Reflection/transmission laws in viscoelastic anisotropic media

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Summary

The reflection/transmission laws of plane waves at a plane interface between two homogeneous anisotropic viscoelastic (dissipative) halfspaces are discussed. Algorithms for determining the slowness vectors of reflected/transmitted plane waves from the known slowness vector of the incident wave are proposed. In viscoelastic media, the slowness vectors are complex-valued, $\mathbf{p} = \mathbf{P} + \mathrm{i}\mathbf{A}$, where $\mathbf{P}$ is the propagation vector, and $\mathbf{A}$ the attenuation vector. The proposed algorithms may be applied to bulk plane waves ($\mathbf{A} = \mathbf{0}$), homogeneous plane waves ($\mathbf{P}$ and $\mathbf{A}$ parallel), and inhomogeneous plane waves ($\mathbf{P}$ and $\mathbf{A}$ non-parallel). The manner, in which the slowness vector is specified, plays an important role in the algorithms. For unrestricted anisotropy and viscoelasticity, the algorithms require an algebraic equation of the sixth degree to be solved in each halfspace. The degree of the algebraic equation decreases to four or two for simpler cases (isotropic media, plane waves in symmetry planes of anisotropic media). The physical consequences of the proposed algorithms are discussed in detail.

Keywords
Viscoelastic anisotropic media, reflection and transmission laws, Snell's law

1 Introduction

Consider a planar structural interface Σ between two homogeneous anisotropic viscoelastic halfspaces, and a time-harmonic plane wave incident from one halfspace at it. Denote by $\mathbf{p}^{inc}$ the slowness vector of the incident wave. This paper deals with the determination of slowness vectors $\mathbf{p}^r$ of reflected/transmitted waves from known $\mathbf{p}^{inc}$. Both $\mathbf{p}^r$ and $\mathbf{p}^{inc}$ may be complex-valued.

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For an interface between two perfectly elastic, isotropic media, the analytical solutions of the above formulated problem are well-known. It will be instructive to recapitulate briefly the well-known solutions for perfectly elastic isotropic media, and only then to treat the viscoelastic anisotropic media.

We denote by $\mathbf{n}$ the unit vector normal to interface Σ, oriented to any side of it. Denote also by i^{inc} the angle of incidence, which specifies the acute angle between $\mathbf{n}$ and the real-valued slowness vector of the incident wave $\mathbf{p}^{inc}$. Snell's law, in its classical form, relates the angle i^r of any reflected/transmitted wave to the angle of incidence i^{inc}. It reads

$$\sin i^{inc}/V^{inc} = \sin i^r/V^r \ . \tag{1}$$

The reflection/transmission (R/T) angle i^r is again defined as the acute angle between $\mathbf{n}$ and the slowness vector $\mathbf{p}^r$ of the R/T wave. Symbols V^{inc} and V^r denote the real-valued velocities of the incident wave and of the selected R/T wave, respectively. They may correspond to P wave velocities α, or S wave velocities β, depending on the modes of the incident and R/T waves being considered. For monotypic reflected waves ($P \to P, S \to S$), $V^{inc} = V^r$, and, consequently, $i^r = i^{inc}$.

Equation (1) is valid for any incident (P or S), reflected (P or S) and transmitted (P or S) waves. It actually expresses the fact that the tangential component of the slowness vector of any R/T wave generated at interface Σ is the same as the tangential component of the slowness vector of the incident wave at Σ. This is the well-known condition which must be satisfied at any structural interface.

For given i^{inc}, V^{inc} and V^r, angle i^r may be complex-valued. This occurs if $\sin i^{inc} > V^{inc}/V^r$ (postcritical incidence). The R/T plane wave corresponding to velocity V^r is then inhomogeneous, and its amplitudes decay exponentially in the direction perpendicular to Σ, away from the interface.

Snell's law is one of the basic laws in the seismic ray method in inhomogeneous, isotropic, layered media. Nevertheless, it may be very useful to consider it in a more general form, formulated at the beginning of this section, in which the slowness vector $\mathbf{p}^r$ of any R/T wave is to be computed from the slowness vector $\mathbf{p}^{inc}$ of incident wave. For isotropic perfectly elastic halfspaces, it is easy to derive the solution, both analytically and geometrically. It reads:

$$\mathbf{p}^r = \mathbf{p}^{inc} - \mathbf{n}\Big[(\mathbf{p}^{inc}\cdot\mathbf{n}) \mp \sqrt{(1/V^r)^2 - (1/V^{inc})^2 + (\mathbf{p}^{inc}\cdot\mathbf{n})^2}\Big] \ . \tag{2}$$

This equation has also been broadly used in the seismic ray method of laterally varying layered structures, see Červený, Molotkov and Pšenčík (1977, Eq. (3.7)), Červený (2001, Eq. (2.3.181)), Robein (2003, Eq. (1.34)), etc., and is included in most of the 2-D and 3-D ray tracing computer packages. Equation (2) can be applied directly to computing the slowness vectors of R/T waves at an arbitrarily oriented planar interface, from an arbitrarily oriented slowness vector of the incident wave. It is expressed in a suitable vectorial, coordinate independent form. Contrary to (1), Equation (2) yields the complete slowness vectors of R/T waves, including both their components: normal and tangential to Σ. Note that the cross-product of Equation (2) with $\mathbf{n}$ yields $\mathbf{p}^r \times \mathbf{n} = \mathbf{p}^{inc} \times \mathbf{n}$. This equation indicates that the tangential components of the slowness vectors of the incident and all R/T waves are equal, and is analogous to Equation (1). Equations for the normal components of the slowness vectors of R/T waves, contained in Equation (2), follow simply from (2), taking the scalar product of (2) with $\mathbf{n}$.

Let us return from perfectly elastic isotropic to viscoelastic anisotropic media. Formally, the expression for the slowness vector $\mathbf{p}^r$ of the R/T waves remains very similar to (2), only the square root in this expression must be replaced by a root of a complex-valued algebraic equation. For unrestricted anisotropy and viscoelasticity, the degree of the algebraic equation is six, see Section 4. The six roots correspond to individual qP, qS1 and qS2 waves, propagating in the halfspace under consideration outwards from the interface and towards to it. In isotropic media, see (2), the number of waves is reduced to four, due to the degeneracy of S waves.

Thus, in general, we consider twelve waves, six in either halfspace. (The incident wave is one of them.) The tangential components of the slowness vectors of all these waves must be the same. Consequently, if we wish to study some general properties of the R/T laws, we can study them using these general equations. They remain valid even if we do not specify the type of incident wave and the halfspace, from which the incident wave is approaching; it is actually sufficient to consider only the tangential component of the slowness vector of this wave.

Only if we wish to solve a specific case of a selected incident wave, the number of R/T waves must be reduced to six: three reflected and three transmitted. Only such R/T waves must be selected, which propagate outwards from the interface. Analogously, the incident wave under consideration must propagate towards interface Σ.

Thus, a proper selection criterion must be used to choose the R/T waves propagating outwards from Σ, and the incident wave propagating towards Σ correctly. For isotropic perfectly elastic media and real-valued $\mathbf{p}^r$, the selection criterion is simple: For reflected waves, the signs of $\mathbf{p}^{refl} \cdot \mathbf{n}$ and $\mathbf{p}^{inc} \cdot \mathbf{n}$ must be opposite, and for the transmitted waves, the signs of $\mathbf{p}^{transm} \cdot \mathbf{n}$ and $\mathbf{p}^{inc} \cdot \mathbf{n}$ must be the same. For anisotropic perfectly elastic media and real-valued $\mathbf{p}^r$, analogous selection criteria are based on the directions of the time-averaged energy flux, not on the directions of the slowness vector. For complex-valued $\mathbf{p}^r$ in perfectly elastic media (postcritical R/T waves), the selection criterion requires that the amplitudes of R/T waves to decay exponentially in the direction away from the interface. See Červený (2001, p. 51).

For viscoelastic media, however, the selection criteria are still a subject of research. In this paper, the selection criteria are not treated at all; we consider the slowness vectors of all 12 waves (qP, qS1 and qS2 waves, propagating towards Σ and outwards from Σ), which have the same tangential components of slowness vectors as the incident wave.

2 Plane waves in viscoelastic anisotropic media

Consider a time-harmonic plane wave propagating in a homogeneous viscoelastic anisotropic unbounded medium, specified by complex-valued density-normalised viscoelastic moduli

a_{ijkl} $(i,j,k,l = 1,2,3)$, satisfying the symmetry relations $a_{ijkl} = a_{jikl} = a_{ijlk} = a_{klij}$. Alternatively to a_{ijkl}, we also use the complex-valued density-normalised viscoelastic moduli $A_{\alpha\beta}$ $(\alpha,\beta = 1,...,6)$ in the Voigt notation, and assume that the 6×6 matrix $\mathrm{Re}(\mathbf{A})$ is positive definite, and the 6×6 matrix $\mathrm{Im}(\mathbf{A})$ negative definite, or zero. Any time-harmonic plane wave is given by the relation

$$u_j(x_k,t) = U_j \exp[-\mathrm{i}\omega(t - p_n x_n)] \ , \tag{3}$$

where x_n are Cartesian coordinates, t the running time and ω a fixed positive circular frequency. Further, u_j, p_j and U_j are Cartesian components of the complex-valued displacement vector $\mathbf{u}$, slowness vector $\mathbf{p}$ and polarization vector $\mathbf{U}$, respectively. It follows from the elastodynamic equation that (3) represents a plane wave only if the following constraint relations are satisfied:

$$\det[a_{ijkl}p_j p_l - \delta_{ik}] = 0 \ . \tag{4}$$

Slowness vector $\mathbf{p}$ is complex-valued,

$$\mathbf{p} = \mathbf{P} + \mathrm{i}\mathbf{A} \ , \tag{5}$$

where the real-valued propagation vector $\mathbf{P}$ is perpendicular to the plane of constant phase, and is oriented in the direction of propagation of the wavefront, and the real-valued attenuation vector $\mathbf{A}$ is perpendicular to the plane of constant amplitude, and oriented in the direction of maximum decay of amplitudes.

Propagation vector $\mathbf{P}$ and attenuation vector $\mathbf{A}$ cannot be chosen arbitrarily; they must satisfy the constraint relation (4). We now explain two specifications of the slowness vector, which will be useful in the discussion of the reflection/transmission laws: the componental specification and the mixed specification of the slowness vector.

In the **componental specification**, we consider a known reference plane Σ, with a real-valued unit vector $\mathbf{n}$ perpendicular to it, and along which the complex-valued tangential components $\mathbf{p}^\Sigma$ of the slowness vector are known. We wish to determine the unknown normal component $\mathbf{p}^n$ of the slowness vector, and, consequently, the complete slowness vector $\mathbf{p}$. Reference plane Σ may represent a structural interface, the surface of the Earth, or a formal plane in a homogeneous medium. The componental specification of slowness vector $\mathbf{p}$ then reads,

$$\mathbf{p} = \sigma\mathbf{n} + \mathbf{p}^\Sigma \ , \qquad \text{with } \mathbf{p}^\Sigma \cdot \mathbf{n} = 0 \ . \tag{6}$$

The unknown complex-valued quantity σ must be determined by inserting (6) into the constraint relation (4). Consider a halfspace bounded by Σ, described by complex-valued viscoelastic moduli a_{ijkl}. Inserting (6) into (4) yields an equation for σ:

$$\det[a_{ijkl}(\sigma n_j + p_j^\Sigma)(\sigma n_l + p_l^\Sigma) - \delta_{ik}] = 0 \ . \tag{7}$$

Given a_{ijkl}, n_j and p_j^Σ, Equation (7) represents a complex-valued algebraic equation of the sixth degree. Consequently, six complex-valued roots are obtained. They correspond to the qP, qS1 and qS2 waves, propagating in the halfspace under consideration, towards

Σ and outwards from Σ. All these six waves have the same tangential components $\mathbf{p}^\Sigma$ of the slowness vectors at reference plane Σ.

The situation in the second halfspace bounded by Σ is quite analogous but appropriate a_{ijkl} must be used. Again, we obtain six roots, corresponding to the qP, qS1 and qS2 waves, propagating in this halfspace, towards and outwards from Σ. All these waves again have the same tangential component $\mathbf{p}^\Sigma$ of the slowness vectors at Σ.

Once the six roots σ in the halfspace under consideration have been found, the relevant real-valued propagation vector $\mathbf{P}$, real-valued attenuation vector $\mathbf{A}$, unit propagation vector $\mathbf{N}$, unit attenuation vector $\mathbf{M}$, phase velocity $\mathcal{C}$ and attenuation angle γ ($\cos\gamma = \mathbf{N}\cdot\mathbf{M}$) can be determined from (6)–(7):

$$\begin{array}{ll}
\mathbf{P} = \mathbf{n}\,(\mathrm{Re}\sigma) + \mathrm{Re}\mathbf{p}^\Sigma \ , & \mathbf{A} = \mathbf{n}\,(\mathrm{Im}\sigma) + \mathrm{Im}\mathbf{p}^\Sigma \ , \\
|\mathbf{P}| = [(\mathrm{Re}\sigma)^2 + (\mathrm{Re}\mathbf{p}^\Sigma)(\mathrm{Re}\mathbf{p}^\Sigma)]^{1/2} \ , & |\mathbf{A}| = [(\mathrm{Im}\sigma)^2 + (\mathrm{Im}\mathbf{p}^\Sigma)(\mathrm{Im}\mathbf{p}^\Sigma)]^{1/2} \ , \\
\mathbf{N} = \mathbf{P}/|\mathbf{P}| \ , & \mathbf{M} = \mathbf{A}/|\mathbf{A}| \ , \\
\cos\gamma = [(\mathrm{Re}\mathbf{p}^\Sigma)(\mathrm{Im}\mathbf{p}^\Sigma) + (\mathrm{Re}\sigma)(\mathrm{Im}\sigma)]/|\mathbf{P}||\mathbf{A}| \ , & \mathcal{C} = 1/|\mathbf{P}| \ .
\end{array} \tag{8}$$

The plane wave with $\mathbf{A} = \mathbf{0}$ (real-valued slowness vector) is referred to here as the bulk plane wave, the plane wave with parallel $\mathbf{P}$ and $\mathbf{A}$ as the homogeneous plane wave, and the plane wave with non-parallel $\mathbf{P}$ and $\mathbf{A}$ as the inhomogeneous plane wave.

The mixed specification is a special case of the componental specification, in which reference plane Σ is a wavefront (T=const.). As $\mathrm{Re}(\mathbf{p}^\Sigma)$ vanishes along wavefront Σ, the unit propagation vectors $\mathbf{N}$ of all six waves equal $\pm\mathbf{n}$. Thus, all the plane waves propagating in a selected direction $\pm\mathbf{n}$ can be studied. The reader is reminded that the propagation directions of the six plane waves are mutually different, differ from $\mathbf{n}$ and are not known in advance in the componental specification.

As $\mathbf{p}^\Sigma$ is purely imaginary, we can introduce it by a simple relation:

$$\mathbf{p}^\Sigma = \mathrm{i}\mathbf{d} \ , \qquad \text{with } \ \mathbf{d}\cdot\mathbf{n} = 0 \ . \tag{9}$$

Here $\mathbf{d}$ is a real-valued unit vector, situated in the wavefront, and called the wavefront attenuation vector. Moreover, $d = |\mathbf{d}|$ is the nonnegative **inhomogeneity strength**. The mixed specification of the slowness vector then reads

$$\mathbf{p} = \sigma\mathbf{n} + \mathrm{i}\mathbf{d} \ , \qquad \text{with } \ \mathbf{d}\cdot\mathbf{n} = 0 \ . \tag{10}$$

The unknown complex-valued quantity σ must again be determined from the constraint relation (4):

$$\det[a_{ijkl}(\sigma n_j + \mathrm{i}d_j)(\sigma n_l + \mathrm{i}d_j) - \delta_{ik}] = 0 \ . \tag{11}$$

Given a_{ijkl}, n_j, and d_j, Equation (11) again yields six complex-valued roots, which correspond to qP, qS1 and qS2, propagating in the direction of $\mathbf{N} = \pm\mathbf{n}$ (towards and outwards from Σ).

Any choice of mutually perpendicular vectors $\mathbf{n}$ and $\mathbf{d}$ specifies the propagation-attenuation half-plane $\Sigma^{||}$, which contains both propagation vector $\mathbf{P}$ and attenuation vector $\mathbf{A}$. The unit propagation vector $\mathbf{N} = \pm\mathbf{n}$ is the same (or opposite) for all waves, and the attenuation vector is related to $\mathbf{d}$ as follows: $\mathbf{d} = \mathbf{n} \times (\mathbf{A} \times \mathbf{n})$.

Once quantities σ have been determined, the equations analogous to (8) read:

$$
\begin{array}{ll}
\mathbf{P} = \mathbf{n}\,\mathrm{Re}\sigma \ , & \mathbf{A} = \mathbf{n}\,\mathrm{Im}\sigma + \mathbf{d} \\
|\mathbf{P}| = |\mathrm{Re}\sigma| \ , & |\mathbf{A}| = [(\mathrm{Im}\sigma)^2 + d^2]^{1/2} \\
\mathbf{N} = \epsilon\,\mathbf{n} \ . & \mathbf{M} = \mathbf{A}/|\mathbf{A}| \ . \\
\mathcal{C} = 1/|\mathrm{Re}\sigma| \ , & \cos\gamma = \epsilon\,\mathrm{Im}\sigma/[(\mathrm{Im}\sigma)^2 + d^2]^{1/2} \\
\mathbf{d} = \mathbf{N} \times (\mathbf{A} \times \mathbf{N}) \ , & \epsilon = \mathrm{Re}\sigma/|\mathrm{Re}\sigma| = \pm 1 \ .
\end{array}
\tag{12}
$$

3 R/T laws: a general version

Again consider reference plane Σ, the normal unit vector $\mathbf{n}$ perpendicular to Σ, oriented to either side of Σ, and the slowness vector of incident wave $\mathbf{p}^{inc}$. We assume that $\mathbf{p}^{inc}$ satisfy the constraint relation (4), where a_{ijkl} correspond to the appropriate halfspace. Here we do not specify the slowness vector of the incident wave in more detail; the incident wave may propagate in any of the two halfspaces. We wish to determine the slowness vectors $\mathbf{p}^r$ of reflected/transmitted waves in the halfspace, described by the density-normalised viscoelastic moduli a^r_{ijkl}. The procedure in the second halfspace would be quite analogous.

First we determine $\mathbf{p}^\Sigma$, the component of $\mathbf{p}^{inc}$ into reference plane Σ:

$$
\mathbf{p}^\Sigma = \mathbf{n} \times (\mathbf{p}^{inc} \times \mathbf{n}) = \mathbf{p}^{inc} - \mathbf{n}(\mathbf{p}^{inc} \cdot \mathbf{n}) \ . \tag{13}
$$

Using (6) and (7), we obtain the slowness vector $\mathbf{p}^r$ of the reflected/transmitted wave:

$$
\begin{aligned}
\mathbf{p}^r &= \sigma^r\mathbf{n} + \mathbf{n} \times (\mathbf{p}^{inc} \times \mathbf{n}) \\
&= \mathbf{p}^{inc} - \mathbf{n}[\mathbf{p}^{inc} \cdot \mathbf{n} - \sigma^r] \ .
\end{aligned}
\tag{14}
$$

Here $\sigma^r = \sigma^r(a^r_{aijkl}, \mathbf{p}^{inc}, \ , \mathbf{n})$ is any of the six roots of the algebraic equation (7), in which a^r_{ijkl} have been substituted for a_{ijkl},

$$
\det[a^r_{ijkl}(\sigma^r n_j + p^\Sigma_j)(\sigma^r n_l + p^\Sigma_l) - \delta_{ik}] = 0 \ , \tag{15}
$$

and $\mathbf{p}^\Sigma$ is given by (13).

We separate the real-valued and imaginary-valued parts of (14), and obtain:

$$
\begin{aligned}
\mathbf{P}^r &= \mathbf{P}^{inc} - \mathbf{n}[(\mathbf{P}^{inc} \cdot \mathbf{n}) - \mathrm{Re}\sigma^r] \ , \\
\mathbf{A}^r &= \mathbf{A}^{inc} - \mathbf{n}[(\mathbf{A}^{inc} \cdot \mathbf{n}) - \mathrm{Im}\sigma^r] \ .
\end{aligned}
\tag{16}
$$

Equations (14) and (16), with (13) and (15), are quite general. They are valid for isotropic and anisotropic, viscoelastic and perfectly elastic media. They are valid for bulk plane waves, homogeneous and inhomogeneous plane waves, for reflected and transmitted waves (we only specify properly the viscoelastic moduli a^r_{ijkl} in the halfspace under consideration), and for subcritically and postcritically R/T waves.

For different approaches to the R/T laws in viscoelastic anisotropic media see Caviglia and Morro (1998), Carcione (2001), etc.

Now define the propagation plane of incidence Σ^{inc} by vectors $\mathbf{n}$ and $\mathbf{P}^{inc}$, and the attenuation plane of incidence Σ^{att} by $\mathbf{n}$ and $\mathbf{A}^{inc}$. For a general inhomogeneous incident wave, planes Σ^{inc} and Σ^{att} do not coincide. Certain general rules, however, are valid even in this case:

1) The propagation vectors of incident wave $\mathbf{P}^{inc}$ and of all R/T waves $\mathbf{P}^{r}$ are situated in the plane of incidence Σ^{inc}.

2) The attenuation vectors of incident wave $\mathbf{A}^{inc}$ and of all R/T waves $\mathbf{A}^{r}$ are situated in the attenuation plane of incidence Σ^{att}.

3) Taking the cross-product of (14) with $\mathbf{n}$ confirms that the tangential components of slowness vectors of all R/T waves are the same as the tangential component of the incident plane wave:

$$\mathbf{p}^{r} \times \mathbf{n} = \mathbf{p}^{inc} \times \mathbf{n} \ . \tag{17}$$

This implies

$$\mathbf{P}^{r} \times \mathbf{n} = \mathbf{P}^{inc} \times \mathbf{n} \ , \qquad \mathbf{A}^{r} \times \mathbf{n} = \mathbf{A}^{inc} \times \mathbf{n} \ . \tag{18}$$

4) If we introduce the angle of incidence θ^{inc} and the R/T angle θ^{r} by relations

$$\sin\theta^{inc} = |\mathbf{N}^{inc} \times \mathbf{n}| \ , \qquad \sin\theta^{r} = |\mathbf{N}^{r} \times \mathbf{n}| \ , \tag{19}$$

we obtain from (18)

$$\sin\theta^{inc}/\mathcal{C}^{inc} = \sin\theta^{r}/\mathcal{C}^{r} \ . \tag{20}$$

This form of Snell's law is valid quite generally; viscoelastic anisotropic media and inhomogeneous plane waves included. For bulk plane waves propagating in isotropic perfectly elastic media, it coincides with the classical version of Snell's law (as $\mathcal{C}^{inc} = V^{inc}$, $\mathcal{C}^{r} = V^{r}$, $\theta^{inc} = i^{inc}$ and $\theta^{r} = i^{r}$ in this case).

5) The second equation of (18) yields an analogous "Snell's law" for attenuation, $|\mathbf{A}^{r}|$ and $|\mathbf{A}^{inc}|$. If we introduce angles ζ^{inc} and ζ^{r} by relations:

$$\sin\zeta^{inc} = |\mathbf{M}^{inc} \times \mathbf{n}| \ , \quad \sin\zeta^{r} = |\mathbf{M}^{r} \times \mathbf{n}| \ , \tag{21}$$

we obtain

$$|\mathbf{A}^{r}|\sin\zeta^{r} = |\mathbf{A}^{inc}|\sin\zeta^{inc} \ . \tag{22}$$

6) Taking the scalar product of (14) and (16) with $\mathbf{n}$, we obtain simple expressions for the normal components of the slowness vectors of all R/T waves:

$$\mathbf{p}^{r} \cdot \mathbf{n} = \sigma^{r} \ , \quad \mathbf{P}^{r} \cdot \mathbf{n} = \mathrm{Re}\sigma^{r} \ , \quad \mathbf{A}^{r} \cdot \mathbf{n} = \mathrm{Im}\sigma^{r} \ . \tag{23}$$

4 R/T laws: special cases

In this section, we consider special cases of Equations (14)–(16).

1) **Normal incidence**. Consider an incident wave with the propagation vector $\mathbf{P}^{inc}$ perpendicular to reference plane Σ, $\mathbf{P}^{inc} = |\mathbf{P}^{inc}|\mathbf{n}$. The attenuation vector $\mathbf{A}^{inc}$ may be arbitrary. Equation (16) then yields:

$$\begin{aligned} \mathbf{P}^r &= \mathbf{n}\,\mathrm{Re}\sigma^r \ , \\ \mathbf{A}^r &= \mathbf{A}^{inc} - \mathbf{n}[(\mathbf{A}^{inc}\cdot\mathbf{n}) - \mathrm{Im}\sigma^r] \ . \end{aligned} \tag{24}$$

Consequently, the propagation vectors $\mathbf{P}^r$ of all R/T waves are also perpendicular to Σ. The attenuation vectors, however, transform across Σ in a standard way.

Note that reference plane Σ actually represents the wavefront in this case, and σ^r can be determined by simpler equations of mixed specification (11), with $a_{ijkl} = a^r_{ijkl}$. They depend on the choice of the wavefront attenuation vector $\mathbf{d}$.

2) **Coplanar case**. Consider an incident plane wave, for which the three real-valued vectors $\mathbf{P}^{inc}$, $\mathbf{A}^{inc}$ and $\mathbf{n}$ are coplanar. Planes Σ^{inc} and Σ^{att} coincide in this case, $\Sigma^{inc} = \Sigma^{att}$. It follows from the first two rules of Section 3 that vectors $\mathbf{P}^r$, $\mathbf{A}^r$ and $\mathbf{n}$ are also coplanar for any reflected/transmitted wave, and all these planes coincide. The R/T laws then constitute a planar problem.

3) **Bulk incident plane wave** ($\mathbf{A}^{inc} = \mathbf{0}$). Assume that the incident halfspace is perfectly elastic (isotropic or anisotropic), and that the incident plane is bulk. The slowness vectors of R/T waves are then described by equations:

$$\begin{aligned} \mathbf{P}^r &= \mathbf{P}^{inc} - \mathbf{n}[(\mathbf{P}^{inc}\cdot\mathbf{n}) - \mathrm{Re}\sigma^r] \ , \\ \mathbf{A}^r &= \pm\mathbf{n}\,\mathrm{Im}\sigma^r \ . \end{aligned} \tag{25}$$

The R/T waves are then either bulk waves (for $\mathrm{Im}\sigma^r = 0$), or inhomogeneous plane waves (for $\mathrm{Im}\sigma^r \neq 0$). In the latter case, however, $\mathbf{A}^r$ is always perpendicular to the interface. Homogeneous R/T plane waves cannot exist in this case.

4) **Incident homogeneous plane wave**. For homogeneous plane waves, $\mathbf{P}^{inc}$ and $\mathbf{A}^{inc}$ have the same direction, and $\mathbf{P}^{inc}$, $\mathbf{A}^{inc}$, $\mathbf{n}$ are coplanar. Consequently, $\mathbf{P}^r$, $\mathbf{A}^r$ and $\mathbf{n}$ are also coplanar, for any R/T wave. This, however, does not mean that the R/T waves are also homogeneous. Thus, the homogeneous incident plane wave generates inhomogeneous R/T plane waves, with $\mathbf{P}^r$ and $\mathbf{A}^r$ situated in the plane of incidence. Homogeneous R/T plane waves are only exceptional.

Consequently, the solution of the R/T problem in terms of homogeneous plane waves solely is not possible; the existence of inhomogeneous plane waves must be allowed.

5) **Isotropic viscoelastic media**. In this case, Equation (7) can be solved analytically for σ^r. We obtain a simple relation

$$\begin{aligned} \sigma^r &= \pm[(1/\mathcal{V}^r)^2 - \mathbf{p}^\Sigma\mathbf{p}^\Sigma]^{1/2} \\ &= \pm[(1/\mathcal{V}^r)^2 - (1/\mathcal{V}^{inc})^2 + (\mathbf{p}^{inc}\cdot\mathbf{n})^2]^{1/2} \ . \end{aligned} \tag{26}$$

Here $\mathcal{V}^r$ and $\mathcal{V}^{inc}$ are complex-valued velocities of P waves ($\mathcal{V} = \alpha$) or S waves ($\mathcal{V} = \beta$). Equation (14) then yields:

$$\mathbf{p}^r = \mathbf{p}^{inc} - \mathbf{n}\Big[(\mathbf{p}^{inc}\cdot\mathbf{n}) \mp \sqrt{(1/\mathcal{V}^r)^2 - (1/\mathcal{V}^{inc})^2 + (\mathbf{p}^{inc}\cdot\mathbf{n})^2}\Big] \ . \tag{27}$$

This equation is exactly the same as (2), known for isotropic perfectly elastic media. The only difference is that $\mathbf{p}^{inc}$, $\mathcal{V}^r$ and $\mathcal{V}^{inc}$ are complex-valued in this case. If we use Q^r and Q^{inc} to denote the quality factors of the R/T waves and of the incident wave, we can insert the following expressions into (27),

$$\left(\frac{1}{\mathcal{V}^r}\right)^2 = \frac{1+\mathrm{i}/Q^r}{(V^r)^2(1+(Q^r)^{-2})} \ , \qquad \left(\frac{1}{\mathcal{V}^{inc}}\right)^2 = \frac{1+\mathrm{i}/Q^{inc}}{(V^{inc})^2(1+(Q^{inc})^2)} \ . \tag{28}$$

Quantities V^r and V^{inc} on the R.H.S.'s of (28) correspond to the real-valued velocities of R/T waves.

Equation (27) can also be simply expressed in terms of propagation vectors $\mathbf{P}^{inc}$, $\mathbf{P}^r$, and attenuation vectors $\mathbf{A}^{inc}$, $\mathbf{A}^r$. It is also valid for non-coplanar $\mathbf{P}^{inc}$, $\mathbf{A}^{inc}$ and $\mathbf{n}$.

In the seismological literature, considerable attention has been devoted to the R/T laws in viscoelastic isotropic media, assuming the coplanar case. See Borcherdt (1977, 1982), Aki and Richards (1980), Krebes (1983), Wennerberg (1985), Caviglia and Morro (1992), where many other references can be found.

6) **Isotropic media, perfectly elastic incident halfspace**

Here we again consider an interface between two isotropic media; the incident halfspace is perfectly elastic and the R/T halfspace viscoelastic. The incident wave is a bulk wave ($\mathbf{A}^{inc} = \mathbf{0}$). This is the simplest case of the R/T problem for viscoelastic media studied here. All equations can be expressed in analytical form.

The slowness vector of the incident bulk wave is given by the relation

$$\mathbf{p}^{inc} = \mathbf{P}^{inc} = \mathbf{N}^{inc}/V^{inc} = \cos\theta^{inc}/V^{inc} \ , \tag{29}$$

where θ^{inc} is the angle of incidence given by (19). Propagation vector $\mathbf{P}^r$ and attenuation vector $\mathbf{A}^r$ of the R/T waves are then given by (25), where σ^r is given by (26). We can express (26) in the following form

$$\sigma^r = \pm[a+\mathrm{i}b]^{1/2} \ , \tag{30}$$

where

$$a = \frac{1}{(V^r)^2(1+(Q^r)^{-2})} - \frac{\sin^2\theta^{inc}}{(V^{inc})^2} \ , \quad b = \frac{1}{Q^r(V^r)^2(1+(Q^r)^{-2})} \ . \tag{31}$$

Quantities a and b are always real-valued, a is arbitrary and b non-negative. Consequently,

$$\mathrm{Re}\sigma^r = \pm\sqrt{\sqrt{a^2+b^2}+a}\Big/\sqrt{2} \ , \quad \mathrm{Im}\sigma^r = \pm\sqrt{\sqrt{a^2+b^2}-a}\Big/\sqrt{2} \ . \tag{32}$$

Equations (25) and (32) yield the final expressions for $\mathbf{P}^r$ and $\mathbf{A}^r$:

$$\begin{aligned}\mathbf{P}^r &= \mathbf{N}^{inc}/V^{inc} - \mathbf{n}\left[\cos\theta^{inc}/V^{inc} \mp \sqrt{\sqrt{a^2+b^2}+a}\Big/\sqrt{2}\right] \ , \\ \mathbf{A}^r &= \pm\mathbf{n}\sqrt{\sqrt{a^2+b^2}-a}\Big/\sqrt{2} \ .\end{aligned} \tag{33}$$

We can also compute the R/T angle θ^r, defined as the angle between $\mathbf{P}^r$ and $\mathbf{n}$:

$$\cos\theta^r = \mathbf{P}^r \cdot \mathbf{n}/|\mathbf{P}^r| = \pm\sqrt{\sqrt{a^2+b^2}+a}\Big/\sqrt{2}|\mathbf{P}^r| \ . \tag{34}$$

These are the final expressions for $\mathbf{P}^r$, $\mathbf{A}^r$ and $\cos\theta^r$ of any R/T wave. Using (33) and (8), we can simply compute $|\mathbf{P}^r|$, $|\mathbf{A}^r|$, $\mathbf{N}^r$, $\mathbf{M}^r$, $\mathcal{C}^r$, $\cos\gamma^r$, etc. We explicitly present only the important expressions for phase velocity $\mathcal{C}^r$ and attenuation angle γ^r:

$$\begin{aligned} \mathcal{C}^r &= 1/|\mathbf{P}^r| = 1/[\sin^2\theta^{inc}/(V^{inc})^2 + \tfrac{1}{2}(\sqrt{a^2+b^2}+a)]^{1/2} \ , \\ \cos\gamma^r &= b/|\mathbf{P}^r||\mathbf{A}^r| \ . \end{aligned} \tag{35}$$

The viscoelastic properties of R/T waves are fully controlled by quality factor Q^r.

7) **Perfectly elastic isotropic media**

The results presented in this paragraph are mostly known, although certain interesting aspects have not been often sufficiently explained. The derivation is simple, we only insert $Q^r \to \infty$ into the equations derived in paragraph 5.

For $Q^r \to \infty$, we obtain

$$a = (1/V^r)^2 - \sin^2\theta^{inc}/(V^{inc})^2 \ , \qquad b = 0 \ . \tag{36}$$

As $b = 0$, Equation (32) for $\mathrm{Re}\sigma^r$ and $\mathrm{Im}\sigma^r$ can be expressed in the following form:

$$\mathrm{Re}\sigma^r = \sqrt{|a|+a}\Big/\sqrt{2} \ , \qquad \mathrm{Im}\sigma^r = \sqrt{|a|-a}\Big/\sqrt{2} \ . \tag{37}$$

It is simple to see that the behaviour of the R/T waves is quite different for positive a (subcritical region), $a = 0$ (critical case), and negative a (postcritical region):

a) *Subcritical region*, $\sin\theta^{inc} < V^{inc}/V^r$: In this case $a > 0$. We obtain $\mathbf{A} = \mathbf{0}$, so that the R/T wave is a bulk wave. Phase velocity $\mathcal{C}^r$ is given by the relation

$$\mathcal{C}^r = 1/[\sin^2\theta^{inc}/(V^{inc})^2 + (1/V^r)^2 - \sin^2\theta^{inc}/(V^{inc})^2]^{1/2} = V^r \ . \tag{38}$$

We further insert $b = 0$ into (34) and obtain the R/T angle:

$$\cos\theta^r = \pm\sqrt{|a|+a}\Big/\sqrt{2}|\mathbf{P}^r| = \pm[1-(V^r/V^{inc})^2\sin^2\theta^{inc}]^{1/2} \ . \tag{39}$$

Equations (38) and (39) fully correspond to the familiar form of Snell's law (1), valid for bulk waves.

b) *Critical incidence*, $\sin\theta^{inc} = V^{inc}/V^r$. In this case, $a = 0$, $\theta^r = 90^0$, $\mathcal{C}^r = V^r$. The propagation vector of R/T wave is oriented along interface Σ, and the phase velocity corresponds to V^r.

c) *Postcritical region*, $\sin\theta^{inc} > V^{inc}/V^r$: In this case $a < 0$. We further obtain

$$\begin{aligned} &\mathbf{A}^r = \pm\mathbf{n}\sqrt{|a|}, = \pm\mathbf{n}\sqrt{\sin^2\theta^{inc}/(V^{inc})^2 - 1/(V^r)^2} \ , && \mathcal{C}^r = V^{inc}/\sin\theta^{inc} \ , \\ &\theta^r = \tfrac{1}{2}\pi \ , && \gamma^r = \tfrac{1}{2}\pi \ . \end{aligned} \tag{40}$$

Thus, the propagation vector $\mathbf{P}^r$ of the R/T wave is always parallel to interface Σ in the postcritical region, for any angle of incidence, $V^{inc}/V^r < \sin\theta^{inc} < 1$. Attenuation vector $\mathbf{A}^r$ is always perpendicular to the interface, so that the R/T plane wave is inhomogeneous. The phase velocity of R/T waves in the postcritical region, however, is not constant, but depends on the angle of incidence. It decreases with increasing θ^{inc}. For the critical angle of incidence, it equals V^r, and for the grazing angle of incidence ($\theta^{inc} = 90^0$) it equals V^{inc}.

8) **Viscoelastic monoclinic anisotropic media: SH waves**

Equation (11) represents an algebraic equation of the sixth degree in σ, and must be solved numerically. For SH waves, propagating in a plane of symmetry of a monoclinic (orthorhombic, hexagonal) anisotropic viscoelastic medium, however, the algebraic equation of the sixth degree reduces to a quadratic equation, and may be solved analytically, similarly as in viscoelastic isotropic media. See Červený and Pšenčík (2004).

We choose the Cartesian coordinate system x_i so that the plane of symmetry corresponds to plane x_1x_3, and assume $p_2^{inc} = 0$. The two quantities σ^r corresponding to R/T SH waves are then given by relations, see Červený and Pšenčík (2004, Equation (49)):

$$\sigma^r = -\frac{E_{22}}{\Gamma_{22}} \pm \left[\frac{1}{\Gamma_{22}} - \frac{(p_1^\Sigma n_3 - n_1 p_3^\Sigma)^2\Delta}{\Gamma_{22}^2}\right]^{1/2} , \tag{41}$$

where

$$\begin{aligned}
\Gamma_{22} &= A_{66}n_1^2 + A_{44}n_3^2 + 2A_{46}n_1n_3 , \\
E_{22} &= A_{66}n_1p_1^\Sigma + A_{44}n_3p_3^\Sigma + A_{46}(n_1p_3^\Sigma + p_1^\Sigma n_3) , \\
\Delta &= A_{44}A_{66} - A_{46}^2 ,
\end{aligned} \tag{42}$$

and where A_{44}, A_{66} and A_{46} are the Voigt complex-valued density-normalised viscoelastic moduli, corresponding to the R/T halfspace under consideration. Quantity Δ does not depend on $\mathbf{n}$ and $\mathbf{p}^\Sigma$. Using (14), we then obtain the complex-valued slowness vector of R/T SH waves:

$$\mathbf{p}^r = \mathbf{p}^{inc} - \mathbf{n}\Big\{(\mathbf{p}^{inc}\cdot\mathbf{n}) - \frac{E_{22}}{\Gamma_{22}} \pm \left[\frac{1}{\Gamma_{22}} - \frac{(p_1^\Sigma n_3 - n_1p_3^\Sigma)^2\Delta}{\Gamma_{22}^2}\right]^{1/2}\Big\} . \tag{43}$$

Equation (43) represents the only analytical solution of the R/T problem for an anisotropic viscoelastic media. All other situations require at least the solutions of the algebraic equation of the fourth degree (qP and qSV waves in an analogous plane of symmetry).

We shall not discuss Equation (43) for general viscoelastic media, but only consider a simpler case of perfectly elastic anisotropic media (A_{44}, A_{66} and A_{46} real-valued, and incident bulk waves ($\mathbf{A}^{inc} = \mathbf{0}$). Equation (43) then yields:

$$\begin{aligned}
\mathbf{P}^r &= \mathbf{P}^{inc} - \mathbf{n}\Big\{(\mathbf{P}^{inc}\cdot\mathbf{n}) - \frac{E_{22}}{\Gamma_{22}} \pm \mathrm{Re}\left[\frac{1}{\Gamma_{22}} - (p_1^\Sigma n_3 - p_3^\Sigma n_1)^2\frac{\Delta}{\Gamma_{22}^2}\right]^{1/2}\Big\} , \\
\mathbf{A}^r &= \pm\mathbf{n}\,\mathrm{Im}\left[1/\Gamma_{22} - (p_1^\Sigma n_3 - n_1p_3^\Sigma)^2\Delta/\Gamma_{22}^2\right]^{1/2} .
\end{aligned} \tag{44}$$

The expression under the square root is real-valued, and may be positive, zero or negative. The positive case (corresponding to subcritical incidence) yields bulk waves. This is the standard situation, well-known from applications. For the negative case (postcritical incidence), we obtain

$$\begin{aligned} \mathbf{P}^r &= \mathbf{P}^{inc} - \mathbf{n}[(\mathbf{P}^{inc} \cdot \mathbf{n}) - E_{22}/\Gamma_{22}] \ , \\ \mathbf{A}^r &= \pm\mathbf{n}\,\mathrm{Im}[1/\Gamma_{22} - (p_1^{\Sigma} n_3 - n_1 p_3^{\Sigma})^2 \Delta/\Gamma_{22}^2]^{1/2} \ . \end{aligned} \tag{45}$$

Thus, the attenuation vector is perpendicular to reference plane Σ. Propagation vector $\mathbf{P}^r$, however, is not generally parallel to the reference plane. This is easy to see if we compute $\mathbf{P}^r \cdot \mathbf{n}$:

$$\mathbf{P}^r \cdot \mathbf{n} = E_{22}/\Gamma_{22} \ . \tag{46}$$

Thus, $\mathbf{P}^r$ is parallel to the interface in the postcritical region only for $E_{22} = 0$. This is a great difference with respect to isotropic media.

5 Specification of incident wave

In the preceding sections, we have specified the slowness vector of the incident wave only in a general form, $\mathbf{p}^{inc}$. Moreover, we have not tested whether $\mathbf{p}^{inc}$ satisfies the constraint relation (4). In such a case, we have not needed to know the viscoelastic moduli of the incident halfspace at all. It has been sufficient to compute the tangential components $\mathbf{p}^{\Sigma}$ in reference plane Σ from $\mathbf{p}^{inc}$, see (13).

In this section, we discuss the R/T problem in more specific terms. We denote the complex-valued density-normalised viscoelastic moduli in the incident halfspace by a_{ijkl}^{inc}, and specify the plane incident wave by two vectors $\mathbf{N}^{inc}$ and $\mathbf{d}^{inc}$, where:

a) $\mathbf{N}^{inc}$ is a unit real-valued propagation vector, specifying the direction of propagation ($\mathbf{N} = \mathbf{P}/|\mathbf{P}|$).

b) $\mathbf{d}^{inc}$ is a real-valued wavefront attenuation vector, perpendicular to $\mathbf{N}^{inc}$, specifying the propagation-attenuation halfplane. This halfplane contains both the propagation and attenuation vectors $\mathbf{P}^{inc}$ and $\mathbf{A}^{inc}$. The length of $\mathbf{d}^{inc}$, $|\mathbf{d}^{inc}|$, specifies the inhomogeneity strength of the incident wave.

The specification of $\mathbf{N}^{inc}$ and $\mathbf{d}^{inc}$, for known a_{ijkl}^{inc}, is quite sufficient to determine slowness vector $\mathbf{p}^{inc}$ completely:

$$\mathbf{p}^{inc} = \sigma^{inc}\mathbf{N}^{inc} \pm \mathrm{i}\mathbf{d}^{inc} \ , \tag{47}$$

see (10). Quantity σ^{inc} is a solution of the algebraic equation of the sixth degree:

$$\det[a_{ijkl}^{inc}(\sigma^{inc} N_j^{inc} + \mathrm{i}dj^{inc})(\sigma^{inc} N_l^{inc} + \mathrm{i}d_l^{inc}) - \delta_{ik}] = 0 \ . \tag{48}$$

Consequently, the slowness vector $\mathbf{p}^{inc}$, given by (47) with (48), satisfies the constraint equation (4).

There are six roots σ^{inc} of Equation (48), corresponding to the qP, qS1 and qS2 plane waves, propagating towards and outwards from reference plane Σ in the incident halfspace. We have to select the root corresponding to the incident wave we require, propagating towards Σ. The problem of the selection criteria, however, is not treated here, see Introduction.

Inserting (47) into (14) or (16), we obtain the final expressions for $\mathbf{N}^r$ and $\mathbf{d}^r$, corresponding to the R/T waves. We obtain

$$\mathbf{N}^r = \mathbf{P}^r/|\mathbf{P}^r| \ , \qquad \mathbf{d}^r = \mathbf{A}^r - \mathbf{N}^r(\mathbf{A}^r \cdot \mathbf{N}^r) \ , \tag{49}$$

where

$$\begin{aligned} \mathbf{P}^r &= \mathbf{P}^{inc} - \mathbf{n}[(\mathbf{P}^{inc} \cdot \mathbf{n}) - \mathrm{Re}\,\sigma^r] \ , \\ \mathbf{A}^r &= \mathbf{A}^{inc} - \mathbf{n}[(\mathbf{A}^{inc} \cdot \mathbf{n}) - \mathrm{Im}\,\sigma^r] \ , \end{aligned} \tag{50}$$

and

$$\begin{aligned} \mathbf{P}^{inc} &= \mathbf{N}^{inc}(\mathrm{Re}\,\sigma^{inc}) \ , \\ \mathbf{A}^{inc} &= \mathbf{N}^{inc}(\mathrm{Im}\,\sigma^{inc}) + \mathbf{d}^{inc} \ . \end{aligned} \tag{51}$$

Here σ^{inc} is any root of (48) and σ^r any root of (15), with (13).

For completeness, we also present the equations required to determine other useful parameters of incident and R/T waves, assuming $\mathbf{P}^{inc}$, $\mathbf{A}^{inc}$ and $\mathbf{P}^r$, $\mathbf{A}^r$ are known, see (50) and (51):

Incident wave:

$$\begin{aligned} |\mathbf{P}^{inc}| &= |\mathrm{Re}\,\sigma^{inc}| \ , \\ |\mathbf{A}^{inc}| &= [(\mathrm{Im}\,\sigma^{inc})^2 + (d^{inc})^2]^{1/2} \ , \\ \mathbf{M}^{inc} &= \mathbf{A}^{inc}/|\mathbf{A}^{inc}| \ , \\ \mathcal{C}^{inc} &= 1/|\mathrm{Re}\,\sigma^{inc}| \ , \\ \cos\gamma^{inc} &= (\mathrm{Re}\,\sigma^{inc})(\mathrm{Im}\,\sigma^{inc})/|\mathbf{P}^{inc}||\mathbf{A}^{inc}| \ . \end{aligned} \tag{52}$$

R/T waves:

$$\begin{aligned} |\mathbf{P}^r| &= [(\mathrm{Re}\,\sigma^{inc})^2 \sin^2\theta^{inc} + (\mathrm{Re}\,\sigma^r)^2]^{1/2} \ , \\ |\mathbf{A}^r| &= [(\mathrm{Im}\,\sigma^{inc})^2 \sin^2\theta^{inc} + (\mathrm{Im}\,\sigma^r)^2 + (d^{inc})^2 \\ &- (\mathbf{d}^{inc} \cdot \mathbf{n})^2 - 2(\mathbf{d}^{inc} \cdot \mathbf{n})(\mathbf{N}^{inc} \cdot \mathbf{n})(\mathrm{Im}\,\sigma^{inc})]^{1/2} \ , \\ \mathbf{M}^r &= \mathbf{A}^r/|\mathbf{A}^r| \ , \\ \mathcal{C}^r &= 1/[(\mathrm{Re}\,\sigma^{inc})^2 \sin^2\theta^{inc} + (\mathrm{Re}\,\sigma^r)^2]^{1/2} \ , \\ \cos\gamma^r &= [(\mathrm{Re}\,\sigma^{inc})(\mathrm{Im}\,\sigma^{inc}) \sin^2\theta^{inc} + (\mathrm{Re}\,\sigma^r)(\mathrm{Im}\,\sigma^r) \\ &- (\mathrm{Re}\,\sigma^{inc})(\mathbf{n} \cdot \mathbf{N}^{inc})(\mathbf{n} \cdot \mathbf{d}^{inc})]/|\mathbf{P}^r||\mathbf{A}^r| \ . \end{aligned} \tag{53}$$

In (53), we have denoted $\sin^2\theta^{inc} = 1 - (\mathbf{n} \cdot \mathbf{N}^{inc})^2$.

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Appendix 1
Snell's law in viscoelastic anisotropic media

We have presented two forms of Snell's law:

1) The **first form** of Snell's law is represented by its classical version (1),

$$\sin i^{inc}/V^{inc} = \sin i^{r}/V^{r} \ . \tag{A1}$$

(A1) is valid for isotropic perfectly elastic media. V^{inc} and V^{r} are real-valued velocities (α or β) in both halfspaces, i^{inc} is the angle of incidence (i.e. the acute angle between the normal to the interface and the slowness vector of the incident wave), and i^{r} is the reflection/transmission angle, defined in an analogous way. Angle i^{r}, however, may be complex-valued for $\sin i^{inc} > V^{inc}/V^{r}$ (postcritical incidence).

2) The **second form** of the Snell's law, see (20),

$$\sin \theta^{inc}/\mathcal{C}^{inc} = \sin \theta^{r}/\mathcal{C}^{r} \ , \tag{A2}$$

is valid quite generally, for viscoelastic and perfectly elastic media, for anisotropic and isotropic media, for inhomogeneous, homogeneous and bulk plane waves. In certain cases, the individual quantities in (A2) have a different meaning than in (A1). θ^{inc} and θ^{r} are always real-valued and determine the directions of the propagation vectors $\mathbf{P}^{inc}$ and $\mathbf{P}^{r}$ of the incident and R/T waves, see (19), $\mathcal{C}^{inc}$ and $\mathcal{C}^{r}$ are the relevant phase velocities of both waves. Phase velocities $\mathcal{C}^{inc}$ and $\mathcal{C}^{r}$ are not constants, but they depend on the directions of $\mathbf{P}$ and $\mathbf{A}$, and on inhomogeneity strength d.

The second form of Snell's law (A2) is considerably more general than the first form. Unfortunately, it is not constructive as (A1). In (A1), three quantities are often known, and the fourth may be computed. For example, V^{inc}, i^{inc} and i^{r} are known, and V^{r} is computed. This is, in general, impossible in (A2). Even if we specify θ^{inc} and compute the corresponding $\mathcal{C}^{inc}$, both θ^{r} and $\mathcal{C}^{r}$ remain unknown. To compute them, the procedures proposed in Sections 3 and 4 must be used. Snell's law (A2) is not sufficient for this purpose. In fact, Snell's law (2) is not needed in this procedure at all, it is only obtained as a by-product.

In isotropic perfectly elastic media, however, both (A1) and (A2) yield the same results. For subcritically incident waves the equivalence of (A1) and (A2) is obvious, as $\mathcal{C}^{inc} = V^{inc}$, $\mathcal{C}^{r} = V^{r}$, $\sin \theta^{inc} = \sin i^{inc}$ and $\sin \theta^{r} = \sin i^{r}$ in this case, see Section 4, Paragraph 6. For postcritically incident waves, however, this is not immediately obvious. The reader is reminded that i^{r} is complex-valued in this case, but θ^{r} is real-valued. For this reason, we shall briefly discuss this case, and show that (1) and (2) are equivalent even there.

It is easy to see that complex-valued i^{r} can be expressed for the postcritical incidence in the form:

$$i^{r} = \tfrac{1}{2}\pi + \mathrm{i}\delta \ , \tag{A3}$$

where δ is real-valued. We further obtain

$$\sin i^{r} = \sin \tfrac{1}{2}\pi \cos \mathrm{i}\delta \ , \qquad \cos \mathrm{i}^{r} = -\sin \tfrac{1}{2}\pi \cdot \sin \mathrm{i}\delta \ . \tag{A4}$$

Snell's law (1) then yields,

$$\sin \mathrm{i}^{r} = \sin \tfrac{1}{2}\pi \ \cos \mathrm{i}\delta = V^{r} \sin \delta^{inc}/V^{inc} \ ,$$

$$\cos \mathrm{i}^r = -\sin \tfrac{1}{2}\pi \ \sin \mathrm{i}\delta = -\mathrm{i}\sqrt{\sin^2 i^{inc}(V^r/V^{inc})^2 - 1}\ . \tag{A5}$$

Inserting the first equation of (A5) into Snell's law (A4) yields

$$\frac{\sin\ i^{inc}}{V^{inc}} = \frac{\sin \frac{1}{2}\pi}{(V^{inc}/\sin\ i^{inc})}\ . \tag{A6}$$

We now take into account that $\theta^r = \frac{1}{2}\pi$, $\mathcal{C}^r = V^{inc}/\sin i^{inc}$, see Section 4, Paragraph 6. Further, the incident wave is a bulk wave, so that $\theta^{inc} = i^{inc}$, $\mathcal{C}^{inc} = V^{inc}$. Consequently, the classical form of Snell's law (A1) yields exactly Snell's law (A2) for postcritical incidence. This proves that both Snell's laws (A1) and (A2) are fully alternative for isotropic media, including the postcritical region.

We can also simply show that the classical version (A1) of Snell's law yields the same attenuation vector $\mathbf{A}^r$ as version (A2) in the postcritical region of isotropic media. Compare the first equation of (40) with the second equation of (A5).

For viscoelastic anisotropic media, however, the classical version (A1) of Snell's law is not valid.