Decomposition of the wave field into optimized Gaussian packets

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Summary

The decomposition of the wave field into optimized Gaussian packets is of key importance in the Gaussian packet migration. We present the theoretical basics and a set of numerical examples of our method.

Key words

Gaussian packet, Gabor function, coherent-state transform, prestack depth migration, common-shot gather.

1 Introduction

Our long-term project is to explore the properties of a depth migration method based on Gaussian packets. Gaussian packets are waves whose envelopes at any given time are nearly Gaussian in space. They represent high-frequency asymptotic solutions of the elastodynamic equation, which are concentrated close to the central point of the packet (e.g., Babich & Ulin, 1981; Ralston, 1983; Klimeš, 1989a, 2004).

The main advantage of the Gaussian packet migration over the methods based on Gaussian beams is a direct relation between the regions in the common-shot gather and corresponding localized regions in the migrated section.

Like Gaussian beams, Gaussian packets become inaccurate solutions of the elastodynamic equation if they spread excessively as they propagate. This spreading depends on the complexity of the velocity model and on the initial shape of Gaussian packets. Thus, we should use sufficiently smooth velocity model and optimize the initial shape of the packets.

The Gaussian packet migration algorithm consists of four basic steps:
a) Preparation of a suitable smooth velocity model (Žáček, 2002).
b) Optimization of the shape of Gaussian packets (Klimeš, 1989b; Žáček, 2001a, 2001b).
c) Decomposition of the wave field into optimized Gaussian packets, which was already presented by Žáček (2003), and which we thoroughly describe in this paper. The word *optimized* implies that the shape of Gaussian packets, in the plane perpendicular to the central ray of the packet, depends not only on the frequency, but also on the coordinate of the intersection of the central ray of a Gaussian packet with the profile, on its arrival time, and on the component of the slowness vector along the profile.
d) Back-propagation of the wave field using Gaussian packets and application of the imaging functional (Žáček, 2004, 2005).

The algorithm of the decomposition is designed here especially for numerical testing on the 2-D Marmousi data set (Versteeg & Grau, 1991). That is why only 2-D case is dealt with in this paper, although the generalization to 3-D is straightforward.

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2 Decomposition

We would like to approximate the wave field (common-shot gather) $f(x,t)$ in the form of

$$\widetilde{f}(x',t') = \int \mathrm{d}t_{\mathrm{R}} \int \mathrm{d}\omega \, \exp\left[-\mathrm{i}\omega(t'-t_{\mathrm{R}})\right] \int \mathrm{d}x_{\mathrm{R}} \int \mathrm{d}p \, \exp\left[\mathrm{i}\omega p(x'-x_{\mathrm{R}})\right]$$

$$\times\, W(x'-x_{\mathrm{R}}, t'-t_{\mathrm{R}}, x_{\mathrm{R}}, p, t_{\mathrm{R}}, \omega)\; F(x_{\mathrm{R}}, p, t_{\mathrm{R}}, \omega) \quad , \tag{1}$$

where x_{R} is the coordinate of the intersection of the central ray of a Gaussian packet with the profile, t_{R} denotes corresponding arrival time of a Gaussian packet, p is the component of the slowness vector a Gaussian packet along the profile, and ω is the positive circular frequency of a Gaussian packet. The complex-valued function W determines the shape of the envelope of a Gaussian packet, and the complex-valued function $F(x_{\mathrm{R}}, p, t_{\mathrm{R}}, \omega)$ specifies the amplitude of a Gaussian packet. Hereafter, the limits of integration with respect to x_{R}, t_{R}, x, t and p are $-\infty$ and $+\infty$, and the limits of integration with respect to ω are 0 and $+\infty$.

Let us express the amplitude of a Gaussian packet in the form of an integral transform similar to the forward 2-D *coherent-state transform*,

$$F(x_{\mathrm{R}}, p, t_{\mathrm{R}}, \omega) = \frac{\omega}{2\pi^2} \int \mathrm{d}x \, \exp\left[-\mathrm{i}\omega p(x-x_{\mathrm{R}})\right] \int \mathrm{d}t \, \exp\left[\mathrm{i}\omega(t-t_{\mathrm{R}})\right]$$

$$\times\, w(x_{\mathrm{R}}-x, t_{\mathrm{R}}-t, x_{\mathrm{R}}, p, t_{\mathrm{R}}, \omega)\; f(x,t) \quad , \tag{2}$$

where w denotes a complex-valued analyzing function. Then, equation (1) can be expressed as

$$\widetilde{f}(x',t') = \int \mathrm{d}t \int \mathrm{d}\omega \, \exp\left[\mathrm{i}\omega(t-t')\right] \int \mathrm{d}x \int \mathrm{d}p \, \exp\left[-\mathrm{i}\omega p(x-x')\right]$$

$$\times\, \frac{\omega}{2\pi^2}\; f(x,t)\; I(x'-x, t'-t, p, \omega) \quad , \tag{3}$$

where

$$I(x'-x, t'-t, p, \omega)$$

$$= \int \mathrm{d}t_{\mathrm{R}} \int \mathrm{d}x_{\mathrm{R}} \; W(x'-x_{\mathrm{R}}, t'-t_{\mathrm{R}}, x_{\mathrm{R}}, p, t_{\mathrm{R}}, \omega)\; w(x_{\mathrm{R}}-x, t_{\mathrm{R}}-t, x_{\mathrm{R}}, p, t_{\mathrm{R}}, \omega) \quad . \tag{4}$$

Considering the convolution (4) independent of p, integration with respect to p in (3) yields a Dirac distribution $\delta(x'-x)$,

$$\widetilde{f}(x',t') = \int \mathrm{d}t \int \mathrm{d}\omega \, \exp\left[\mathrm{i}\omega(t-t')\right] \int \mathrm{d}x \; \delta(x'-x)\; \frac{1}{\pi}\; f(x,t)\; I(x'-x, t'-t, 0, \omega) \quad , \tag{5}$$

which can be integrated with respect to x,

$$\widetilde{f}(x',t') = \int \mathrm{d}t \int \mathrm{d}\omega \, \exp\left[\mathrm{i}\omega(t-t')\right] \; \frac{1}{\pi}\; f(x',t)\; I(0, t'-t, 0, \omega) \quad . \tag{6}$$

Considering $I(0,t'-t,0,\omega)$ independent of ω, integration with respect to ω in (6) yields a complex-valued analytic Dirac distribution $\delta(t'-t) - \frac{\mathrm{i}}{\pi(t'-t)}$. Moreover, let us consider real-valued $I(0,t'-t,0,\omega)$, which leads to

$$\mathrm{Re}\left(\widetilde{f}(x',t')\right) = \int \mathrm{d}t \; \delta(t'-t)\; f(x',t)\; I(0, t'-t, 0, \omega_0) \quad , \tag{7}$$

which can be integrated with respect to t,

$$\mathrm{Re}\left(\widetilde{f}(x',t')\right) = f(x',t')\ I(0,0,0,\omega_0) \quad . \tag{8}$$

For

$$I(0,0,0,\omega_0) = 1 \quad , \tag{9}$$

the transformation would be exact,

$$\mathrm{Re}\left(\widetilde{f}(x,t)\right) = f(x,t) \quad . \tag{10}$$

The envelope of a trace of a Gaussian packet along the profile may be expressed as

$$W(x'-x_\mathrm{R},t'-t_\mathrm{R},x_\mathrm{R},p,t_\mathrm{R},\omega) = \exp\left[\mathrm{i}\omega\frac{1}{2}(\mathbf{y}'-\mathbf{y}_\mathrm{R})^\mathrm{T}\mathbf{K}(\mathbf{y}'-\mathbf{y}_\mathrm{R})\right] \quad , \tag{11}$$

where

$$\mathbf{y}' = \begin{pmatrix} x' \\ t' \end{pmatrix} \quad , \tag{12}$$

$$\mathbf{y}_\mathrm{R} = \begin{pmatrix} x_\mathrm{R} \\ t_\mathrm{R} \end{pmatrix} \tag{13}$$

and

$$\mathbf{K} = N \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + N_{44} \begin{pmatrix} p^2 & -p \\ -p & 1 \end{pmatrix} \quad . \tag{14}$$

Please note that parameters N and N_{44} correspond to N^0 and N^0_{44}, respectively, in Žáček (2005, eq. 65).

Let us write the analyzing function w in an analogous form,

$$w(x'-x_\mathrm{R},t'-t_\mathrm{R},x_\mathrm{R},p,t_\mathrm{R},\omega) = \widetilde{a}\ \exp\left[\mathrm{i}\omega\frac{1}{2}(\mathbf{y}-\mathbf{y}_\mathrm{R})^\mathrm{T}\widetilde{\mathbf{K}}(\mathbf{y}-\mathbf{y}_\mathrm{R})\right] \quad , \tag{15}$$

where

$$\mathbf{y} = \begin{pmatrix} x \\ t \end{pmatrix} \tag{16}$$

and

$$\widetilde{\mathbf{K}} = \widetilde{N} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \widetilde{N}_{44} \begin{pmatrix} \widetilde{p}^2 & -\widetilde{p} \\ -\widetilde{p} & 1 \end{pmatrix} \quad . \tag{17}$$

The meaning and choice of parameters $\widetilde{a}$, $\widetilde{p}$, $\widetilde{N}$ and $\widetilde{N}_{44}$ is discussed below.

Neglecting the dependence of parameters N, N_{44}, $\widetilde{a}$, $\widetilde{p}$, $\widetilde{N}$ and $\widetilde{N}_{44}$ on x_R and t_R, we may calculate the convolution (4),

$$\begin{aligned} & I(x'-x,t'-t,p,\omega) \\ & \quad = \widetilde{a}\int \mathrm{d}t_\mathrm{R} \int \mathrm{d}x_\mathrm{R}\ \exp\left[\mathrm{i}\omega\frac{1}{2}\left((\mathbf{y}'-\mathbf{y}_\mathrm{R})^\mathrm{T}\mathbf{K}(\mathbf{y}'-\mathbf{y}_\mathrm{R}) + (\mathbf{y}-\mathbf{y}_\mathrm{R})^\mathrm{T}\widetilde{\mathbf{K}}(\mathbf{y}-\mathbf{y}_\mathrm{R})\right)\right] \quad , \end{aligned} \tag{18}$$

which leads to

$$
\begin{aligned}
&I(x'-x,t'-t,p,\omega) \\
&\quad = \frac{2\pi\widetilde{a}}{\omega\sqrt{\det[-\mathrm{i}(\mathbf{K}+\widetilde{\mathbf{K}})]}} \exp\left[\mathrm{i}\omega\frac{1}{2}\left((\mathbf{y}'-\mathbf{y})^{\mathrm{T}}(\mathbf{K}^{-1}+\widetilde{\mathbf{K}}^{-1})^{-1}(\mathbf{y}'-\mathbf{y})\right)\right] \quad . \qquad (19)
\end{aligned}
$$

Inverse matrices to (14) and (17) read

$$\mathbf{K}^{-1} = \frac{1}{N_{44}}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{N}\begin{pmatrix} 1 & p \\ p & p^2 \end{pmatrix} \qquad (20)$$

and

$$\widetilde{\mathbf{K}}^{-1} = \frac{1}{\widetilde{N}_{44}}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{\widetilde{N}}\begin{pmatrix} 1 & \widetilde{p} \\ \widetilde{p} & \widetilde{p}^2 \end{pmatrix} \quad . \qquad (21)$$

In order to proceed from (3) to (5) and (6), we require $I(x'-x,t'-t,p,\omega)$ to be independent of p. Therefore, the sum $\mathbf{K}^{-1}+\widetilde{\mathbf{K}}^{-1}$ of matrices (20) and (21) in (19) should be independent of p. To proceed from (6) to (7), we need $I(0,t'-t,0,\omega)$ to be independent of ω. This means that the rightmost diagonal component of matrix $(\mathbf{K}^{-1}+\widetilde{\mathbf{K}}^{-1})^{-1}$ in (19) should be proportional to ω^{-1}.

To meet these requirements, we choose the sum of matrices (20) and (21) in the form of

$$\mathbf{K}^{-1}+\widetilde{\mathbf{K}}^{-1} = \begin{pmatrix} \frac{2}{N_0} & 0 \\ 0 & \frac{\omega}{K_0}+\frac{\omega}{\widetilde{K}_0} \end{pmatrix} \quad , \qquad (22)$$

where N_0, K_0 and $\widetilde{K}_0$ are constants. We introduce two generally different constants K_0 and $\widetilde{K}_0$, because we would like to choose $N_{44}=N_{44}(K_0)$ and $\widetilde{N}_{44}=\widetilde{N}_{44}(\widetilde{K}_0)$. Inserting expressions (20) and (21) into equation (22), we obtain three conditions,

$$\frac{1}{N}+\frac{1}{\widetilde{N}} = \frac{2}{N_0} \quad , \qquad (23)$$

$$\frac{p}{N}+\frac{\widetilde{p}}{\widetilde{N}} = 0 \qquad (24)$$

and

$$\frac{p^2}{N}+\frac{1}{N_{44}}+\frac{\widetilde{p}^2}{\widetilde{N}}+\frac{1}{\widetilde{N}_{44}} = \frac{\omega}{K_0}+\frac{\omega}{\widetilde{K}_0} \quad , \qquad (25)$$

where N_0, K_0 and $\widetilde{K}_0$ are free complex-valued constants that must be selected.

In 2-D, the initial shape of a Gaussian packet is defined by two complex-valued parameters — N and K_0. The imaginary part of N determines the Gaussian packet width along the profile and the real part of N defines the curvature of the phase-front. We choose parameter K_0, which describes the shape of a Gaussian packet along the central ray, and which is connected with parameter N_{44} by equation (25), pure imaginary and uniform for all Gaussian packets.

In order to minimize the spreading of Gaussian packets, parameter N should be determined by the optimization of the shape of corresponding Gaussian beams (Klimeš, 1989b; Žáček, 2001a, 2001b). Consequently, we should consider *optimized* Gaussian packets,

$$N = N(x_{\mathrm{R}},t_{\mathrm{R}},p) \quad . \qquad (26)$$

In this place, we must point out that the transformation is exact,

$$\widetilde{f}(x,t) = f(x,t) \quad , \tag{27}$$

only for N independent of x_{R} and t_{R}, see equations (18) and (19). Nevertheless, for a reasonably smooth distribution of N with respect to x_{R} and t_{R}, we could achieve satisfactory results. Therefore, we have developed a procedure, which allows us to smooth iteratively the distribution of parameter N on a *Hamiltonian hypersurface* in the phase-space (Žáček, 2001a, 2001b).

Using equations (23) and (24), we may write

$$\widetilde{N} = \frac{NN_0}{2N - N_0} \tag{28}$$

and

$$\widetilde{p} = -\frac{\widetilde{N}}{N}p \quad . \tag{29}$$

Please note that parameter p is real-valued, but $\widetilde{p}$ is generally complex-valued.

Equation (25) may be rearranged in several ways to determine parameters N_{44} and $\widetilde{N}_{44}$. For instance, we can get two symmetrical expresions,

$$\widetilde{N}_{44} = \frac{\widetilde{N}\widetilde{K}_0}{\omega\widetilde{N} - \widetilde{p}^2\widetilde{K}_0} \tag{30}$$

and

$$N_{44} = \frac{NK_0}{\omega N - p^2 K_0} \quad . \tag{31}$$

Then, pure imaginary constants $\widetilde{K}_0$ and K_0 must obey

$$\mathrm{Im}(\widetilde{K}_0) < \mathrm{Im}\left(\frac{\omega\widetilde{N}}{\widetilde{p}^2}\right) \tag{32}$$

and

$$\mathrm{Im}(K_0) < \mathrm{Im}\left(\frac{\omega N}{p^2}\right) \quad . \tag{33}$$

Another approach would allow the use of the fast Fourier transform (FFT) in the decomposition (but not in the re-composition) of the wave field,

$$\widetilde{N}_{44} = \frac{\widetilde{K}_0}{\omega} \quad , \tag{34}$$

$$N_{44} = \left(\frac{\omega}{K_0} - \frac{p^2}{N} - \frac{\widetilde{p}^2}{\widetilde{N}}\right)^{-1} \tag{35}$$

and

$$\mathrm{Im}(K_0) < \mathrm{Im}\left(\frac{\omega\,(2N - N_0)}{p^2}\right) \quad . \tag{36}$$

Equation (9) requires

$$\widetilde{a} = \frac{\omega}{2\pi}\sqrt{\det[-\mathrm{i}(\mathbf{K} + \widetilde{\mathbf{K}})]} \quad , \tag{37}$$

where matrix $\mathbf{K}$ is given by (14) and matrix $\widetilde{\mathbf{K}}$ by (17). Equation (37) with (14) and (17) can be converted using equations (29) and (25) into

$$\widetilde{a} = \frac{\omega}{2\pi}\sqrt{-\mathrm{i}(N+\widetilde{N})}\sqrt{-\mathrm{i}N_{44}\widetilde{N}_{44}\left(\frac{\omega}{K_0}+\frac{\omega}{\widetilde{K}_0}\right)} \quad , \tag{38}$$

where the square roots should be taken with positive real parts.

3 Discretization

A proper choice of the discretization of integral (1) is not only of key importance in the decomposition of the wave field into Gaussian packets, but it also affects the accuracy and efficiency of the Gaussian packet migration.

Let us consider a 2-D integral

$$\int \mathrm{d}\xi_1 \int \mathrm{d}\xi_2 \, \exp\left(\mathrm{i}\,\xi_J D_{JK}\xi_K\right) \tag{39}$$

discretized on a regular rectangular grid with intervals $\Delta\xi_1$ and $\Delta\xi_2$. Sufficient conditions for the discretization were derived by Klimeš (1986),

$$\Delta\xi_1 \le \kappa\sqrt{-[\mathrm{Im}(\mathbf{D}^{-1})]_{11}} \tag{40}$$

and

$$\Delta\xi_2 \le \kappa\sqrt{-[\mathrm{Im}(\mathbf{D}^{-1})]_{22}} \quad . \tag{41}$$

Matrix $\mathbf{D}$ must satisfy

$$\left|[\mathrm{Im}(\mathbf{D}^{-1})]_{12}\right| \le \frac{1}{2}\sqrt{[\mathrm{Im}(\mathbf{D}^{-1})]_{11}[\mathrm{Im}(\mathbf{D}^{-1})]_{22}} \quad . \tag{42}$$

The error of discretization depends on the properties of the wave field. For a simple wave field with linear amplitudes and quadratic arrival times, the relative maximum discretization error is about

$$\delta_{\mathrm{MAX}} = 4\,\exp(-\pi^2\kappa^{-2}) \quad . \tag{43}$$

Then, the maximum relative error for $\kappa^2=\frac{\pi}{2}$ would be about 0.7 %. Unfortunately, the error may raise rapidly for a more complex wave field. In the worst case, the maximum relative error could reach up to 21 % for $\kappa^2=\frac{\pi}{2}$, see Table 3.3 in Daubechies (1992). Nevertheless, we believe that a reasonable value of κ for a realistic wave field would be close to $\kappa^2=\frac{\pi}{2}$.

The error of discretization of integral (18) with respect to x_{R} and t_{R} is controlled by matrix

$$\mathbf{D} = \frac{1}{2}\omega(\mathbf{K}+\widetilde{\mathbf{K}}) \quad . \tag{44}$$

According to conditions (40) and (41), maximum step in x_{R} for replacement of integral (18) by its discretized counterpart is

$$\Delta x_{\mathrm{R}} = \kappa\sqrt{-\frac{2}{\omega}\left[\mathrm{Im}\left((\mathbf{K}+\widetilde{\mathbf{K}})^{-1}\right)\right]_{11}} \quad , \tag{45}$$

maximum step in t_{R} is

$$\Delta t_{\mathrm{R}} = \kappa\sqrt{-\frac{2}{\omega}\left[\mathrm{Im}\left((\mathbf{K}+\widetilde{\mathbf{K}})^{-1}\right)\right]_{22}} \quad , \tag{46}$$

and we require

$$\left|\left[\mathrm{Im}\left((\mathbf{K}+\widetilde{\mathbf{K}})^{-1}\right)\right]_{12}\right| \le \frac{1}{2}\sqrt{\left[\mathrm{Im}\left((\mathbf{K}+\widetilde{\mathbf{K}})^{-1}\right)\right]_{11}\left[\mathrm{Im}\left((\mathbf{K}+\widetilde{\mathbf{K}})^{-1}\right)\right]_{22}} \quad . \tag{47}$$

Equation (3) can be understood as a convolution of functions

$$J(x,t,p,\omega) = \frac{\omega}{2\pi^2} I(x,t,p,\omega)\, \exp(\mathrm{i}\omega p x)\, \exp(-\mathrm{i}\omega t) \tag{48}$$

and $f(x,t)$ with respect to x and t, integrated over p and ω. Fourier transform $\widetilde{f}(k',\omega')$ of $\widetilde{f}(x',t')$ is then the product of Fourier transforms $J(k',\omega',p,\omega)$ and $f(k',\omega')$, integrated over p and ω. Inserting equation (19) into (48) and using the Fourier transform, we obtain $J(k',\omega',p,\omega)$ in the form similar to (39) with $\xi_1 = k'$ and $\xi_2 = \omega'$. The error of discretization of the integral of $J(k',\omega',p,\omega)$ with respect to $k = \omega p$ and ω is then controlled by matrix

$$\mathbf{D} = -\frac{1}{2}\omega^{-1}(\mathbf{K}^{-1} + \widetilde{\mathbf{K}}^{-1}) \quad . \tag{49}$$

According to conditions (40) and (41), maximum step in p is

$$\Delta p = \kappa\sqrt{\frac{2}{\omega}\left[\mathrm{Im}\left((\mathbf{K}^{-1}+\widetilde{\mathbf{K}}^{-1})^{-1}\right)\right]_{11}} \quad , \tag{50}$$

maximum step in ω is

$$\Delta\omega = \kappa\sqrt{2\omega\left[\mathrm{Im}\left((\mathbf{K}^{-1}+\widetilde{\mathbf{K}}^{-1})^{-1}\right)\right]_{22}} \quad , \tag{51}$$

and we require

$$\left|\left[\mathrm{Im}\left((\mathbf{K}^{-1}+\widetilde{\mathbf{K}}^{-1})^{-1}\right)\right]_{12}\right| \le \frac{1}{2}\sqrt{\left[\mathrm{Im}\left((\mathbf{K}^{-1}+\widetilde{\mathbf{K}}^{-1})^{-1}\right)\right]_{11}\left[\mathrm{Im}\left((\mathbf{K}^{-1}+\widetilde{\mathbf{K}}^{-1})^{-1}\right)\right]_{22}} \quad . \tag{52}$$

Using conditions (23), (24) and (25), we can simplify expressions (50) and (51) into

$$\Delta p = \kappa\sqrt{\frac{\mathrm{Im}(N_0)}{\omega}} \tag{53}$$

and

$$\Delta\omega = \kappa\sqrt{2\ \mathrm{Im}\left(\frac{K_0\widetilde{K}_0}{K_0+\widetilde{K}_0}\right)} \quad . \tag{54}$$

Presented discretization conditions are valid for a general smooth distribution of parameter N on a *Hamiltonian hypersurface* in the phase-space, see relation (26). However, to simplify the whole migration procedure, it might be useful to smooth the optimized distribution of parameter N to a constant value as described by Žáček (2001a, 2001b). Considering *uniform* Gaussian packets with $N = \widetilde{N} = N_0$, $K_0 = \widetilde{K}_0$, $\widetilde{N}_{44}$ given by relation (30) and N_{44} determined by (31), maximum discretization steps according to (45), (46), (53) and (54) read

$$\Delta x_{\mathrm{R}} = \frac{\kappa}{\omega}\sqrt{-\mathrm{Im}\left[(\omega N_0 - p^2 K_0)\, N_0^{-2}\right]} \quad , \tag{55}$$

$$\Delta t_{\mathrm{R}} = \kappa \sqrt{-\mathrm{Im}\left[\left(\omega N_0 - p^2 K_0\right) N_0^{-1} K_0^{-1}\right]} \quad , \tag{56}$$

$$\Delta p = \kappa \sqrt{\frac{\mathrm{Im}\left(N_0\right)}{\omega}} \tag{57}$$

and

$$\Delta \omega = \kappa \sqrt{\mathrm{Im}\left(K_0\right)} \quad . \tag{58}$$

Naturally, *uniform* Gaussian packets bring about more rapid spreading, which could jeopardize the accuracy of the migration.

4 Numerical examples

We tested our method on a common-shot gather belonging to the Marmousi data set (Versteeg & Grau, 1991), see Figure 1. We decomposed the gather $f(x,t)$ using equation (2) into individual Gabor functions, see Figure 2. The complex-valued amplitude of a Gaussian packet equals the amplitude of the corresponding Gabor function.

In order to examine the accuracy of our method, we composed the approximated common-shot gathers $\widetilde{f}(x,t)$ according to equation (1), see Figure 3. The differences between the original gather and the approximated gathers are shown in Figure 4. We calculated all the numerical examples with $\kappa^2 = \frac{\pi}{2}$.

In the the first example, we present the decomposition of the wave field into *uniform* Gaussian packets with $\mathrm{Re}(N_0) = -0.25 \times 10^{-6}$ sm^{-2} and $\mathrm{Im}(N_0) = 0.25 \times 10^{-6}$ sm^{-2}. See Figure 3a for the corresponding approximated common-shot gather. Figure 4a indicates that the transformation is almost exact.

The next example demostrates the decomposition of the wave field into Gaussian packets with varying parameter N. In this case, $\mathrm{Re}(N) = -0.25 \times 10^{-6}$ sm^{-2} and $\mathrm{Im}(N)$ ranges from 0.125×10^{-6} sm^{-2} to 0.5×10^{-6} sm^{-2}. The approximated gather is in Figure 3b. According to Figure 4b, the differences between the original and the approximated wave field can be spotted not only at the edges of the gather. This is caused by the dependency of parameter N on x_{R} and t_{R}.

The last example is the decomposition of the wave field with $\mathrm{Re}(N) = -0.25 \times 10^{-6}$ sm^{-2} and $\mathrm{Im}(N)$ varying from 0.75×10^{-7} sm^{-2} to 0.1×10^{-5} sm^{-2}. The approximated gather is displayed in Figure 3b. Although we can note several differences between the original and the approximated gather in Figure 4c, the result seems to be satisfactory.

The only serious problem of our method consists in the computational cost. Unfortunately, we cannot use 2-D FFT. The mathematical formulation of our task allows only a series of 1-D FFTs in the decomposition, but not in the re-composition of the wave field. In order to make the decomposition of the wave field more efficient, we are constantly trying to refine the algorithm and speed up the computations.

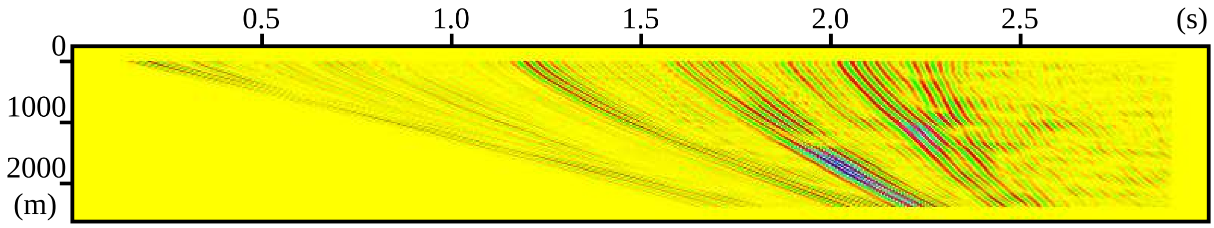

Figure 1. Original common-shot gather $f(x,t)$ of the Marmousi data set (Versteeg & Grau, 1991).

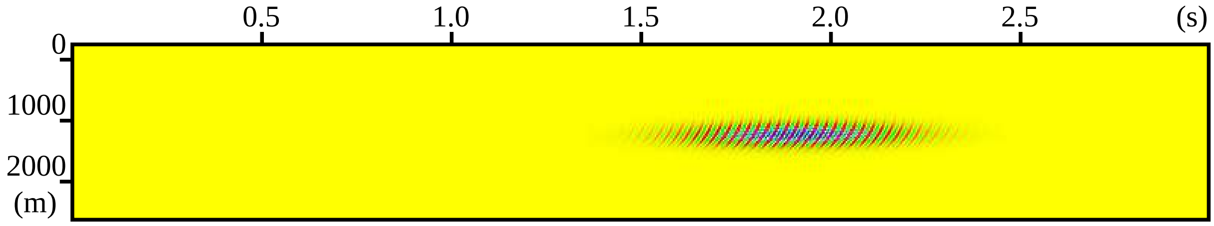

Figure 2. Gabor function.

5 Conclusions

The presented procedure of the decomposition of the wave field into Gaussian packets is fully functional. It might be suitable not only for Gaussian packet migration, but also for many different applications of the Gaussian packet method.

The choice of the discretization of the integral transform plays a crucial role in the decomposition of the wave field. It affects not only the accuracy, but also the efficiency of the method. We are able to control the discretization error of the decomposition into *uniform* Gaussian packets.

Decomposition of the wave field into Gaussian packets, whose shape depends on the coordinate of the intersection of the central ray of a Gaussian packet with the profile and on its arrival time, is strictly mathematically speaking incorrect. In order to achieve satisfactory results even in such a case, we smooth the distribution of the parameter determining the shape of a packet in the plane perpendicular to its central ray (Žáček, 2001a, 2001b).

The main disadvantage of our approach seems to be the computational cost of the decomposition. Nevertheless, once we obtain the amplitudes of Gaussian packets, we can calculate many migrated sections for various reasonably similar velocity models.

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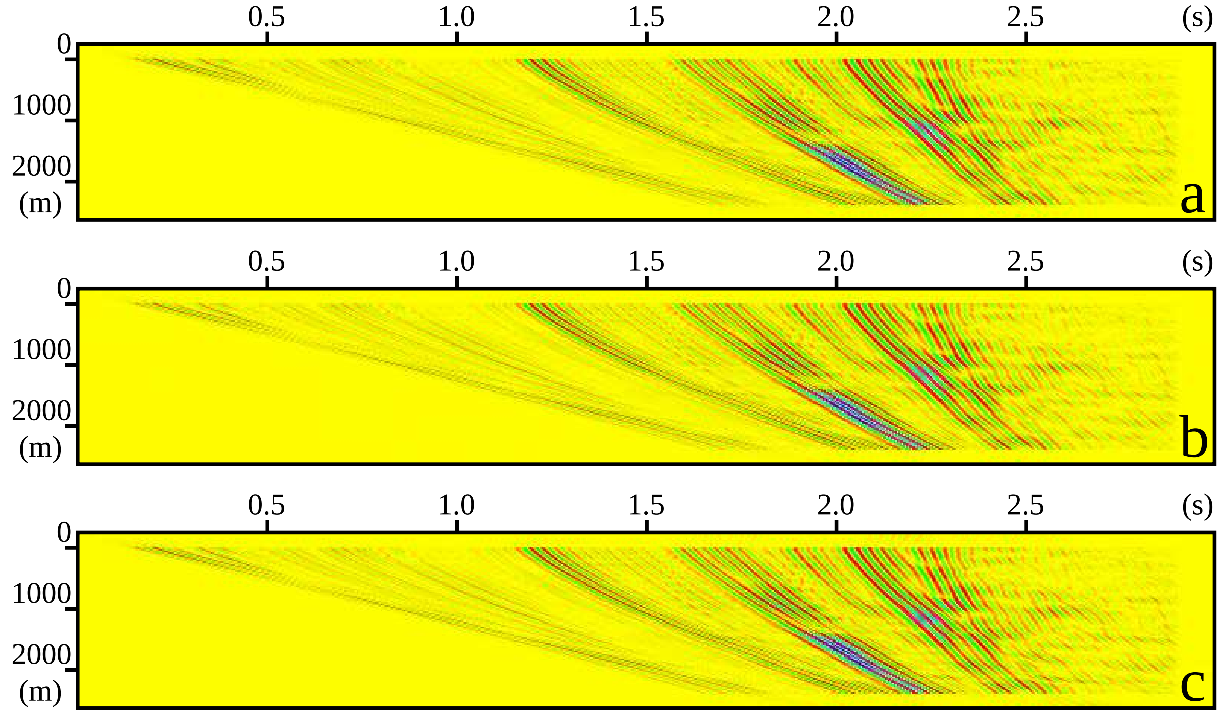

Figure 3. Approximated common-shot gathers $\widetilde{f}(x,t)$ for (a) *uniform* Gaussian packets with $\mathrm{Re}(N_0) = -0.25 \times 10^{-6}$ sm^{-2} and $\mathrm{Im}(N_0) = 0.25 \times 10^{-6}$ sm^{-2}; (b) Gaussian packets with $\mathrm{Re}(N) = -0.25 \times 10^{-6}$ sm^{-2}, and $\mathrm{Im}(N)$ ranging from 0.125×10^{-6} sm^{-2} to 0.5×10^{-6} sm^{-2}; (c) Gaussian packets with $\mathrm{Re}(N) = -0.25 \times 10^{-6}$ sm^{-2}, and $\mathrm{Im}(N)$ ranging from 0.75×10^{-7} sm^{-2} to 0.1×10^{-5} sm^{-2}.

Figure 4. Differences between the original common-shot gather $f(x,t)$ and the approximated common-shot gathers $\widetilde{f}(x,t)$ for (a) *uniform* Gaussian packets with $\mathrm{Re}(N_0) = -0.25 \times 10^{-6}$ sm^{-2} and $\mathrm{Im}(N_0) = 0.25 \times 10^{-6}$ sm^{-2}; (b) Gaussian packets with $\mathrm{Re}(N) = -0.25 \times 10^{-6}$ sm^{-2}, and $\mathrm{Im}(N)$ ranging from 0.125×10^{-6} sm^{-2} to 0.5×10^{-6} sm^{-2}; (c) Gaussian packets with $\mathrm{Re}(N) = -0.25 \times 10^{-6}$ sm^{-2}, and $\mathrm{Im}(N)$ ranging from 0.75×10^{-7} sm^{-2} to 0.1×10^{-5} sm^{-2}.

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