Hamiltonian formulation of the Finsler and Riemann geometries

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1. Introduction

In the Riemann geometry, the metric tensor depends on coordinates. In the Finsler geometry, which is a generalization of the Riemann geometry, the metric tensor depends also on the direction of propagation. Light rays are the zero–length geodesics in Finsler space, determined by the light–propagation metric tensor.

We would like to demonstrate that the Hamiltonian formulation simplifies the equations of the Riemann geometry, and makes the Finsler geometry no more difficult than the Riemann geometry.

We shall refer to the case of a metric space, in which the length of all geodesics is positive, as the spatial case, and to the corresponding geodesics as spatial geodesics. We shall refer to the opposite case as the space–time case, and to the corresponding geodesics as space–time geodesics. In the space–time case, we shall distinguish between zero–length and time–like geodesics. In the spatial case, the Riemannian metric tensor is always positive–definite, but the Finslerian metric tensor may also be indefinite. In the space–time case, the metric tensor is always indefinite.

Sections 2 and 3 are the introductory sections only. In Section 2, we briefly explain why we should strictly distinguish the light–propagation metric tensor from the gravitational metric tensor of general relativity. Section 3 is devoted to a brief outline of the physical background behind the light–propagation Hamiltonian. Then we shall start to speak about geodesics and proper time in the Finsler geometry.

2. Gravitation

Gravitational field is described in terms of the components

$$G^{\alpha\beta} = G^{\alpha\beta}(x^\kappa) \tag{1}$$

of the Riemannian *gravitational metric tensor*, which depends on coordinates x^κ. The gravitational metric tensor determines the gravitational Christoffel symbols

$$\Gamma^\gamma_{\alpha\beta} = \Gamma^\gamma_{\alpha\beta}(x^\kappa) \quad , \tag{2}$$

and the gravitational Riemann curvature tensor

$$R_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta}(x^\kappa) \quad . \tag{3}$$

The corresponding Ricci curvature tensor

$$R_{\alpha\gamma} = R_{\alpha\beta\gamma\delta} G^{\beta\delta} \tag{4}$$

satisfies the Einstein equation for gravitational field.

In vacuum, without other interactions than gravitational, both light and matter would propagate along geodesics determined by the gravitational metric tensor.

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In reality, both light and satellites interact with surrounding matter and radiation, and propagate differently. Propagation of light may be approximated by the light–propagation metric tensor, which differs from the gravitational metric tensor and is rather Finslerian than Riemannian. The motion of satellites could also be approximately described by satellite metric tensors, but each satellite would have its own metric depending on the shape and other characteristics of the satellite. Fortunately, a satellite may be equipped with an accelerometer, which enables us to discriminate between the satellite and gravitational metric tensors. We thus do not need to introduce the satellite metric tensors. On the other hand, we cannot equip the light with an accelerometer. We thus have to strictly distinguish the light–propagation metric tensor from the gravitational metric tensor.

3. Maxwell equations and ray theory (geometrical optics)

The propagation of light is determined by the Maxwell equations and by the material relations, which describe the mutual dependence between electric field intensity E_i, magnetic flux density B_i, electric flux density D_i and magnetic field intensity H_i. These material relations approximately describe the effective properties of the medium in which the light propagates. For infinitesimally weak electromagnetic field and a given frequency range, the material relations can be approximated by a linear dependence of D_i and H_i on E_i and B_i.

Electric field strength E_i and magnetic induction B_i can conveniently be expressed in terms of electromagnetic vector potential $A_\alpha = A_\alpha(x^\kappa)$. The propagation of light is then approximately determined by the covariant equation

$$[C^{\alpha\beta\gamma\delta} A_{\gamma;\delta}]_{;\beta} = -J^\alpha \tag{5}$$

for the electromagnetic vector potential. Equation (5) follows from the Maxwell equations and from the approximate linearized material relations. The covariant derivatives in equation (5) are defined with respect to the gravitational metric tensor $G^{\alpha\beta}$.

Tensor

$$C^{\alpha\beta\gamma\delta} = C^{\alpha\beta\gamma\delta}(x^\kappa) \quad , \tag{6}$$

describes both the linearized material relations and the gravitational interaction. It is skew both with respect to the first pair of indices and with respect to the second pair of indices,

$$C^{\alpha\beta\gamma\delta} = -C^{\beta\alpha\gamma\delta} = -C^{\alpha\beta\delta\gamma} \quad . \tag{7}$$

Due to this property, covariant equation (5) may also be expressed in terms of partial derivatives,

$$[-\det(G^{\kappa\lambda})]^{\frac{1}{2}} \{[-\det(G^{\kappa\lambda})]^{-\frac{1}{2}} C^{\alpha\beta\gamma\delta} A_{\gamma,\delta}\}_{,\beta} = -J^\alpha \quad . \tag{8}$$

This equation has the form of the Lagrange equations for the electromagnetic field.

For example, in vacuum, where there are no interactions other than gravitational, we have

$$C^{\alpha\beta\gamma\delta} = \frac{1}{\mu_0\, c^4} \left(G^{\alpha\gamma} G^{\beta\delta} - G^{\alpha\delta} G^{\beta\gamma} \right) \quad , \tag{9}$$

where μ_0 is the magnetic permeability of vacuum and c is the light speed in vacuum. Otherwise, tensor $C^{\alpha\beta\gamma\delta}$ depends not only on the properties of the surrounding medium, but also on its movement with respect to the coordinate system, and expressions for $C^{\alpha\beta\gamma\delta}$ are not simple.

We now express the time–harmonic electromagnetic vector potential of circular frequency ω in terms of vectorial amplitude $a_\alpha = a_\alpha(x^\kappa)$ and proper time (phase function) $\tau = \tau(x^\lambda)$,

$$A_\alpha(x^\kappa) = a_\alpha(x^\kappa) \exp[\mathrm{i}\omega\tau(x^\lambda)] \quad . \tag{10}$$

In the high–frequency approximation for $\omega \to +\infty$ and with vanishing electric current density J^α, equation (8) yields approximately the *Christoffel equation*

$$\Gamma_\alpha^\gamma(x^\kappa, \tau_{,\lambda})\, a_\gamma(x^\kappa) = 0 \quad , \tag{11}$$

where

$$\Gamma_\alpha^\gamma(x^\kappa, \tau_{,\lambda}) = f_{\alpha\sigma}(x^\mu)\, C^{\sigma\beta\gamma\delta}(x^\kappa)\, \tau_{,\beta}\tau_{,\delta} \tag{12}$$

is the *Christoffel matrix.* Here $f_{\alpha\beta}(x^\mu)$ is an arbitrary regular scaling matrix.

This light–propagation Christoffel matrix has one eigenvalue (solution of the characteristic equation) identical to zero. This identically zero eigenvalue is due to the skew property of matrix $C^{\alpha\beta\gamma\delta}$, the corresponding eigenvector is parallel with $\tau_{,\alpha}$, and the corresponding electric and magnetic field intensities and flux densities vanish.

Christoffel equation (11) has a non–trivial solution if another eigenvalue G of the Christoffel matrix is set to zero, i.e., if the eigenvalue satisfies the *eikonal equation*

$$G(x^\kappa, \tau_{,\lambda}) = 0 \quad . \tag{13}$$

The vectors from the null space of Christoffel matrix (12) then generate the light polarization vectors corresponding to the selected eigenvalue G.

Eikonal equation (13) has the form of the stationary Hamilton–Jacobi equation, and can be solved using the Hamilton equations. Note that the eigenvalues $G(x^\kappa, \tau_{,\lambda})$ are homogeneous functions of the second degree with respect to $\tau_{,\alpha}$.

4. Hamiltonian formulation of the Finsler geometry

The Finsler space is a generalization of the Riemann space. This generalization is appropriate for the description of the propagation of light and of many other waves. Geodesics and proper time in Finsler spaces can efficiently be calculated by Hamilton's method. All equations applicable to the Finsler space are equally applicable to the Riemann space.

4.1. Hamilton equations

Proper time τ satisfies the *stationary Hamilton–Jacobi equation*

$$H(x^\kappa, \tau_{,\mu}) = C \quad , \tag{14}$$

where $H(x^\kappa, p_\mu)$ is the Hamiltonian, and C is a constant. For the propagation of light (zero–length geodesics), the constant is usually zero, $C = 0$. For spatial geodesics and time–like space–time geodesics of particles, the constant is often non–zero, $C \neq 0$.

The Hamiltonian $H(x^\kappa, p_\mu)$ should be many times differentiable function of *phase-space coordinates*, which are composed of coordinates x^α and *slowness–vector components p_β*. In particular, this smoothness requirement considerably restricts a possible form of the Hamiltonian for zero–length space–time geodesics.

The geodesics corresponding to stationary Hamilton–Jacobi equation (14) are described by the *Hamilton equations*

$$\frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda} = \frac{\partial H}{\partial p_\alpha} \quad , \tag{15}$$

$$\frac{\mathrm{d}p_\alpha}{\mathrm{d}\lambda} = -\frac{\partial H}{\partial x^\alpha} \quad , \tag{16}$$

with initial conditions

$$x^\alpha(\lambda_0) = x_0^\alpha \quad , \qquad p_\alpha(\lambda_0) = \tau_{,\alpha}(x_0^\mu) \quad . \tag{17}$$

The initial conditions should satisfy Hamilton–Jacobi equation (14). Then

$$\tau_{,\alpha}[x^\mu(\lambda)] = p_\alpha(\lambda) \tag{18}$$

along the geodesics. Parameter λ along a geodesic is determined by the form of the Hamiltonian and by initial conditions (17) for the geodesic. Proper time τ along the geodesic is given by

$$\tau(\lambda) = \tau(\lambda_0) + \int_{\lambda_0}^{\lambda} \mathrm{d}\lambda \; p_\alpha \frac{\partial H}{\partial p_\alpha} \quad , \tag{19}$$

which follows from (15) and (18). This proper time satisfies Hamilton–Jacobi equation (14).

Note that equal geodesics and equal proper time may be generated by various Hamiltonians. For example, Hamiltonian $\tilde{H}(x^\kappa, p_\mu) = F[H(x^\kappa, p_\mu)]$ with $\tilde{C} = F(C)$, where $F(x)$ is an arbitrary function with a non–vanishing finite derivative at $x = C$, yields equal geodesics as Hamiltonian $H(x^\kappa, p_\mu)$. Transform $\tilde{H}(x^\kappa, p_\mu) = H(x^\kappa, p_\mu) - C$ results in the Hamilton–Jacobi equation with $\tilde{C} = 0$. If $C = 0$, the Hamiltonian may be multiplied by any non–zero smooth function $f = f(x^\kappa, p_\lambda)$ of phase–space coordinates, $\tilde{H}(x^\kappa, p_\mu) = f(x^\kappa, p_\lambda) H(x^\mu, p_\nu)$. All these transformations do influence neither geodesics nor proper time, but may result in different parameters λ along geodesics, in different Finslerian metric tensors, and in different perturbation expansions of different accuracy.

The Hamiltonian is often chosen as a homogeneous function of degree N in p_α. For a homogeneous Hamiltonian of degree N, parameter λ along a geodesic is proportional to proper time,

$$\mathrm{d}\tau = NC \, \mathrm{d}\lambda \quad . \tag{20}$$

Degree N may be arbitrary for spatial or time–like geodesics. In this case, the constant in Hamilton–Jacobi equation (14) is usually chosen as

$$C = \frac{1}{N} \quad . \tag{21}$$

Especially, homogeneous Hamiltonians of degrees $N = 2$, $N = 1$ or $N = -1$ are frequently used for spatial and time–like space–time geodesics.

For zero–length geodesics, a homogeneous Hamiltonian of degree $N = 2$ is usually chosen in order to satisfy the smoothness condition and to keep the right–hand sides of Hamilton equations (15) and (16) finite and non–vanishing.

4.2. Finslerian metric tensor

If the Hamiltonian is chosen as a *homogeneous function of the second degree* in p_α, and is properly normalized, then the *contravariant Finslerian metric tensor* is defined by equation

$$g^{\alpha\beta}(x^\kappa, p_\mu) = \frac{\partial^2 H}{\partial p_\alpha \partial p_\beta}(x^\kappa, p_\mu) \tag{22}$$

(Rund, 1959, eq. 1.5.4). If metric tensor (22) is independent of p_μ,

$$g^{\alpha\beta}(x^\kappa, p_\mu) = g^{\alpha\beta}(x^\kappa) \quad , \tag{23}$$

i.e., if the *Cartan torsion tensor*

$$C^{\alpha\beta\gamma}(x^\kappa, p_\mu) = \frac{\partial^3 H}{\partial p_\alpha \partial p_\beta \partial p_\gamma}(x^\kappa, p_\mu) \tag{24}$$

is identical to zero, the Finsler space reduces to the *Riemann space.*

On the other hand, if we know the contravariant metric tensor $g^{\alpha\beta}(x^\kappa, p_\mu)$, we may construct a homogeneous Hamiltonian of the second degree in p_α as

$$H(x^\kappa, p_\mu) = \frac{1}{2}\, p_\alpha\, g^{\alpha\beta}(x^\kappa, p_\mu)\, p_\beta \quad . \tag{25}$$

In this section, we shall consider homogeneous Hamiltonian (25) of the second degree.

For zero–length geodesics we choose the constant in Hamilton–Jacobi equation (22) $C = 0$, for spatial and time–like space–time geodesics $C = \frac{1}{2}$. For homogeneous Hamiltonian (25), equation (19) reads

$$\tau(\lambda) = \tau(\lambda_0) + 2C\,(\lambda - \lambda_0) \quad . \tag{26}$$

The contravariant *velocity vector* $p^\alpha = p^\alpha(x^\kappa, p_\lambda)$ is defined by equation (15),

$$p^\alpha = \frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda} = \frac{\partial H}{\partial p_\alpha} \quad . \tag{27}$$

Euler's theorem on homogeneous functions then yields $p^\alpha p_\alpha = 2H$ and

$$p^\alpha = g^{\alpha\beta}\, p_\beta \quad . \tag{28}$$

Relation $p^\alpha p_\alpha = 2H$ reads

$$p^\alpha p_\alpha = 2C \tag{29}$$

on all geometrical objects which satisfy equation (14).

The *covariant metric tensor* $g_{\alpha\beta}$ is the inverse matrix to the contravariant metric tensor,

$$g_{\alpha\beta}\, g^{\beta\gamma} = \delta^\gamma_\alpha \tag{30}$$

(Rund, 1959, eq. 1.5.9). Equation (28) then yields

$$p_\alpha = g_{\alpha\beta}\, p^\beta \quad . \tag{31}$$

The Finslerian Christoffel symbols

$$\gamma^\gamma_{\alpha\beta} = \gamma^\gamma_{\alpha\beta}(x^\kappa, p_\lambda) \tag{32}$$

are defined by the same rule as in the Riemannian case (Rund, 1959, eq. 2.2.3). Hamilton equations (15) and (16) are then equivalent to the equations of geodesics

$$\frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda} = p^\alpha \quad , \tag{33}$$

$$\frac{\mathrm{d}p^\alpha}{\mathrm{d}\lambda} = -\gamma^\alpha_{\beta\gamma}\, p^\beta p^\gamma \quad . \tag{34}$$

In Finsler geometry, the metric tensor, especially the covariant metric tensor, is often expressed as the function of contravariant velocity vector p^α, instead of slowness vector p_α. Since mapping $p^\alpha = p^\alpha(x^\kappa, p_\lambda)$ defined by equation (27) may not be uniquely invertible with respect to p_α, metric tensor $g_{\alpha\beta}$ may be, in general, a multivalued function of p^α. We shall thus stick to the metric tensor $g_{\alpha\beta} = g_{\alpha\beta}(x^\kappa, p_\lambda)$, which is a unique function of p_λ for a selected polarization.

4.3. Geodesic deviation and second–order derivatives of proper time

We shall refer to the system of geodesics corresponding to distribution $\tau = \tau(x^\alpha)$ of proper time as the *orthonomic system of geodesics*. Initial conditions (17) for the orthonomic system of geodesics can be parametrized by 3 geodesic parameters γ_i, $i = 1,2,3$ in 4-D space–time.

We define the matrices of the partial derivatives of coordinates x^α and of the slowness–vector components $p_\alpha = \tau_{,\alpha}$ with respect to 4 *geodesic coordinates*: 3 parameters γ_i describing the initial conditions for the system of geodesics, and independent parameter $\gamma_4 = \lambda$ along geodesics,

$$Q^\alpha_\kappa = \frac{\partial x^\alpha}{\partial \gamma_\kappa} \quad , \qquad P_{\alpha\kappa} = \frac{\partial \tau_{,\alpha}}{\partial \gamma_\kappa} \quad . \tag{35}$$

Matrices Q^α_κ and $P_{\alpha\kappa}$ describe, by definition, the properties of the orthonomic system of geodesics corresponding to the proper time under consideration. Let us emphasize that the definition of matrices Q^α_κ and $P_{\alpha\kappa}$ depends on the kind of parameter λ along geodesics, which is in turn determined by the form of the Hamiltonian. The use of matrices Q^α_κ and $P_{\alpha\kappa}$, defined for a particular kind of λ, in the equations for the perturbations and partial derivatives of proper time, derived from the Hamiltonian corresponding to different λ, has to be avoided.

Equation

$$P_{\alpha\kappa} = \tau_{,\alpha\beta}\, Q^\beta_\kappa \tag{36}$$

is a direct consequence of definitions (35).

The "*dynamic ray tracing system*"

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\lambda} Q^\alpha_\kappa &= \quad H^{,\alpha}_{,\beta}\, Q^\beta_\kappa + H^{,\alpha\beta}\, P_{\beta\kappa} \quad , \\ \frac{\mathrm{d}}{\mathrm{d}\lambda} P_{\alpha\kappa} &= -H_{,\alpha\beta}\, Q^\beta_\kappa - H^{,\beta}_{,\alpha}\, P_{\beta\kappa} \end{aligned} \tag{37}$$

can be obtained by differentiating the Hamilton equations (15) and (16) with respect to γ_κ (Červený, 1972). Here

$$\begin{aligned} H^{,\alpha\beta} &= \frac{\partial^2 H}{\partial p_\alpha \partial p_\beta} = g^{\alpha\beta} \\ H^{,\alpha}_{,\beta} &= \frac{\partial^2 H}{\partial p_\alpha \partial x^\beta} = g^{\alpha\sigma}_{,\beta}\, p_\sigma \\ H_{,\alpha\beta} &= \frac{\partial^2 H}{\partial x^\alpha \partial x^\beta} = \frac{1}{2}\, g^{\sigma\rho}_{,\alpha\beta}\, p_\sigma p_\rho \quad . \end{aligned} \tag{38}$$

Note that the right–hand sides of equations (38) are much simpler than the expression of the Riemann curvature tensor in terms of the Riemannian metric tensor.

Once we have solved equations (37), we may calculate the second–order derivatives of proper time using equation

$$\tau_{,\alpha\beta} = P_{\alpha\kappa}\, Q^{-1}_{\kappa\beta} \quad , \tag{39}$$

where $Q^{-1}_{\kappa\beta}$ denotes the components of matrix inverse to Q^{β}_{κ},

$$Q^{-1}_{\kappa\beta}\, Q^{\beta}_{\lambda} = \delta_{\kappa\lambda} \quad . \tag{40}$$

Equations (37) are equivalent to the equations

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}\lambda} Q^{\alpha}_{\kappa} + \gamma^{\alpha}_{\sigma\beta}\, p^{\sigma} Q^{\beta}_{\kappa} = g^{\alpha\beta}\, \tilde{P}_{\beta\kappa} \quad , \\ &\frac{\mathrm{d}}{\mathrm{d}\lambda} \tilde{P}_{\alpha\kappa} - \gamma^{\beta}_{\sigma\alpha}\, p^{\sigma}\, \tilde{P}_{\beta\kappa} = r_{\alpha\sigma\rho\beta}\, p^{\sigma} p^{\rho}\, Q^{\beta}_{\kappa} \end{aligned} \tag{41}$$

of geodesic deviation. Here

$$\tilde{P}_{\alpha\kappa} = P_{\alpha\kappa} - \gamma^{\sigma}_{\alpha\beta}\, p_{\sigma}\, Q^{\beta}_{\kappa} \tag{42}$$

is the covariant version of matrix $P_{\alpha\kappa}$ defined by (35). The expressions at the left–hand side of equations (41) are the covariant derivatives with respect to λ. Note that the definition and calculation of the curvature tensor

$$r_{\alpha\beta\gamma\delta} \tag{43}$$

in the Finsler geometry is more involved than the definition and calculation of the Riemann curvature tensor.

In the Hamiltonian formulation, the Finsler geometry is no more complex than the Riemann geometry. Moreover, equations (37) are much simpler than equations (41) of geodesic deviation even in the Riemann geometry.

4.4. Third–order and higher–order derivatives of proper time

The third–order and higher–order partial derivatives of proper time can be calculated by simple numerical quadratures along geodesics (Babich, Buldyrev & Molotkov, 1985; Klimeš, 2002).

5. Perturbations

5.1. Model parameters

We assume that the Hamiltonian is a function of phase–space coordinates, composed of coordinates x^α and slowness–vector components p_β, and of any number of *model parameters* f^k,

$$H = H(x^\alpha, p_\beta, f^k) \quad . \tag{44}$$

Then the proper time and other quantities calculated in the model using this Hamiltonian are functions of coordinates x^α and of model parameters f^k,

$$\tau = \tau(x^\alpha, f^k) \quad . \tag{45}$$

In this section, we index the model parameters by lower–case Roman subscripts. Model parameters are the parameters to be perturbed. They may have various meaning. For example, model parameters f^k may parametrize the model. However, the applicability of model parameters is more general. For example, there are two light waves with the two respective Hamiltonians in anisotropic media. Instead of selecting one of these two Hamiltonians, we may consider a one–parametric set of Hamiltonians, parametrized by a parameter f^1, such that we obtain the Hamiltonian corresponding to one light wave for $f^1 = -1$, the Hamiltonian corresponding to the other light wave for $f^1 = +1$, and the averaged Hamiltonian of both light waves corresponding to the anisotropic common ray tracing for $f^1 = 0$. Another example: Instead of using Hamiltonian $H = H(x^\alpha, p_\beta, f^k)$ and perturbation expansion from $f^k = f_1^k$, to $f^k = f_2^k$, we may define new one–parametric Hamiltonian $\tilde{H} = \tilde{H}(x^\alpha, p_\beta, f)$,

$$\tilde{H}(x^\alpha, p_\beta, f) = H(x^\alpha, p_\beta, f_1^k) + [H(x^\alpha, p_\beta, f_2^k) - H(x^\alpha, p_\beta, f_1^k)]\, f \tag{46}$$

and apply more accurate perturbation expansion from $f = 0$ to $f = 1$.

We shall refer to the partial derivatives with respect to model parameters f^k as the *perturbation derivatives* in order to distinguish them from *partial derivatives* with respect to coordinates x^α.

We denote the partial derivatives with respect to coordinates x^α by lower–case Greek subscripts following a comma, and the *perturbation derivatives by lower–case Roman subscripts* following a comma. In the phase space, we denote the partial derivatives with respect to components p_β of the slowness vector by lower–case Greek superscripts following a comma. For instance,

$$H^{,\kappa\lambda\ldots\mu}_{,\alpha\beta\ldots\nu i\ldots n} = \frac{\partial}{\partial f^i} \cdots \frac{\partial}{\partial f^n} \frac{\partial}{\partial x^\alpha} \frac{\partial}{\partial x^\beta} \cdots \frac{\partial}{\partial x^\nu} \frac{\partial}{\partial p_\kappa} \frac{\partial}{\partial p_\lambda} \cdots \frac{\partial}{\partial p_\mu} H \tag{47}$$

denotes the perturbation derivatives of the partial phase–space derivatives of the Hamiltonian. The perturbation derivatives are taken at fixed phase–space coordinates x^α, p_β, and the partial phase–space derivatives are calculated at fixed model parameters f^k. Similarly,

$$\tau_{,\alpha\beta\ldots\nu i\ldots n} = \frac{\partial}{\partial f^i} \cdots \frac{\partial}{\partial f^n} \frac{\partial}{\partial x^\alpha} \frac{\partial}{\partial x^\beta} \cdots \frac{\partial}{\partial x^\nu} \tau \tag{48}$$

denotes the perturbation derivatives of the partial derivatives of proper time τ. The perturbation derivatives are taken at fixed coordinates x^α, and the partial derivatives are calculated at fixed model parameters f^k.

5.2. Perturbations of proper time

Perturbation derivatives of proper time of all orders can be calculated by simple numerical quadratures along the unperturbed geodesic. The perturbation derivatives of the partial derivatives of proper time can also be calculated by analogous numerical quadratures along the unperturbed geodesic. For the general equations for the perturbation derivatives of all orders refer to Klimeš (2002). Here we only present the equations for the first–order and second–order perturbation derivatives of proper time, expressed in terms of a general Hamiltonian.

Individual perturbation derivatives of proper time depend on the form of the Hamiltonian. For different Hamiltonians, we obtain different perturbation expansions of proper time. Some of these perturbation expansions may be more accurate for some applications than other perturbation expansions.

5.3. First–order perturbation derivatives of proper time

Equation

$$\tau_{,i}(\lambda) = \tau_{,i}(\lambda_0) + \int_{\lambda_0}^{\lambda} \mathrm{d}\lambda \; (-H_{,i}) \tag{49}$$

for the first–order perturbation derivatives of proper time is well–known. The integration variable λ along the geodesic is determined by the form of the Hamiltonian.

5.4. First–order perturbation derivatives of proper–time gradient

The first–order perturbation derivatives of proper–time gradient may be calculated using equation

$$\tau_{,\alpha i} = T_{\kappa i} \; Q^{-1}_{\kappa\alpha} \quad , \tag{50}$$

where $Q^{-1}_{\kappa\alpha}$ are the elements of the matrix inverse to matrix Q^{α}_{κ} of geometrical spreading, see (40). The perturbation derivatives of the partial derivatives $T_{\kappa i}$, covariantly transformed to geodesic coordinates, can be obtained by quadrature

$$T_{\kappa i}(\lambda) = T_{\kappa i}(\lambda_0) + \int_{\lambda_0}^{\lambda} \mathrm{d}\lambda \; (-H_{,\alpha i} \; Q^{\alpha}_{\kappa} - H^{,\alpha}_{,i} \; P_{\alpha\kappa}) \quad . \tag{51}$$

The matrix $P_{\alpha\kappa}$ of the transformation from geodesic coordinates to the components of the slowness vector is calculated, together with the matrix Q^{α}_{κ} of geometrical spreading, by dynamic ray tracing (37).

5.5. Second–order perturbation derivatives of proper time

The second–order perturbation derivatives of proper time can be calculated by quadrature

$$\tau_{,ij}(\lambda) = \tau_{,ij}(\lambda_0) + \int_{\lambda_0}^{\lambda} \mathrm{d}\lambda \; K_{ij} \quad , \tag{52}$$

where the integration kernel is given by

$$K_{ij} = -H_{,ij} - T_{\kappa i} \; Q^{-1}_{\kappa\rho} H^{,\rho}_{,j} - T_{\kappa j} \; Q^{-1}_{\kappa\rho} H^{,\rho}_{,i} - T_{\kappa i} \; Q^{-1}_{\kappa\rho} T_{\lambda j} \; Q^{-1}_{\lambda\sigma} H^{,\rho\sigma} \quad . \tag{53}$$

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