Approximate conditions for the off-axis triplication in transversely isotropic media

Václav Vavryčuk

Geophysical Institute, Academy of Sciences of the Czech Republic, Boční II/1401, 141 31 Praha 4, Czech Republic, E-mail: vv@ig.cas.cz

Abstract

The presented approximate formulas yield a critical value of anisotropy parameter σ, for which an incipient off-axis SV-wave triplication occurs in transversely isotropic media. The formulas are simple and approximate the exact solution with a high accuracy. The best results are obtained by the third-order approximation, which yields accuracy at least 20 times higher than the formulas presented by Thomsen & Dellinger (2003). The formula works safely for parameters $\kappa = a_{33}/a_{44} > 2$ and $0.2 > \varepsilon = (a_{11} - a_{33})/2a_{33} > -0.2$ yielding critical values of σ from 0.1 to 0.7. Out of this interval, it is recommended to use an exact solution.

Introduction

In homogeneous transverse isotropy, the triplications can occur for the SV wave only and can be classified into four different types (Musgrave 1970; Payton 1983): (1) off-axis triplication, (2) on-axis triplication near the symmetry plane, (3) on-axis triplication near the symmetry axis, and (4) double on-axis triplication (see Fig. 1). The triplications are delimited by two cuspidal edge lines on the wave surface (see Fig. 2). These lines (also called caustics) produce energy focusing (Kravtsov & Orlov 1990; Wolfe 1998) and phase shifting of signals (Bakker 1998; Červený 2001). The cuspidal lines (shown as cuspidal points in Fig. 2) correspond to two lines of inflection points on the slowness surface characterized by the zero Gaussian curvature (see Fig. 3). The lines of inflection points are also called parabolic lines (Vavryčuk 2003). If the two parabolic lines are not separated but coalesce into one, we speak of the incipient triplication (Helbig 1994, p. 231). In this case, the medium represents a borderline between media with and without a triplication.

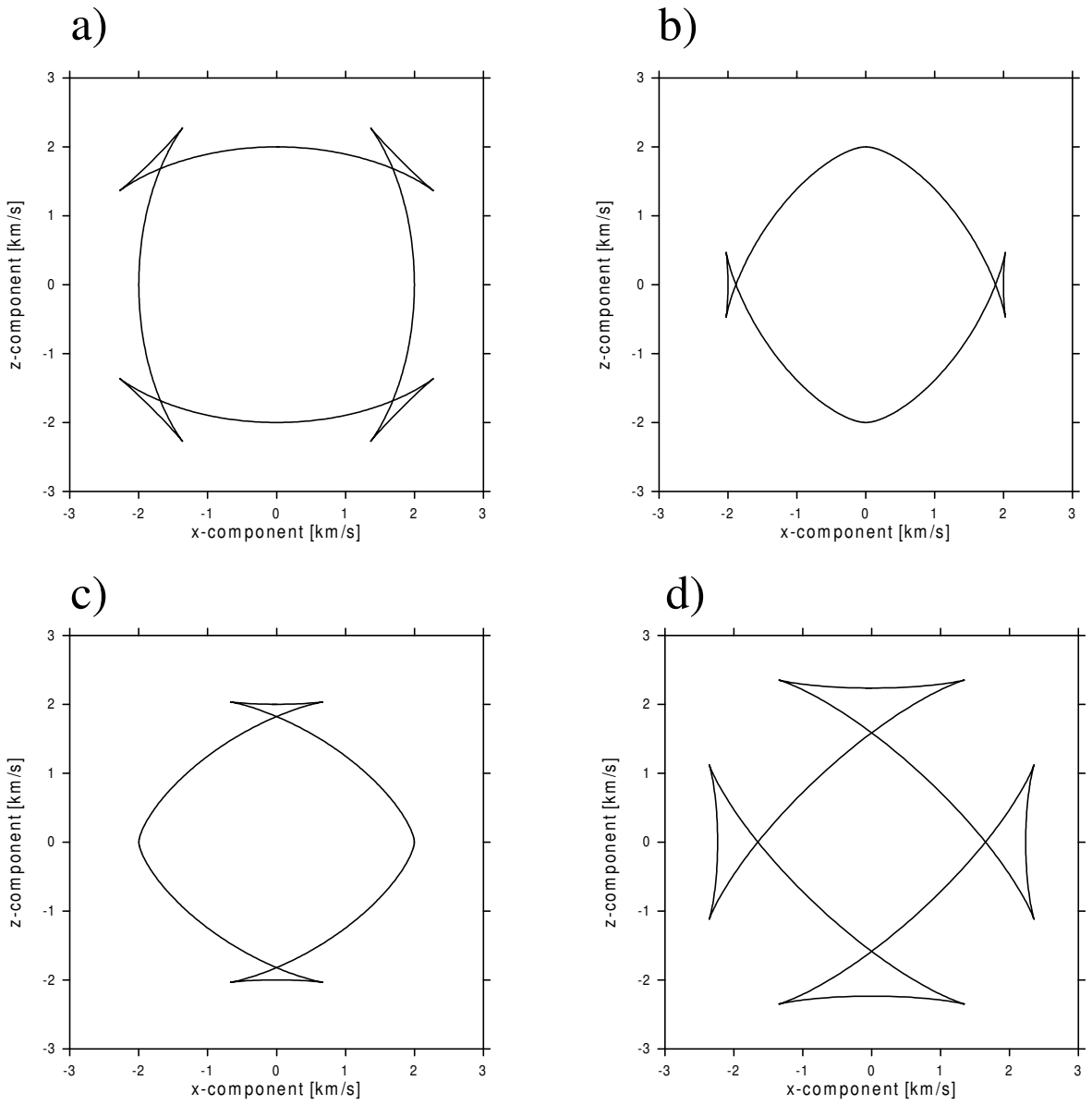

Figure 1. Types of triplications in transverse isotropy. Vertical sections of wave surfaces are shown for (a) off-axis triplication, (b) on-axis triplication near the symmetry plane, (c) on-axis triplication near the symmetry axis, and (d) double on-axis triplication.

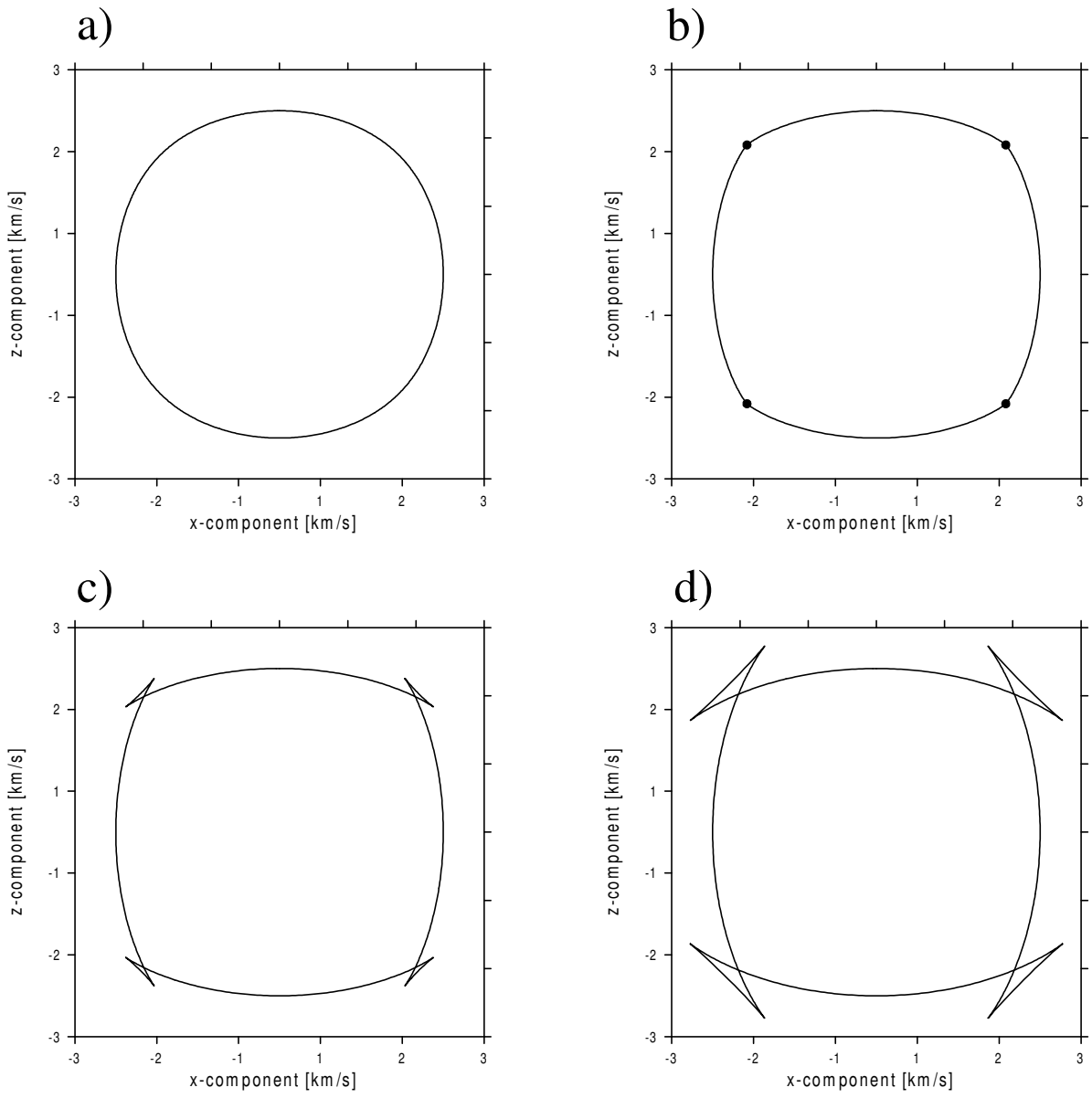

Figure 2. Off-axis triplications. (a) No triplication, (b) incipient triplication, (c) small triplication (d) distinct triplication. Elastic parameters are (in km^2/s^2): $a_{11} = a_{22} = a_{33} = 10$, $a_{44} = a_{55} = 4$, $a_{66} = 5$, $a_{12} = a_{11} - 2a_{66} = 0$, and (a) $a_{13} = 1.5$, (b) $a_{13} = 0$, (c) $a_{13} = -1.5$, (d) $a_{13} = -3$. The dots in plot (b) mark the directions of the incipient triplication.

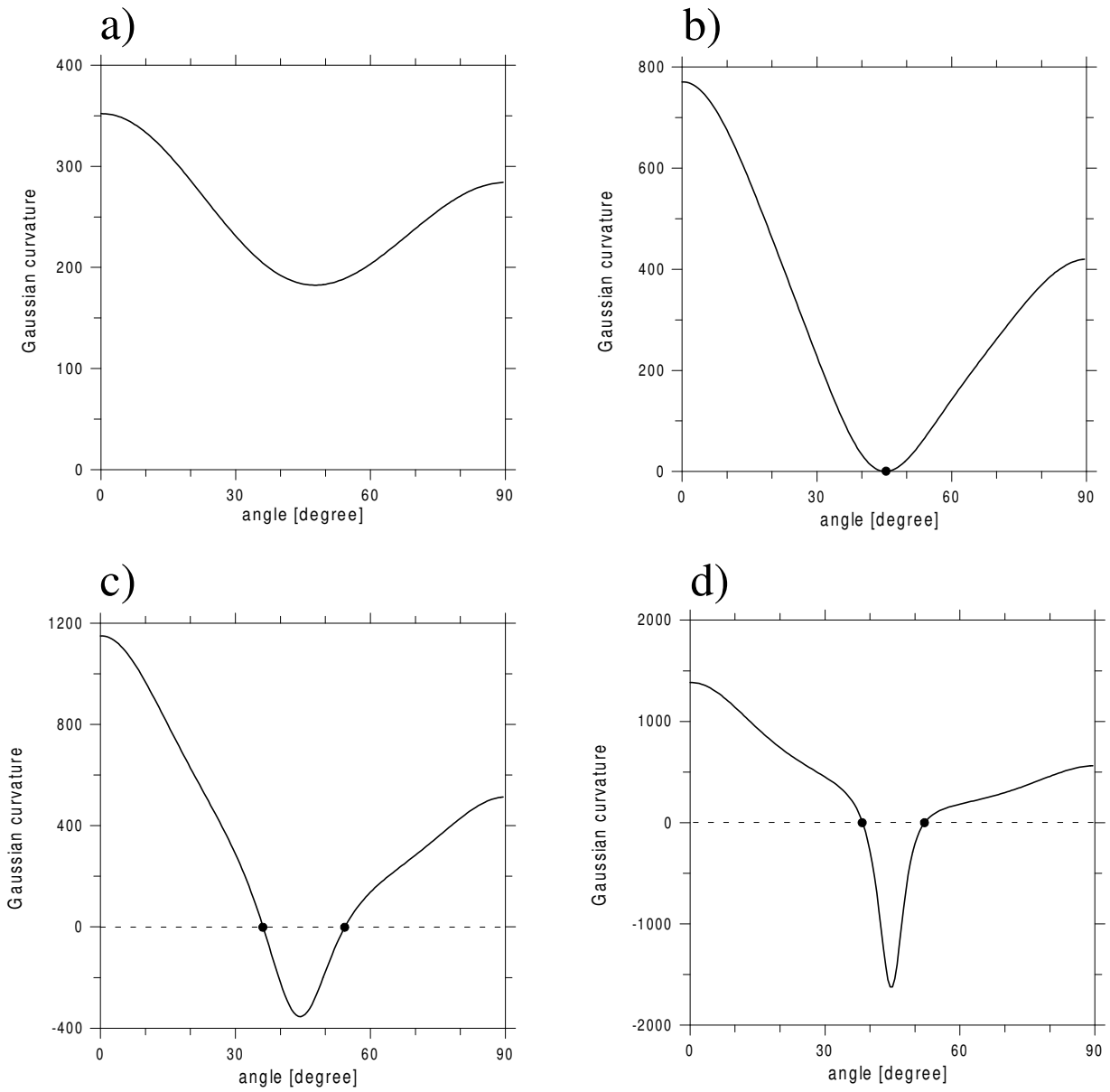

Figure 3. The Gaussian curvature. (a) No triplication, (b) incipient triplication, (c) small triplication (d) distinct triplication. The angle measures the deviation of the slowness vector from the symmetry axis. The dots mark inflection points on the slowness surface. For elastic parameters of the media, see the caption of Fig. 1.

The conditions under which the triplications occur in transverse isotropy are well known (Helbig 1958; Musgrave 1970; Payton 1983; Dellinger 1991). They are elementary except for the off-axis triplication, which is algebraically more involved. For this case, several authors proposed simpler but approximate triplication conditions (McCurdy 1974; Musgrave 1979; Musgrave & Payton 1984; Alshits & Chadwick 1997; Thomsen & Dellinger 2003). From these conditions, the condition proposed by Thomsen & Dellinger (2003) is particularly interesting, because it is far simpler than the other approximations and it gives even an insight into which parameters control the off-axis triplication. The validity of the formula is, however, restricted to a limited interval of anisotropy parameters. Out of this interval, the accuracy of the formula is rather low. The aim of this study is to present an approximation which is more accurate and applicable to a broader interval of anisotropy parameters.

Exact triplication condition

In this analysis we shall consider transversely isotropic media, which satisfy the "stability" conditions (Backus 1962, Eq. 20; Helbig 1994, Eqs 5.3 and 5.23),

$$a_{33} > 0\,,\; a_{44} > 0\,,\; a_{66} > 0\,,\; a_{11} - a_{66} > 0\,,\; a_{33}(a_{11} - a_{66}) > a_{13}^2\,, \tag{1}$$

the condition for "normal polarization" (Helbig & Schoenberg 1987),

$$a_{13} + a_{44} > 0\,, \tag{2}$$

and the "separation" conditions, which prevent the P and SV slowness or phase-velocity surfaces mutually to intersect,

$$a_{11} - a_{44} > 0\,,\; a_{33} - a_{44} > 0\,, \tag{3}$$

where a_{kl} are the density normalized elastic parameters in the Voigt notation. These conditions are met by overwhelming majority of real materials. For analysis under less restrictive conditions, see Payton (1983) and Alshits & Chadwick (1997).

The condition for the off-axis triplication is expressed as follows (Dellinger 1991, Eq. 2.19; Thomsen & Dellinger 2003, Eq. 9):

$$\left(a_{13}+a_{44}\right)^2-3a_{44}^2+a_{44}\left(a_{33}+a_{11}\right)-3a_{11}a_{33}+2\sqrt{\left(a_{33}-a_{44}\right)\left(a_{11}-a_{44}\right)}\,\frac{a_{11}a_{33}-a_{44}^2}{a_{13}+a_{44}}\geq 0\ , \tag{4}$$

where the equality sign stands for the incipient triplication that occurs at the slowness angle θ_i defined by the following equations:

$$\sin^2\theta_i=\frac{a_{33}-a_{44}}{a_{11}+a_{33}-2a_{44}},\ \ \cos^2\theta_i=\frac{a_{11}-a_{44}}{a_{11}+a_{33}-2a_{44}}\ . \tag{5}$$

The condition (4) can also be expressed using anisotropy parameters σ and ε or σ and δ, which represent an alternative parameterizations of transverse isotropy (Thomsen 1986; Tsvankin & Thomsen 1994):

$$\varepsilon=\frac{a_{11}-a_{33}}{2a_{33}},\ \delta=\frac{\left(a_{13}+a_{44}\right)^2-\left(a_{33}-a_{44}\right)^2}{2a_{33}\left(a_{33}-a_{44}\right)},\ \sigma=\frac{a_{33}}{a_{44}}\left(\varepsilon-\delta\right)\ . \tag{6}$$

These parameters are frequently used in describing transverse isotropy. They become zero in isotropy and can serve as a measure of strength of transverse isotropy. Therefore, expressing the triplication condition using these parameters, we can better understand how strong anisotropy must be to generate triplications. Among these parameters, the crucial role is played by parameter σ in the triplication condition, because it controls the variation of the phase velocity of the SV wave in the weak-anisotropy approximation

$$c_{SV}^2 = a_{44}\left(1+2\sigma\sin^2\theta\,\cos^2\theta\right), \tag{7}$$

where θ is the slowness angle measured from the symmetry axis.

The condition for the off-axis triplication (4) expressed in terms of anisotropy parameters yields the following cubic equation for critical value σ_c of parameter σ under which an incipient triplication occurs

$$\sigma_c^3 + A\sigma_c^2 + B\sigma_c + C = 0\ , \tag{8}$$

where

$$A = \frac{1}{\kappa-1}\left[\frac{3}{2}\kappa^2(2\varepsilon+1)+\kappa(\varepsilon+1)+\frac{3}{2}\right],\quad B = 2\frac{\kappa}{(\kappa-1)^2}(\varepsilon+1)\left[\kappa^2(2\varepsilon+1)+1\right],$$

$$C = -2\frac{\kappa^2}{(\kappa-1)^2}\left[\kappa(2\varepsilon+1)^2-2\varepsilon-1\right],$$

where

$$\kappa = \frac{a_{33}}{a_{44}}.$$

The solution of Eq. (8) can be found without problems and yields an interval of values of the σ parameter the medium triplicates for:

$$\sigma \geq \sigma_c, \tag{10}$$

where

$$\sigma_c = u + v - \frac{1}{3}\ , \tag{11}$$

$$u=\left[-\frac{q}{2}+\sqrt{\left(\frac{q}{2}\right)^2+\left(\frac{p}{3}\right)^3}\right]^{\frac{1}{3}} \quad , \quad v=\left[-\frac{q}{2}-\sqrt{\left(\frac{q}{2}\right)^2+\left(\frac{p}{3}\right)^3}\right]^{\frac{1}{3}}, \tag{12}$$

$$p=B-\frac{1}{3}A^2 \quad , \quad q=-\frac{1}{3}AB+\frac{2}{27}A^3+C \ . \tag{13}$$

The behaviour of σ_c is shown in Fig 4. The figure shows that σ_c is significantly influenced by κ and ε. Interestingly, low values of κ result in low values of σ_c. For example, if we assume $\kappa=2$ and $\varepsilon=-0.12$, the triplication occurs for $\sigma\geq\sigma_c=0.2$, This represents a medium, in which triplication occurs under the SV anisotropy of 8% only. For value $\kappa=3$, corresponding to P-to-S velocity ratio $v_P/v_S=\sqrt{3}$, and for ε ranging from -0.2 to 0.2, we obtain σ_c in the interval $\sigma_c\in\langle 0.24,\ 0.62\rangle$. These values correspond to the SV anisotropy in the range of 9% to 12% .

Approximate triplication conditions

The exact solution (11) of the cubic equation (8) is still rather complicated. Therefore, we shall try to simplify it further. The exact solution (11) can be expanded into a power series in parameters ε and κ^{-1}. Utilizing only several first terms of the expansion, we obtain the approximate formula for σ_c in the form:

$$\sigma_c=\sigma^{(0)}+\sigma^{(1)}+\sigma^{(2)}+\sigma^{(3)}+\ldots \ , \tag{14}$$

$$\sigma^{(0)}=\frac{2}{3} \ , \tag{15}$$

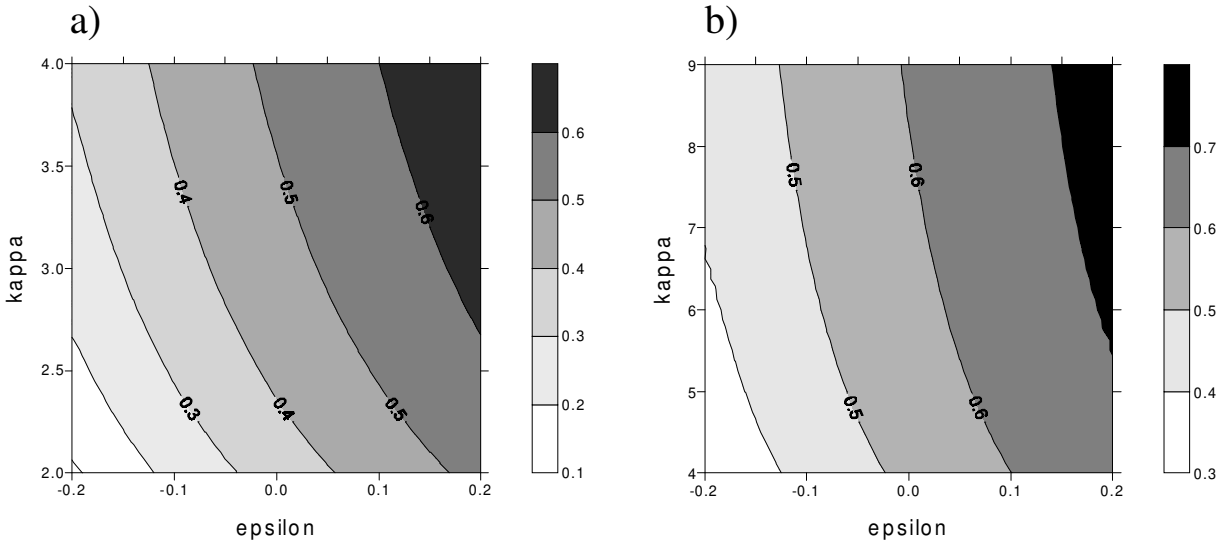

Figure 4. The σ parameter for the incipient off-axis triplication as a function of parameters κ and ε. (a) $2 \le \kappa \le 4$, (b) $4 \le \kappa \le 9$.

$$\sigma^{(1)} = \frac{2}{3}\left(\varepsilon - \frac{7}{9}\kappa^{-1} \right), \tag{16}$$

$$\sigma^{(2)} = \frac{2}{3}\left(-\frac{3}{4}\varepsilon^2 + \varepsilon\kappa^{-1} - \frac{31}{81}\kappa^{-2} \right), \tag{17}$$

$$\sigma^{(3)} = \frac{2}{3}\left(\frac{3}{4}\varepsilon^3 - \frac{43}{36}\varepsilon^2\kappa^{-1} + \frac{49}{81}\varepsilon\kappa^{-2} - \frac{71}{729}\kappa^{-3} \right). \tag{18}$$

Generally, the higher order of the expansion is considered, the higher accuracy of the formula for σ_c should be obtained. The first two terms of the expansion, $\sigma^{(0)}$ and $\sigma^{(1)}$, were also derived

by Thomsen & Dellinger (2003). They obtained the following approximate formulas for σ_c (Thomsen & Dellinger 2003, Eq. 15):

$$\sigma_c = \frac{2}{3}\left(1+\delta-\frac{1}{9\kappa}\right), \tag{19}$$

or alternatively (Thomsen & Dellinger 2003, Eq. 23)

$$\sigma_c = \frac{2}{3}\,\frac{1+\varepsilon-1/9\kappa^{-1}}{1+2/3\kappa^{-1}}\ . \tag{20}$$

Eq. (19) utilizes the δ parameter instead of ε. This significantly lowers the accuracy of this approximation. Eq. (20) is equivalent to the first-order approximation (16). In the next section, we shall examine accuracy of the derived formulas together with formula (19) presented by Thomsen & Dellinger (2003).

Tests of accuracy

We test the accuracy of approximate formulas for σ_c by calculating relative errors of approximations defined as follows

$$e = \frac{\left|\sigma_c^{exact}-\sigma_c^{aprox}\right|}{\sigma_c^{exact}}\cdot 100\%\ . \tag{20}$$

The accuracy is examined for the first-, second- and third-order approximations (14)-(18) derived in the previous section and compared with the accuracy of Eq. (19). Equation (20) is equivalent to (16) and, therefore, it is not tested. We use the following intervals of parameters: $\varepsilon \in \langle -0.2,\ 0.2\rangle$ and $\kappa \in \langle 2,9\rangle$. These intervals are rather broad and should include a majority of geophysically interesting materials. Fig. 5 shows the relative errors of approximations calculated

for $\kappa \in \langle 2,4 \rangle$, and Fig. 6 for $\kappa \in \langle 4,9 \rangle$, respectively. The figures indicate that all formulas work better for higher values of κ and for positive values of ε. The accuracy is significantly lowered if values of κ and ε decrease. Comparing different approximations, the worse accuracy is obtained using the approximation (19). For example, this approximation yields an error of about 20% for values $\kappa = 3$ and $\varepsilon = 0$. Comparing with this approximation, the first-order approximation is approximately two times more accurate. The second- and third-order approximations are even 10 and 40 times more accurate than the approximation (19).

Conclusion

The proposed approximate formulas yield a critical value σ_c of parameter σ, for which an incipient off-axis SV-wave triplication occurs in transversely isotropic media. Although the formulas are simple, they approximate the exact solution with a high accuracy. The best results are obtained by the third-order approximation (14)-(18), which yields accuracy 40 or 20 times higher than formulas (19) and (20). The formula works safely for parameters $\kappa > 2$ and $\varepsilon > -0.2$, being thus applicable to a broad interval of TI parameters. Out of this interval, it is recommended to use the exact solution.

The behaviour of σ_c indicates that triplications do not occur for a fixed strength of anisotropy. The critical strength of anisotropy, i.e. the strength of anisotropy for which the medium triplicates, varies with variation of parameters κ and ε. High values of κ and positive ε produce high values of σ_c, implying that the critical strength of the SV-wave anisotropy is rather high. On the contrary, low values of κ and negative ε cause that wave fronts in TI tend to triplicate more easily, meaning that the critical strength of the SV-wave anisotropy is low. For a typical value of κ, $\kappa = 3$, which corresponds to $v_P / v_S = \sqrt{3}$, and for ε ranging from -0.2 to 0.2, parameter σ_c lies in the interval $\sigma_c \in \langle 0.24, 0.62 \rangle$ (see Fig. 4). These values correspond to the SV anisotropy in the range of 9% to 12% .

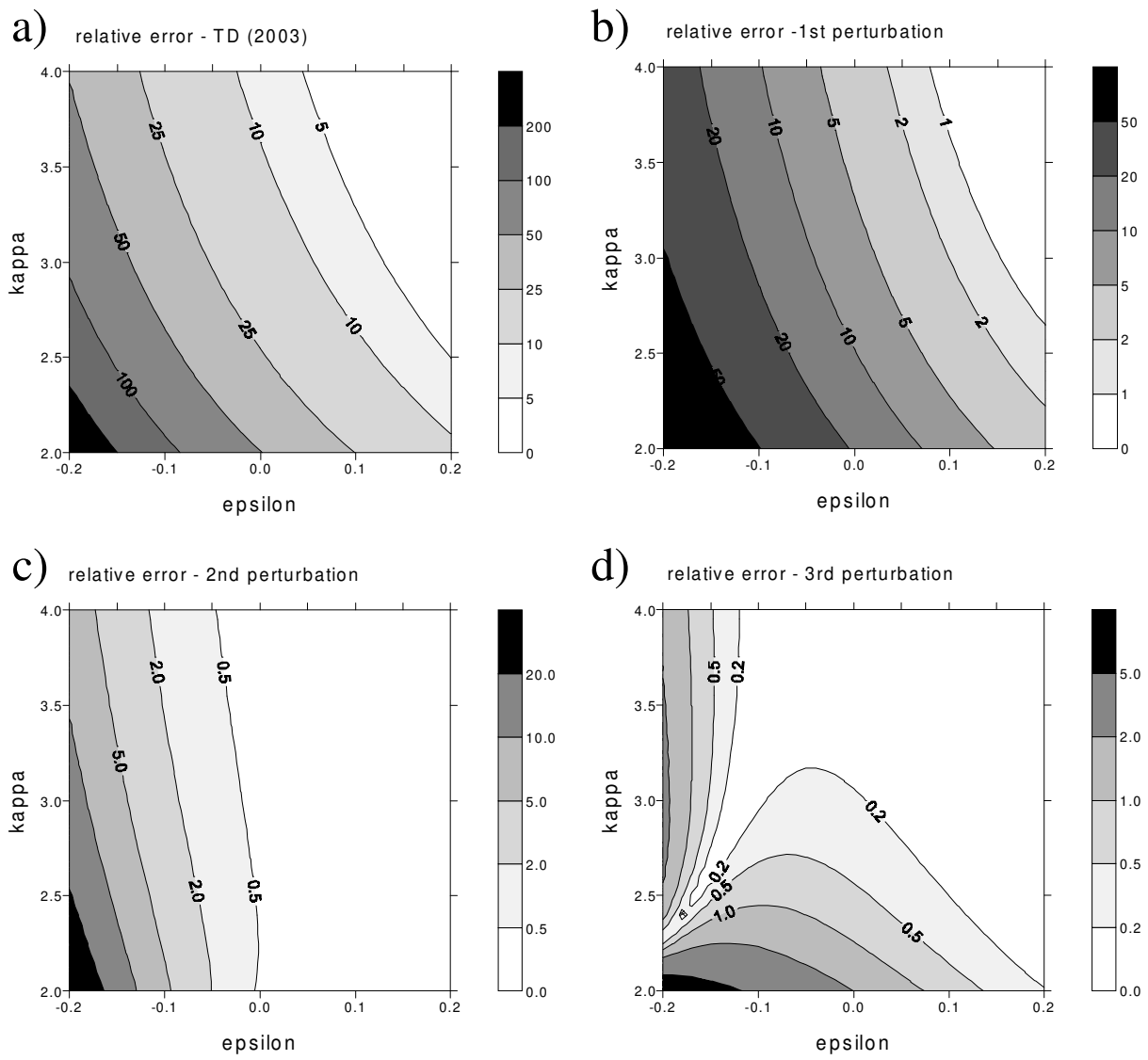

Figure 5. Relative errors (in per cent) of approximate formulas for σ_c, $2 \le \kappa \le 4$. (a) Formula by Thomsen & Dellinger (2003, Eq. 15), (b) first-order approximation, (c) second-order approximation, (d) third-order approximation.

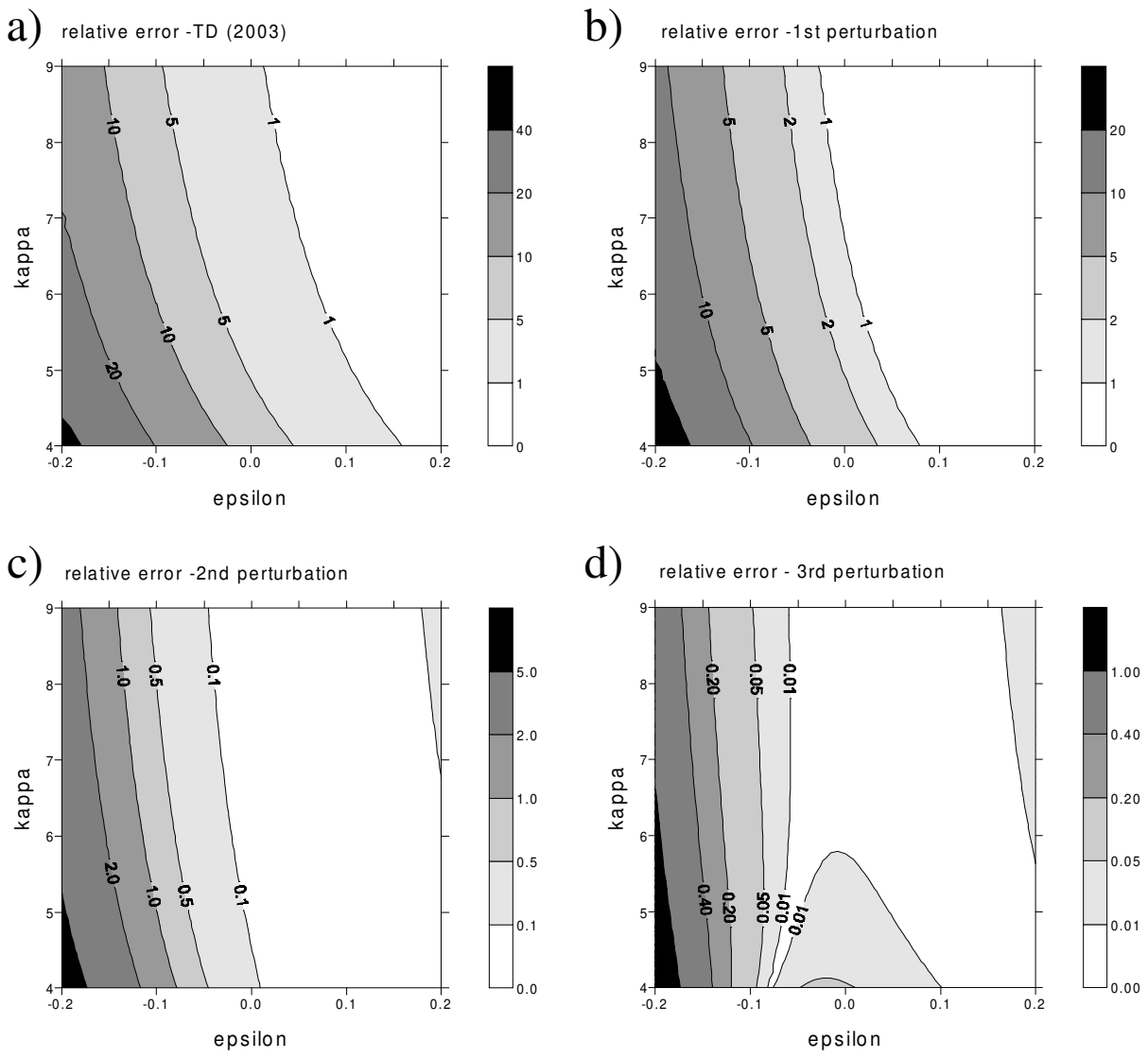

Figure 6. Relative errors (in per cent) of approximate formulas for σ_c, $4 \le \kappa \le 9$. (a) Formula by Thomsen & Dellinger (2003, Eq. 15), (b) first-order approximation, (c) second-order approximation, (d) third-order approximation.

Acknowledgements

The work was supported by the Consortium Project "Seismic waves in complex 3-D structures", and by the Grant Agency of the Academy of Sciences of the Czech Republic, grant No. A3012309.

References

Alshits, V.I. & Chadwick, P., 1997. Concavities on the zonal slowness section of a transversely isotropic material, *Wave Motion*, **25**, 347-359.

Backus, G.E., 1962. Long-wave anisotropy produced by horizontal layering, *J. Geophys. Res.*, **66**, 4427-4440.

Bakker, P.M., 1998. Phase shift at caustics along rays in anisotropic media, *Geophys. J. Int.*, **134**, 515-518.

Červený, V., 2001. *Seismic Ray Theory*, Cambridge University Press, Cambridge.

Dellinger, J., 1991. Anisotropic seismic wave propagation, PhD Thesis, Stanford University.

Helbig, K., 1958. Elastischen wellen in anisotropen medien, Teil I, *Gerlands Beiträge zur Geophysik*, **67**, 177-211.

Helbig, K., 1994. *Foundations of Anisotropy for Exploration Seismics*, Pergamon, N.Y.

Helbig, K. & Schoenberg, M., 1987. Anomalous polarization of elastic waves in transversely isotropic media, *J. Acoust. Soc. Am.*, **81**, 1235-1245.

Kravtsov, Yu.A. & Orlov, Yu.I., 1990. *Geometrical Optics of Inhomogeneous Media*, Springer-Verlag, New York.

McCurdy, A.K., 1974. Phonon focusing and phonon conduction in hexagonal crystals in the boundary scattering regime, *Phys. Rev. B*, **9**, 466-480.

Musgrave, M.J.P., 1970. *Crystal Acoustics*, Holden-Day, London.

Musgrave, M.J.P., 1979. Further criteria for elastic waves in anisotropic media, *J. Elasticity*, **9**, 105-112.

Musgrave, M.J.P & Payton, R.G., 1984. Criteria for elastic waves in anisotropic media – a consolidation, *J. Elasticity*, **14**, 269-285.

Payton, R. G., 1983. *Elastic Wave Propagation in Transversely Isotropic Media*, Martinus Nijhoff Publishers, The Hague.

Thomsen, L., 1986. Weak elastic anisotropy, *Geophysics*, **51**, 1954-1966.

Thomsen, L. & Dellinger, J., 2003. On shear-wave triplication in transversely isotropic media, *J. Appl. Geophys*., accepted.

Tsvankin, I., & Thomsen, L., 1994. Nonhyperbolic reflection moveout in anisotropic media, *Geophysics*, **59**, 1290-1304.

Vavryčuk, V., 2003. Parabolic lines and caustics in homogeneous weakly anisotropic solids, *Geophys. J. Int.*, in print.

Wolfe, J.P., 1998. *Imaging Phonons. Acoustic Wave Propagation in Solids*, Cambridge University Press, Cambridge.