approximation of the anisotropic–ray–theory travel times. The *ICR remaining terms* stand for the differences between the exact anisotropic–ray–theory travel times calculated by the program ANRAY version 4.40 and the second–order perturbation expansion from the isotropic common ray, in order to illustrate the reliability of the error estimates. The ICR remaining terms represent both the inaccuracy of numerical ray tracing and the third–order and higher–order terms in the perturbation expansion. The *ACR quadratic terms* stand for the estimate of the equal quadratic terms in the perturbation expansion from the anisotropic common ray, traced using the reference Hamiltonian (4) averaged with degree $N = -1$, to the anisotropic–ray–theory rays. The quadratic terms represent the estimate of the error due to the anisotropic–common–ray approximation of the anisotropic–ray–theory travel times. Note that these estimated ACR quadratic terms are approximately calculated along the isotropic common rays, i.e. without tracing the anisotropic common rays. The estimated ACR quadratic terms may be compared with the actual errors of the anisotropic–common–ray approximation (Tables 7, 8 and 9, columns ACR quadratic terms and ACR remaining terms). The differences between the ACR quadratic terms calculated along the isotropic common rays and the ACR quadratic terms calculated along the anisotropic common rays are mostly due to the neglected third–order terms in the perturbation expansion.

Table 7. Perturbation expansion of travel time from the anisotropic common ray to the anisotropic–ray–theory rays in model QI. The homogeneous Hamiltonians of the minus first degree are averaged.

Rec. dep.	ACR time	ACR linear terms		ACR quadratic terms		ACR remaining terms	
0.01	0.441850	-0.003061	0.003061	0.000066	0.000066	-0.000003	-0.000002
0.15	0.438880	-0.003135	0.003135	0.000066	0.000066	0.000004	-0.000001
0.29	0.444183	-0.003415	0.003415	0.000070	0.000070	0.000003	0.000000
0.43	0.456721	-0.003861	0.003861	0.000075	0.000075	0.000004	-0.000001
0.57	0.475296	-0.004430	0.004430	0.000079	0.000079	0.000010	0.000004

Table 8. Perturbation expansion of travel time from the anisotropic common ray to the anisotropic–ray–theory rays in model QI2. The homogeneous Hamiltonians of the minus first degree are averaged.

Rec. dep.	ACR time	ACR linear terms		ACR quadratic terms		ACR remaining terms	
0.01	0.444667	-0.007689	0.007689	0.000245	0.000245	0.000002	-0.000011
0.15	0.441634	-0.007818	0.007818	0.000244	0.000244	0.000008	-0.000004
0.29	0.446788	-0.008366	0.008366	0.000245	0.000245	0.000006	-0.000007
0.43	0.459093	-0.009240	0.009240	0.000246	0.000246	0.000017	-0.000003
0.57	0.477392	-0.010344	0.010344	0.000244	0.000244	0.000014	-0.000009

Table 9. Perturbation expansion of travel time from the anisotropic common ray to the anisotropic–ray–theory rays in model QI4. The homogeneous Hamiltonians of the minus first degree are averaged.

Rec. dep.	ACR time	ACR linear terms		ACR quadratic terms		ACR remaining terms	
0.01	0.458572	-0.023856	0.023856	0.000588	0.000588	0.000110	-0.000090
0.15	0.455345	-0.024001	0.024001	0.000560	0.000560	0.000111	-0.000079
0.29	0.460171	-0.024947	0.024947	0.000505	0.000505	0.000108	-0.000067
0.43	0.472022	-0.026489	0.026489	0.000433	0.000433	0.000092	-0.000059
0.57	0.489715	-0.028424	0.028424	0.000355	0.000355	0.000072	-0.000047

Tables 7, 8 and 9. *Rec. dep.* stands for the receiver depth along the vertical profile, see Figure 1. The *ACR time* is the reference travel time τ along the anisotropic common ray. The *ACR linear terms* are the linear terms $\tau_{,1}$ and $\tau_{,2}$ in perturbation expansion (19) of travel time from the anisotropic common ray to the anisotropic–ray–theory rays, and represent the travel–time corrections considered in the anisotropic–common–ray approximation of the coupling ray theory. The *ACR quadratic terms* stand for quadratic terms (20) in the perturbation expansion, and represent the estimates of the errors due to the anisotropic–common–ray approximation of the anisotropic–ray–theory travel times. The *ACR remaining terms* stand for the differences between the exact anisotropic–ray–theory travel times calculated by the program ANRAY version 4.40 and the second–order perturbation expansion from the anisotropic common ray, in order to illustrate the reliability of the error estimates. The ACR remaining terms represent both the inaccuracy of numerical ray tracing and the third–order and higher–order terms in the perturbation expansion.

Table 10. Perturbation expansion of travel time from the anisotropic common ray to the anisotropic–ray–theory rays in model QI. The homogeneous Hamiltonians of the second degree are averaged.

Rec. dep.	ACR time	ACR linear terms		ACR quadratic terms		ACR remaining terms	
0.01	0.441814	-0.003027	0.003090	0.000063	0.000068	0.000001	0.000002
0.15	0.438846	-0.003098	0.003165	0.000064	0.000069	0.000005	-0.000001
0.29	0.444146	-0.003372	0.003451	0.000067	0.000073	0.000000	-0.000003
0.43	0.456674	-0.003808	0.003906	0.000071	0.000078	0.000002	-0.000002
0.57	0.475234	-0.004363	0.004487	0.000075	0.000083	0.000010	0.000004

Table 11. Perturbation expansion of travel time from the anisotropic common ray to the anisotropic–ray–theory rays in model QI2. The homogeneous Hamiltonians of the second degree are averaged.

Rec. dep.	ACR time	ACR linear terms		ACR quadratic terms		ACR remaining terms	
0.01	0.444469	-0.007465	0.007861	0.000220	0.000271	0.000002	-0.000011
0.15	0.441425	-0.007586	0.007998	0.000218	0.000270	0.000010	-0.000002
0.29	0.446554	-0.008105	0.008572	0.000219	0.000273	0.000005	-0.000007
0.43	0.458815	-0.008932	0.009487	0.000218	0.000277	0.000016	-0.000004
0.57	0.477058	-0.009977	0.010647	0.000214	0.000277	0.000012	-0.000011

Table 12. Perturbation expansion of travel time from the anisotropic common ray to the anisotropic–ray–theory rays in model QI4. The homogeneous Hamiltonians of the second degree are averaged.

Rec. dep.	ACR time	ACR linear terms		ACR quadratic terms		ACR remaining terms	
0.01	0.456727	-0.021838	0.025497	0.000450	0.000836	0.000075	-0.000135
0.15	0.453460	-0.021953	0.025686	0.000427	0.000800	0.000080	-0.000120
0.29	0.458162	-0.022780	0.026778	0.000383	0.000730	0.000071	-0.000114
0.43	0.469812	-0.024138	0.028541	0.000325	0.000634	0.000059	-0.000103
0.57	0.487256	-0.025848	0.030746	0.000264	0.000527	0.000047	-0.000081

Tables 10, 11 and 12. Refer to the description of Tables 7, 8 and 9.

4. Conclusions

In using any kind of the common–ray approximation of the coupling ray theory, the travel–time errors due to the applied common–ray approximation should be evaluated. The travel–time errors due to the common–ray approximations can be estimated by numerical quadrature along the isotropic common rays using the equations proposed by Klimeš & Bulant (2004), or by numerical quadrature along the anisotropic common rays using the equations proposed here. The numerical results suggest that the anisotropic–common–ray approximation by Bakker (2002) and Klimeš (2003) is considerably more accurate than the isotropic–common–ray approximation.

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