Perturbations and spatial derivatives of amplitude in isotropic and anisotropic media

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Summary

The partial derivatives of travel time or amplitude with respect to model parameters are referred to as perturbations. Explicit equations for the spatial derivatives and perturbations of amplitude in both isotropic and anisotropic media are derived. The spatial derivatives and perturbations of the logarithm of amplitude can be calculated by numerical quadratures along rays.

Keywords

Ray theory, amplitude, transport equation, paraxial approximation, perturbation methods, inhomogeneous media, seismic anisotropy.

1. Introduction

The model of a smooth elastic medium for ray tracing is assumed to be specified by a set of smooth functions. This set of smooth functions is assumed to be an element of a linear normed space, hereinafter called the *model space*. We shall refer to the coordinates of the model with respect to a selected basis in the model space as *model parameters*. The set of model parameters may also include additional parameters, parametrizing the continuous system of Hamiltonians in the continuous system of eikonal equations (stationary Hamilton–Jacobi equations) for travel time. We shall refer to the partial derivatives of travel time, amplitude or other quantities with respect to the model parameters as *perturbations* of travel time, amplitude or other quantities. Note that the partial derivatives are generally defined as the Gâteaux variations in the directions of the corresponding basis functions.

We have introduced the term "perturbations" in order to distinguish briefly the partial derivatives with respect to model parameters from the partial derivatives with respect to spatial or phase–space coordinates. In this way, we shall distinguish between the *partial phase–space derivatives* and perturbations of the Hamiltonian, and between the *spatial derivatives* and perturbations of travel time, amplitude or other quantities.

This paper is devoted to the calculation of the perturbations and spatial derivatives of amplitude, which is the solution of the *transport equation*. In this paper, the transport equation is formally attached to an orthonomic system of rays corresponding to a given eikonal equation (stationary Hamilton–Jacobi equation) for travel time.

The travel time and its first–order spatial derivatives in a general smoothly heterogeneous model can be calculated by solving numerically a set of nonlinear ordinary differential equations called ray tracing equations. The amplitude and the second–order spatial derivatives of travel time can be calculated by solving numerically a set of linear ordinary differential equations called dynamic ray tracing equations, both in isotropic

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media (Babich & Buldyrev 1972; Arnaud 1973; Červený & Hron 1980) and anisotropic media (Červený 1972). For an overview of the ray theory and paraxial ray methods with their applications and relevant references refer to the book by Červený (2001).

The third–order and higher–order spatial derivatives of travel time can be computed by simple numerical quadratures along rays, both in isotropic media (Babich, Buldyrev & Molotkov 1985; Červený 2001) and anisotropic media (Klimeš 1997; 1999; 2002a). The first–order and higher–order perturbations of travel time and of its spatial derivatives can also be computed by analogous numerical quadratures along rays, both in isotropic media and anisotropic media (Klimeš 2002a).

We present the explicit equations for the first–order and higher–order perturbations and for the first–order and higher–order spatial derivatives of amplitude in Cartesian coordinates. The perturbations and spatial derivatives of the logarithm of amplitude are expressed in the form of integrals along the ray, with the proper initial values. The integrands are composed of spatial derivatives and perturbations of travel time, of spatial derivatives and perturbations of the logarithm of amplitude, and of the partial phase–space derivatives and perturbations of the Hamiltonian. The aggregate orders of spatial derivatives and perturbations of amplitude in the integrands are equal to or lower than the aggregate order of spatial derivatives and perturbations of amplitude being computed. Simultaneously, the orders of perturbations of amplitude in the integrands are lower than the order of perturbations of amplitude or its spatial derivative being computed. The aggregate orders of spatial derivatives and perturbations of travel time and of the Hamiltonian in the integrands do not exceed the aggregate order of spatial derivatives and perturbations of amplitude being computed by more than 2. The order of perturbations of travel time and of the Hamiltonian is equal to or lower than the order of perturbations of amplitude being computed. The perturbations and spatial derivatives of the Hamiltonian contain perturbations and spatial derivatives of density–normalized elastic moduli (or of a power of the phase velocity in isotropic media) up to the order corresponding to that of the Hamiltonian.

Note that the equations for the polarization vectors depend considerably on the kind of the high–frequency asymptotic ray theory. The equations for the spatial derivatives and perturbations of anisotropic ray theory polarization vectors and of isotropic ray theory P–wave polarization vector can be derived directly from the Christoffel equation (Červený 2001, eq. 2.2.20) for the polarization vectors. The equations for the first–order spatial derivatives and first–order perturbations of isotropic ray theory S–wave polarization vectors have been derived by Coates & Chapman (1990) and Klimeš(2002b).

2. Equations for the derivatives and perturbations of amplitude

In *Section 2.1*, we introduce the definitions and notation used throughout the paper. In *Section 2.2*, we express the eikonal equation in the general form of the stationary Hamilton–Jacobi equation, define the transport equation for amplitude, and derive its consequence, identity (16), which is of principal importance in deriving the equations for the spatial derivatives and perturbations of amplitude. In *Section 2.3*, we define ray parameters and recall the basics of dynamic ray tracing. The dynamic ray tracing equations are necessary for calculating the amplitude, and for introducing the covariant derivatives of amplitude with respect to the ray parameters. In *Section 2.4*, we define the covariant derivatives of amplitude with respect to the ray parameters and utilize identity (16) in deriving equations for calculating the covariant derivatives with respect to the ray parameters and perturbations of amplitude. Section 2.4 constitutes the final result of the paper.

2.1. Basic definitions and notation

The model of a smooth elastic medium for calculating travel time and amplitude is assumed to be specified by the set $f = f(x_i)$ of smooth functions. The smooth functions may represent, e.g., the spatial distributions of the individual density–normalized elastic moduli. This set of smooth functions is assumed to be an element of a linear normed space, called the *model space*. Assume that the model space is generated by a linearly independent set $\{b_\kappa\}$ of its elements, each element $b_\kappa = b_\kappa(x_i)$ being a set of smooth functions. Set $\{b_\kappa\}$ then represents the basis of the model space. We shall refer to the coordinates f_κ of *model* $f(x_i) = \sum_\kappa f_\kappa\, b_\kappa(x_i)$ with respect to the selected basis $\{b_\kappa\}$ in the model space as the *model parameters.*

We assume that the Hamiltonian is a function of *phase–space coordinates*, composed of spatial coordinates x_i and slowness–vector components p_j, and of model parameters f_κ,

$$H = H(x_i, p_j, f_\kappa) \quad . \tag{1}$$

Similarly, travel time τ, amplitude A and other quantities calculated in the model are assumed to be functions of *spatial coordinates* x_i and of model parameters f_κ,

$$\tau = \tau(x_i, f_\kappa) \quad , \qquad A = A(x_i, f_\kappa) \quad . \tag{2}$$

Note that, for a given model determined by model parameters f_κ, we may define a continuous set of the Hamiltonians, parametrized by several additional parameters f_κ. For example, there are two S–wave polarizations with the two respective Hamiltonians corresponding to the anisotropic ray theory in anisotropic media. Instead of selecting one of these two Hamiltonians, we may consider a one–parametric set of Hamiltonians, parametrized by f_μ, such that we obtain the Hamiltonian corresponding to one polarization for $f_\mu = -1$, the Hamiltonian corresponding to the other polarization for $f_\mu = +1$, and the averaged Hamiltonian of both S–wave polarizations corresponding to the anisotropic common ray approximation by Bakker (2002) for $f_\mu = 0$. We may also consider a two–parametric set of Hamiltonians, parametrized by f_μ and f_ν, such that we obtain the Hamiltonian corresponding to the isotropic common ray approximation in the reference isotropic medium for $(f_\mu, f_\nu) = (0, 0)$, the Hamiltonian corresponding to one anisotropic S–wave polarization for $(f_\mu, f_\nu) = (1, 0)$, the Hamiltonian corresponding to the other anisotropic S–wave polarization for $(f_\mu, f_\nu) = (0, 1)$, and the averaged

Hamiltonian corresponding to the anisotropic common ray approximation by Bakker (2002) for $(f_\mu, f_\nu) = (\frac{1}{2}, \frac{1}{2})$. For the sake of simplicity, we shall also refer to these additional parameters f_κ, parametrizing the Hamiltonians in a given model, as the *model parameters*.

We shall refer to the partial derivatives with respect to model parameters f_κ as the *perturbations*.

We shall use the Einstein summation convention, and denote the partial derivatives with respect to spatial coordinates x_i by lower–case Roman subscripts following a comma, and the *perturbations by lower–case Greek subscripts* following a comma. In the phase space, we denote the partial derivatives with respect to components p_j of the slowness vector by lower–case Roman superscripts following a comma. For instance,

$$H^{,ab\dots f}_{,ij\dots n\alpha\dots\nu} = \frac{\partial}{\partial f_\alpha}\cdots\frac{\partial}{\partial f_\nu}\,\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\cdots\frac{\partial}{\partial x_n}\,\frac{\partial}{\partial p_a}\frac{\partial}{\partial p_b}\cdots\frac{\partial}{\partial p_f}H \tag{3}$$

denotes the perturbations of the partial *phase–space derivatives* of the Hamiltonian. The perturbations are taken at fixed phase–space coordinates x_i, p_j, and the partial phase–space derivatives are calculated at fixed model parameters f_κ. Similarly,

$$\tau_{,ij\dots n\alpha\dots\nu} = \frac{\partial}{\partial f_\alpha}\cdots\frac{\partial}{\partial f_\nu}\,\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\cdots\frac{\partial}{\partial x_n}\tau \tag{4}$$

denotes the perturbations of the *spatial derivatives* of travel time τ. The perturbations are taken at fixed spatial coordinates x_i, and the partial spatial derivatives are calculated at fixed model parameters f_κ.

We also introduce "multi–indices" in order to make the equations more concise and easy to survey. For example,

$$\begin{matrix}\tau_{,rij} & \tau_{,rk\alpha} \\ {}_{,rik} & {}_{,rj\alpha} \\ {}_{,rjk} & {}_{,ri\alpha}\end{matrix} \tag{5}$$

stands for

$$\tau_{,rij}\,\tau_{,rk\alpha} + \tau_{,rik}\,\tau_{,rj\alpha} + \tau_{,rjk}\,\tau_{,ri\alpha} \quad . \tag{6}$$

2.2. Transport equation

In both isotropic and anisotropic media, we assume the *travel time* (action function) $\tau(x_i, f_\kappa)$ determined by the *eikonal equation* in the general form of the *stationary Hamilton–Jacobi equation*

$$H\big(x_i, \tau_{,j}(x_m, f_\mu), f_\kappa\big) = 0 \quad , \tag{7}$$

where the function $H(x_i, p_j, f_\kappa)$ of the phase–space coordinates and model parameters is the Hamiltonian, see (1). The corresponding *ray tracing* equations then have the form of Hamilton's equations

$$\frac{\mathrm{d}}{\mathrm{d}\gamma_3}x_i = \frac{\partial}{\partial p_i}H(x_q, p_r, f_\kappa) \quad , \qquad \frac{\mathrm{d}}{\mathrm{d}\gamma_3}p_i = -\frac{\partial}{\partial x_i}H(x_q, p_r, f_\kappa) \quad , \tag{8}$$

where parameter γ_3 along rays is determined by the form of the Hamiltonian. The solution of the Hamilton–Jacobi equation (7), travel time τ, is then calculated by the quadrature of equation

$$\frac{\mathrm{d}}{\mathrm{d}\gamma_3}\tau = p_i\,\frac{\partial}{\partial p_i}H(x_q, p_r, f_\kappa) \tag{9}$$

along a ray.

We define the *transport equation* for amplitude $A=A(x_m,f_\kappa)$ as

$$\frac{\partial}{\partial x_i}(A^2\varrho V_i)=0 \quad , \tag{10}$$

where the *ray velocity vector* V_i is defined as the derivative of coordinates x_i with respect to travel time τ along a ray,

$$V_i=\frac{\mathrm{d}x_i}{\mathrm{d}\tau}=\frac{\mathrm{d}x_i}{\mathrm{d}\gamma_3}\left(\frac{\mathrm{d}\tau}{\mathrm{d}\gamma_3}\right)^{-1} \quad . \tag{11}$$

In our abstract formulation, function $\varrho=\varrho(x_m,f_\kappa)$ is a function from the model space, parametrizing the transport equation. If A is the amplitude of displacement of an elastic wavefield, ϱ is the density. We shall thus refer to function $\varrho=\varrho(x_m,f_\kappa)$ as the *density*.

Transport equation (10) is formally attached to an orthonomic system of rays corresponding to stationary Hamilton–Jacobi equation (7). For a particular choice of the Hamiltonian, transport equation (10) may have a relevant physical meaning. For example, if the Hamilton–Jacobi equation defines the anisotropic ray theory travel time corresponding to a selected eigenvalue of the Christoffel matrix, transport equation (10) describes the amplitude A of the zero–order anisotropic ray theory solution of the elastodynamic equation (Červený 2001, eq. 2.4.47).

We express amplitude $A=A(x_i,f_\kappa)$ in terms of *amplitude exponent* $\beta=\beta(x_i,f_\kappa)$,

$$A=\sqrt{\frac{1}{\varrho}\frac{\mathrm{d}\tau}{\mathrm{d}\gamma_3}}\;\exp(\beta) \quad . \tag{12}$$

Transport equation (10) with (11) and (12) reads

$$\frac{\partial}{\partial x_i}\left[\exp(2\beta)\frac{\mathrm{d}x_i}{\mathrm{d}\gamma_3}\right]=0 \quad , \tag{13}$$

and after differentiating the product,

$$\frac{\partial\beta}{\partial x_i}\frac{\mathrm{d}x_i}{\mathrm{d}\gamma_3}+\frac{1}{2}\frac{\partial}{\partial x_i}\frac{\mathrm{d}x_i}{\mathrm{d}\gamma_3}=0 \quad . \tag{14}$$

Inserting $\mathrm{d}x_i/\mathrm{d}\gamma_3$ from ray tracing equations (8), we may express transport equation (14) in terms of the Hamiltonian,

$$\left[\beta_{,i}+\frac{1}{2}\left(\frac{\partial}{\partial x_i}+\tau_{,ik}\frac{\partial}{\partial p_k}\right)\right]\frac{\partial}{\partial p_i}H=0 \quad . \tag{15}$$

The perturbations and spatial derivatives of transport equation (15) yield

$$\begin{aligned}&\left(\frac{\partial}{\partial f_\alpha}+\tau_{,s\alpha}\frac{\partial}{\partial p_s}\right)\cdots\left(\frac{\partial}{\partial f_\nu}+\tau_{,z\nu}\frac{\partial}{\partial p_z}\right)\\ &\times\left(\frac{\partial}{\partial x_i}+\tau_{,ia}\frac{\partial}{\partial p_a}\right)\cdots\left(\frac{\partial}{\partial x_n}+\tau_{,nf}\frac{\partial}{\partial p_f}\right)\left[\beta_{,r}+\frac{1}{2}\left(\frac{\partial}{\partial x_r}+\tau_{,rk}\frac{\partial}{\partial p_k}\right)\right]\frac{\partial}{\partial p_r}H=0 \ .\end{aligned} \tag{16}$$

Here the perturbations and partial phase–space derivatives are computed at point $(x_q,p_r)=\big(x_q,\tau_{,r}(x_s,f_\kappa)\big)$ in the phase space.

2.3. Ray parameters and dynamic ray tracing

We define the matrices of the partial derivatives of coordinates x_i and of the slowness–vector components $\tau_{,i}$ with respect to 3 *ray parameters*: ray take–off parameters γ_1, γ_2, and independent parameter γ_3 along rays,

$$Q_{ia} = \frac{\partial x_i}{\partial \gamma_a} \quad , \qquad P_{ia} = \frac{\partial \tau_{,i}}{\partial \gamma_a} \quad , \tag{17}$$

where $\frac{\partial}{\partial \gamma_3}$ is identical to $\frac{\mathrm{d}}{\mathrm{d}\gamma_3}$ used in other equations. Matrices Q_{ia} and P_{ia} describe, by definition, the properties of the orthonomic system of rays corresponding to the travel time under consideration. They may be expressed in terms of their initial values and the paraxial ray propagator matrix (Červený 2001). Let us emphasize that the definition of matrices Q_{ia} and P_{ia} depends on the kind of parameter γ_3 along rays, which is in turn determined by the form of the Hamiltonian, refer to Section 5 for examples. The use of matrices Q_{ia} and P_{ia}, defined for a particular kind of γ_3, in the equations for the perturbations and spatial derivatives of amplitude, derived from the Hamiltonian corresponding to different γ_3, has to be avoided. Matrices Q_{ia} and P_{ia} are often calculated for $\gamma_3 = \tau$. The transformation of matrix Q_{ia} to the one defined for different γ_3 has been described by Klimeš (1997; 1999).

Equation

$$P_{ia} = \tau_{,ij} \, Q_{ja} \tag{18}$$

is a direct consequence of the above definitions. The *dynamic ray tracing* system

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\gamma_3} Q_{ia} &= \quad H_{,j}^{,i}(x_q, p_r) \, Q_{ja} + H^{,ij}(x_q, p_r) \, P_{ja} \\ \frac{\mathrm{d}}{\mathrm{d}\gamma_3} P_{ia} &= -H_{,ij}(x_q, p_r) \, Q_{ja} - H_{,i}^{,j}(x_q, p_r) \, P_{ja} \end{aligned} \tag{19}$$

can be obtained by differentiating the ray tracing system (8) with respect to γ_a (Červený 1972; 2001; Hanyga 1982; Klimeš 1994). For more details on the dynamic ray tracing system, including the initial conditions, refer to the book by Červený (2001). After insertion of (18), the first equation of the dynamic ray tracing system (19) becomes

$$\frac{\mathrm{d}}{\mathrm{d}\gamma_3} Q_{ia} = Q_{ja} \left(\frac{\partial}{\partial x_j} + \tau_{,jk} \frac{\partial}{\partial p_k} \right) \frac{\partial}{\partial p_i} H \quad . \tag{20}$$

Let us emphasize once again that the matrices Q_{ia} and P_{ia} in this paper are defined by equation (17), and correspond to the orthonomic system of rays under consideration.

2.4. Equations for the spatial derivatives and perturbations of the amplitude exponent

The spatial derivatives and perturbations of amplitude (12) can be composed of the spatial derivatives and perturbations of amplitude exponent $\beta = \beta(x_m, f_\kappa)$, density $\varrho = \varrho(x_m, f_\kappa)$ and function

$$\frac{\mathrm{d}\tau}{\mathrm{d}\gamma_3} = H^{,r}(x_m, \tau_{,n}(x_k, f_\mu), f_\kappa)\; \tau_{,r}(x_n, f_\nu) \quad . \tag{21}$$

The spatial derivatives and perturbations of function $\mathrm{d}\tau/\mathrm{d}\gamma_3$ are composed of the spatial derivatives and perturbations of functions $H^{,i}(x_m, \tau_{,n}(x_k, f_\mu), f_\kappa)$ and $\tau_{,i}(x_n, f_\kappa)$. The spatial derivatives and perturbations of travel time $\tau(x_n, f_\kappa)$ can be calculated using the equations derived by Klimeš(2002a). It is thus sufficient to derive the equations for calculation of the spatial derivatives and perturbations of amplitude exponent $\beta = \beta(x_m, f_\kappa)$.

We introduce the *covariant derivatives*

$$B_{ab\ldots f\alpha\ldots\nu} = \beta_{,ij\ldots n\alpha\ldots\nu}\, Q_{ia} Q_{jb} \cdots Q_{nf} \tag{22}$$

of the amplitude exponent with respect to ray parameters γ_a. Let us differentiate covariant derivatives (22) along a ray with respect to γ_3, and insert equations (8) and (20) for $\frac{\mathrm{d}}{\mathrm{d}\gamma_3} x_i$ and $\frac{\mathrm{d}}{\mathrm{d}\gamma_3} Q_{ia}$, respectively,

$$\frac{\mathrm{d}}{\mathrm{d}\gamma_3}[\beta_{,ijk\ldots mn\alpha\ldots\nu} Q_{ia} Q_{jb} Q_{kc} \cdots Q_{me} Q_{nf}] = C_{ijk\ldots mn\alpha\ldots\nu} Q_{ia} Q_{jb} Q_{kc} \cdots Q_{me} Q_{nf} \ , \tag{23}$$

where

$$\begin{aligned} C_{ijk\ldots mn\alpha\ldots\nu} = \beta_{,rijk\ldots mn\alpha\ldots\nu} \frac{\partial}{\partial p_r} H + \beta_{,rjk\ldots mn\alpha\ldots\nu} & \left(\frac{\partial}{\partial x_i} + \tau_{,is}\frac{\partial}{\partial p_s}\right)\frac{\partial}{\partial p_r} H \\ + \beta_{,irk\ldots mn\alpha\ldots\nu} & \left(\frac{\partial}{\partial x_j} + \tau_{,js}\frac{\partial}{\partial p_s}\right)\frac{\partial}{\partial p_r} H \\ & \vdots \\ + \beta_{,ijk\ldots mr\alpha\ldots\nu} & \left(\frac{\partial}{\partial x_n} + \tau_{,ns}\frac{\partial}{\partial p_s}\right)\frac{\partial}{\partial p_r} H \quad . \end{aligned} \tag{24}$$

The covariant derivatives with respect to the ray parameters and perturbations of the amplitude exponent may thus be expressed in the form of the integral along the ray,

$$\boxed{B_{ab\ldots f\alpha\ldots\nu}(\gamma_3) = B_{ab\ldots f\alpha\ldots\nu}(\gamma_3^0) + \int_{\gamma_3^0}^{\gamma_3} \mathrm{d}\gamma_3\; C_{ij\ldots n\alpha\ldots\nu}\; Q_{ia} Q_{jb} \cdots Q_{nf} \quad ,} \tag{25}$$

with initial conditions $B_{ab\ldots f\alpha\ldots\nu}(\gamma_3^0)$ defined by equation (22). The initial conditions for the covariant derivatives are generally finite for the finite initial value $Q_{ia}(\gamma_3^0)$ of the matrix of geometrical spreading. The initial values $B_{ab\ldots f\alpha\ldots\nu}(\gamma_3^0)$ at a point source are infinite, similarly as the covariant derivatives (22) are infinite at caustics. A point source thus requires a separate treatment.

The spatial derivatives in coordinates x_i and perturbations of the amplitude exponent may then be expressed in terms of covariant derivatives (25),

$$\boxed{\beta_{,ij\ldots n\alpha\ldots\nu} = B_{ab\ldots f\alpha\ldots\nu}\; Q^{-1}_{ai} Q^{-1}_{bj} \cdots Q^{-1}_{fn} \quad ,} \tag{26}$$

where we have used Q^{-1}_{ai} to denote the components of the matrix inverse to matrix Q_{ia}. Matrix Q^{-1}_{ai} exists and is finite off caustics.

The perturbations and spatial derivatives of the amplitude exponent on the right–hand side of equation (24) coincide with the corresponding terms on the left–hand side of equation (16), and are thus cancelled if subtracting (16) from (24),

$$\boxed{\begin{aligned} C_{ijk...mn\alpha...\nu} = \beta_{,rijk...mn\alpha...\nu} \frac{\partial}{\partial p_r} H + \beta_{,rjk...mn\alpha...\nu} \left(\frac{\partial}{\partial x_i} + \tau_{,is} \frac{\partial}{\partial p_s} \right) \frac{\partial}{\partial p_r} H \\ + \beta_{,irk...mn\alpha...\nu} \left(\frac{\partial}{\partial x_j} + \tau_{,js} \frac{\partial}{\partial p_s} \right) \frac{\partial}{\partial p_r} H \\ \vdots \\ + \beta_{,ijk...mr\alpha...\nu} \left(\frac{\partial}{\partial x_n} + \tau_{,ns} \frac{\partial}{\partial p_s} \right) \frac{\partial}{\partial p_r} H \\ - \left(\frac{\partial}{\partial f_\alpha} + \tau_{,u\alpha} \frac{\partial}{\partial p_u} \right) ... \left(\frac{\partial}{\partial f_\nu} + \tau_{,z\nu} \frac{\partial}{\partial p_z} \right) \\ \times \left(\frac{\partial}{\partial x_i} + \tau_{,ia} \frac{\partial}{\partial p_a} \right) ... \left(\frac{\partial}{\partial x_n} + \tau_{,nf} \frac{\partial}{\partial p_f} \right) \left[\beta_{,r} + \frac{1}{2} \left(\frac{\partial}{\partial x_r} + \tau_{,rs} \frac{\partial}{\partial p_s} \right) \right] \frac{\partial}{\partial p_r} H \ , \qquad (27) \end{aligned}}$$

for all integration kernels (27), including C corresponding to amplitude exponent β.

If we calculate the spatial derivatives and perturbations in the order $\beta_{,i}$, $\beta_{,\alpha}$, $\beta_{,ij}$, $\beta_{,i\alpha}$, $\beta_{,\alpha\beta}$, $\beta_{,ijk}$, $\beta_{,ij\alpha}$, $\beta_{,i\alpha\beta}$, $\beta_{,\alpha\beta\gamma}$, $\beta_{,ijkl}$, $\beta_{,ijk\alpha}$, $\beta_{,ij\alpha\beta}$, $\beta_{,i\alpha\beta\gamma}$, $\beta_{,\alpha\beta\gamma\delta}$, etc., the right–hand side of equation (27) contains only the previously calculated spatial derivatives and perturbations of the amplitude exponent. The special cases of equation (27) corresponding to the zero–order, first–order and second–order derivatives and perturbations of the amplitude exponent are presented in Section 3 as examples.

Let us illustrate the order of calculating the spatial derivatives and perturbations of the amplitude exponent graphically:

$$\begin{array}{lllllll} \beta_{,i} & \beta_{,\alpha} & & & & & \\ \beta_{,ij} & \beta_{,i\alpha} & \beta_{,\alpha\beta} & & & & \\ \beta_{,ijk} & \beta_{,ij\alpha} & \beta_{,i\alpha\beta} & \beta_{,\alpha\beta\gamma} & & & \\ \beta_{,ijkl} & \beta_{,ijk\alpha} & \beta_{,ij\alpha\beta} & \beta_{,i\alpha\beta\gamma} & \beta_{,\alpha\beta\gamma\delta} & & \\ \beta_{,ijklm} & \beta_{,ijkl\alpha} & \beta_{,ijk\alpha\beta} & \beta_{,ij\alpha\beta\gamma} & \beta_{,i\alpha\beta\gamma\delta} & \beta_{,i\alpha\beta\gamma\delta\varepsilon} & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

To calculate the derivatives or perturbations of the amplitude exponent of a selected order, the derivatives and perturbations of the amplitude exponent situated on the left and above the selected ones must be known, with exception of $\beta_{,\alpha}$ which is not present in the equations for other perturbations of the amplitude exponent.

Note that covariant derivatives $B_{...3...}$, containing at least one index 3 in place of the lower–case subscripts corresponding to the covariant derivatives with respect to the ray parameters, can be expressed analytically. For instance, for $a = 3$, equation (22) with (17) and (8) reads

$$B_{3b...f\alpha...\nu} = \beta_{,ij...n\alpha...\nu} \, H^{,i} \, Q_{jb} \cdots Q_{nf} \quad . \qquad (28)$$

Equation (24) then yields

$$\boxed{B_{3b\ldots f\alpha\ldots\nu} = [C_{j\ldots n\alpha\ldots\nu} - (H^{,s}_{\substack{,j\\ \vdots\\ ,n}} + H^{,rs}\tau_{\substack{,rj\\ \vdots\\ ,rn}})\,\beta_{\substack{,s\ldots n\alpha\ldots\nu\\ \vdots\\ ,j\ldots s\alpha\ldots\nu}}]\, Q_{jb}\cdots Q_{nf} \quad .} \tag{29}$$

We thus need to calculate only $B_{A\ldots F\alpha\ldots\nu}$ with $A,...,F = 1,2$ (including $B_{A\ldots F}$ and $B_{\alpha\ldots\nu}$) by numerical quadrature (25). Initial conditions $B_{A\ldots F\alpha\ldots\nu}(\gamma_3^0)$ are defined by equation (22).

3. Examples of the integrands and integrals in general media

The special cases of the general results of Section 2 corresponding to the zero–order, first–order and second–order perturbations and spatial derivatives of the amplitude exponent are presented in Sections 3.1, 3.2, and 3.3, respectively, in order to illustrate somewhat abstract final equations (27) and (29).

3.1. Amplitude

For the amplitude exponent itself, integrand (27) of equation (25) takes the form of

$$C = \beta_{,r}\frac{\partial}{\partial p_r}H - \left[\beta_{,r} + \frac{1}{2}\left(\frac{\partial}{\partial x_r} + \tau_{,rs}\frac{\partial}{\partial p_s}\right)\right]\frac{\partial}{\partial p_r}H \quad , \tag{30}$$

which reduces to

$$C = -\tfrac{1}{2}(H^{,r}_{,r} + H^{,rs}\tau_{,rs}) \quad . \tag{31}$$

Inserting equation (20) into the right–hand side of expression

$$\frac{\mathrm{d}}{\mathrm{d}\gamma_3}\ln\big[\det(Q_{ia})\big] = Q^{-1}_{bj}\frac{\mathrm{d}}{\mathrm{d}\gamma_3}Q_{jb} \tag{32}$$

for the derivative of the logarithm of a determinant, and comparing the result with equation (31), we see that

$$C = -\frac{1}{2}\,\frac{\mathrm{d}}{\mathrm{d}\gamma_3}\ln\big[\det(Q_{ia})\big] \quad , \tag{33}$$

and

$$\beta(\gamma_3) = \beta(\gamma_3^0) + \tfrac{1}{2}\ln\big[\det(Q_{ia})\big](\gamma_3^0) - \tfrac{1}{2}\ln\big[\det(Q_{ia})\big](\gamma_3) \quad . \tag{34}$$

This is a generally known expression for the logarithm of amplitude (Babich 1961, eq. 3.7). The amplitude is thus obtained by dynamic ray tracing (19), without any additional numerical quadrature.

3.2. First–order perturbations and spatial derivatives of amplitude

For the first–order spatial derivatives of the amplitude exponent, integrand (27) of equation (25) takes the form of

$$\begin{aligned} C_i =& \beta_{,ri}\,\frac{\partial}{\partial p_r}H + \beta_{,r}\left(\frac{\partial}{\partial x_i}+\tau_{,is}\frac{\partial}{\partial p_s}\right)\frac{\partial}{\partial p_r}H \\ & -\left(\frac{\partial}{\partial x_i}+\tau_{,ia}\frac{\partial}{\partial p_a}\right)\left[\beta_{,r}+\frac{1}{2}\left(\frac{\partial}{\partial x_r}+\tau_{,rs}\frac{\partial}{\partial p_s}\right)\right]\frac{\partial}{\partial p_r}H\ , \end{aligned} \tag{35}$$

which reduces to integrand

$$C_i = -\tfrac{1}{2}\left[H^{,r}_{,ri}+H^{,rs}_{,i}\tau_{,rs}+H^{,rs}_{,r}\tau_{,si}+H^{,rst}\tau_{,rs}\tau_{,ti}+H^{,rs}\tau_{,rsi}\right] \quad . \tag{36}$$

The value of the integral (25) of $C_i Q_{i3}$ is given by equations (29) and (31),

$$B_3 = -\tfrac{1}{2}\left(H^{,r}_{,r}+H^{,rs}\tau_{,rs}\right) \quad . \tag{37}$$

For the first–order perturbations of the amplitude exponent, integrand (27) of equation (25) takes the form of

$$C_\alpha = \beta_{,r\alpha}\,\frac{\partial}{\partial p_r}H - \left(\frac{\partial}{\partial f_\alpha}+\tau_{,a\alpha}\frac{\partial}{\partial p_a}\right)\left[\beta_{,r}+\frac{1}{2}\left(\frac{\partial}{\partial x_r}+\tau_{,rs}\frac{\partial}{\partial p_s}\right)\right]\frac{\partial}{\partial p_r}H\ , \tag{38}$$

which reduces to

$$C_\alpha = -\beta_{,r}\left[H^{,r}_{,\alpha}+H^{,rs}\tau_{,s\alpha}\right] - \tfrac{1}{2}\left[H^{,r}_{,r\alpha}+H^{,rs}_{,\alpha}\tau_{,rs}+H^{,rs}_{,r}\tau_{,s\alpha}+H^{,rst}\tau_{,rs}\tau_{,t\alpha}+H^{,rs}\tau_{,rs\alpha}\right]\ . \tag{39}$$

3.3. Second–order perturbations and spatial derivatives of amplitude

For the second-order spatial derivatives of the amplitude exponent, equation (27) reads

$$\begin{aligned} C_{ij} =& \beta_{,rij}\,\frac{\partial}{\partial p_r}H + \beta_{,rj}\left(\frac{\partial}{\partial x_i}+\tau_{,is}\frac{\partial}{\partial p_s}\right)\frac{\partial}{\partial p_r}H + \beta_{,ir}\left(\frac{\partial}{\partial x_j}+\tau_{,js}\frac{\partial}{\partial p_s}\right)\frac{\partial}{\partial p_r}H \\ & -\left(\frac{\partial}{\partial x_i}+\tau_{,ia}\frac{\partial}{\partial p_a}\right)\left(\frac{\partial}{\partial x_j}+\tau_{,jb}\frac{\partial}{\partial p_b}\right)\left[\beta_{,r}+\frac{1}{2}\left(\frac{\partial}{\partial x_r}+\tau_{,rs}\frac{\partial}{\partial p_s}\right)\right]\frac{\partial}{\partial p_r}H\ , \end{aligned} \tag{40}$$

which reduces to integrand

$$\begin{aligned} C_{ij} =& -\beta_{,r}\left[H^{,r}_{,ij}+H^{,rs}_{\substack{,i\\,j}}\tau_{\substack{,sj\\,si}}+H^{,rst}\tau_{,si}\tau_{,tj}+H^{,rs}\tau_{,sij}\right] \\ & -\tfrac{1}{2}\left[H^{,r}_{,rij}+H^{,rs}_{,ij}\tau_{,rs}+H^{,rs}_{\substack{,ri\\,rj}}\tau_{\substack{,sj\\,si}}+H^{,rst}_{\substack{,i\\,j}}\tau_{,rs}\tau_{\substack{,tj\\,ti}}+H^{,rst}_{,r}\tau_{,si}\tau_{,tj}+H^{,rstu}\tau_{,rs}\tau_{,ti}\tau_{,uj}\right. \\ & \left. +H^{,rs}_{\substack{,i\\,j}}\tau_{\substack{,rsj\\,rsi}}+H^{,rs}_{,r}\tau_{,sij}+H^{,rst}\tau_{,rs}\tau_{,tij}+H^{,rs}\tau_{,rsij}\right] \quad . \end{aligned} \tag{41}$$

The value of the integral (25) of $C_{ij}Q_{i3}Q_{jb}$ is determined by equation (29),

$$B_{3b} = \left[C_j - \beta_{,r}\left(H^{,r}_{,j}+H^{,rs}\tau_{,sj}\right)\right]Q_{jb} \quad , \tag{42}$$

where C_j is given by equation (36). For the first–order perturbations of the first–order spatial derivatives of the amplitude exponent, equation (27) reads

$$\begin{aligned} C_{i\alpha} =& \beta_{,ri\alpha}\,\frac{\partial}{\partial p_r}H + \beta_{,r\alpha}\left(\frac{\partial}{\partial x_i}+\tau_{,is}\frac{\partial}{\partial p_s}\right)\frac{\partial}{\partial p_r}H \\ & -\left(\frac{\partial}{\partial f_\alpha}+\tau_{,a\alpha}\frac{\partial}{\partial p_a}\right)\left(\frac{\partial}{\partial x_i}+\tau_{,ib}\frac{\partial}{\partial p_b}\right)\left[\beta_{,r}+\frac{1}{2}\left(\frac{\partial}{\partial x_r}+\tau_{,rs}\frac{\partial}{\partial p_s}\right)\right]\frac{\partial}{\partial p_r}H\ , \end{aligned} \tag{43}$$

which reduces to

$$
\begin{aligned}
C_{i\alpha} = & -\beta_{,ri}\left[H^{,r}_{,\alpha} + H^{,rs}\tau_{,s\alpha}\right] - \beta_{,r}\left[H^{,r}_{,i\alpha} + H^{,rs}_{\substack{,i\\,\alpha}}\tau_{\substack{,s\alpha\\,si}} + H^{,rst}\tau_{,si}\tau_{,t\alpha} + H^{,rs}\tau_{,si\alpha}\right] \\
& -\tfrac{1}{2}\left[H^{,r}_{,ri\alpha} + H^{,rs}_{,i\alpha}\tau_{,rs} + H^{,rs}_{\substack{,ri\\,r\alpha}}\tau_{\substack{,s\alpha\\,si}} + H^{,rst}_{\substack{,i\\,\alpha}}\tau_{,rs}\tau_{\substack{,t\alpha\\,ti}} + H^{,rst}_{,r}\tau_{,si}\tau_{,t\alpha} + H^{,rstu}\tau_{,rs}\tau_{,ti}\tau_{,u\alpha}\right. \\
& \qquad \left. + H^{,rs}_{\substack{,i\\,\alpha}}\tau_{\substack{,rs\alpha\\,rsi}} + H^{,rs}_{,r}\tau_{,si\alpha} + H^{,rst}\tau_{,rs}\tau_{,ti\alpha} + H^{,rs}\tau_{,rsi\alpha}\right] \quad .
\end{aligned}
\tag{44}
$$

The value of the integral (25) of $C_{i\alpha}Q_{i3}$ is determined by equation (29),

$$
B_{3\alpha} = C_\alpha \quad , \tag{45}
$$

where C_α is given by equation (39). For the second–order perturbations of the amplitude exponent, equation (27) becomes

$$
\begin{aligned}
C_{\alpha\beta} = & \beta_{,r\alpha\beta}\,\frac{\partial}{\partial p_r}H \\
& -\left(\frac{\partial}{\partial f_\alpha} + \tau_{,a\alpha}\frac{\partial}{\partial p_a}\right)\left(\frac{\partial}{\partial f_\beta} + \tau_{,b\beta}\frac{\partial}{\partial p_b}\right)\left[\beta_{,r} + \frac{1}{2}\left(\frac{\partial}{\partial x_r} + \tau_{,rs}\frac{\partial}{\partial p_s}\right)\right]\frac{\partial}{\partial p_r}H \quad ,
\end{aligned}
\tag{46}
$$

which can be expressed as

$$
\begin{aligned}
C_{\alpha\beta} = & -\beta_{\substack{,r\alpha\\,r\beta}}\left[H^{,r}_{\substack{,\beta\\,\alpha}} + H^{,rs}\tau_{\substack{,s\beta\\,s\alpha}}\right] - \beta_{,r}\left[H^{,r}_{,\alpha\beta} + H^{,rs}_{\substack{,\alpha\\,\beta}}\tau_{\substack{,s\beta\\,s\alpha}} + H^{,rst}\tau_{,s\alpha}\tau_{,t\beta} + H^{,rs}\tau_{,s\alpha\beta}\right] \\
& -\tfrac{1}{2}\left[H^{,r}_{,r\alpha\beta} + H^{,rs}_{,\alpha\beta}\tau_{,rs} + H^{,rs}_{\substack{,r\alpha\\,r\beta}}\tau_{\substack{,s\beta\\,s\alpha}} + H^{,rst}_{\substack{,\alpha\\,\beta}}\tau_{,rs}\tau_{\substack{,t\beta\\,t\alpha}} + H^{,rst}_{,r}\tau_{,s\alpha}\tau_{,t\beta} + H^{,rstu}\tau_{,rs}\tau_{,t\alpha}\tau_{,u\beta}\right. \\
& \qquad \left. + H^{,rs}_{\substack{,\alpha\\,\beta}}\tau_{\substack{,rs\beta\\,rs\alpha}} + H^{,rs}_{,r}\tau_{,s\alpha\beta} + H^{,rst}\tau_{,rs}\tau_{,t\alpha\beta} + H^{,rs}\tau_{,rs\alpha\beta}\right] \quad .
\end{aligned}
\tag{47}
$$

4. Conclusions

The amplitude exponent is defined by equation (12). The spatial derivatives and perturbations of amplitude can be composed of the spatial derivatives and perturbations of the Hamiltonian, density, travel time and amplitude exponent. The spatial derivatives and perturbations of the amplitude exponent in isotropic or anisotropic media can be calculated using equation (26), with the covariant derivatives with respect to ray parameters and perturbations of the amplitude exponent given partly explicitly by equation (29) and partly by numerical quadratures (25). The general form of the integration kernels of quadratures (25) is given by equation (27). The special cases of integration kernels (27), corresponding to the individual spatial derivatives and perturbations of the amplitude exponent up to the second order in generally anisotropic media, are shown in Section 3.

The derived equations are applicable only along ray segments touching no caustics. Numerical quadratures (25), required for the calculation of spatial derivatives and perturbations of the amplitude exponent, cannot be performed through caustic points. Simultaneously, equation (22) cannot be used to define the initial conditions at point or line sources. These singularities thus require a separate treatment, postponed for future investigation.

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