Complex two-point rays in anisotropic inhomogeneous weakly dissipative media

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Summary

Ray perturbation theory is used to derive two-point complex ray equations in anisotropic, inhomogeneous, weakly dissipative media. The reference medium $\mathcal{M}^0$ is assumed to be non-dissipative, and the reference ray Ω^0 connecting the points, S and R, to be real-valued. Simple equations for the complex, two-point, perturbed ray Ω, situated in the perturbed, weakly dissipative medium $\mathcal{M}$, connecting points S and R, are derived. The basic step in the derivation consists in the determination of the initial-value attenuation vector at point S (i.e., of the imaginary-valued part of the slowness vector at S).

Key words
Ray perturbation theory, inhomogeneous weakly dissipative anisotropic media, two-point complex rays.

1 Introduction

In dissipative anisotropic inhomogeneous media $\mathcal{M}$, elastic parameters are complex-valued. We consider a dissipative medium $\mathcal{M}$ to represent a small perturbation of a reference elastic (non-dissipative) medium $\mathcal{M}^0$. Consequently, we consider only weakly dissipative media $\mathcal{M}$.

In the whole treatment, we work in the frequency domain, and consider a fixed frequency ω. We broadly exploit the notation of Červený (2002), and the equations in a 6-D real phase space x_i, p_i (where x_i are Cartesian coordinates and p_i the Cartesian components of the slowness vector), derived there. These equations remain valid even in a 12-D complex phase space x_i, p_i. Thus, they may be used to study complex rays in weakly dissipative media. To make this paper self-consistent, we briefly explain certain important terms and equations, discussed in a greater detail in Červený (2002).

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For more details and physical meaning of complex rays in dissipative media, see Hearn and Krebes (1990 a,b), Thomson (1997), and Kravtsov, Forbes and Asatryan (1999), where many other important references can be found.

2 Complex rays in weakly dissipative media

Consider an elementary seismic body wave (qP, qS1 or qS2) propagating in a reference, non-dissipative, anisotropic inhomogeneous medium $\mathcal{M}^0$. We denote the Hamiltonian corresponding to this wave, propagating in $\mathcal{M}^0$, by $\mathcal{H}^0(x_i, p_i)$ (with x_i and p_i real).

We further consider two fixed points, S and R, situated in a 3-D real space, and a presumably known reference ray Ω^0 of the elementary wave under consideration, connecting the points S and R. The real ray Ω^0 is parameterized by a monotonic parameter u, related to the Hamiltonian $\mathcal{H}^0(x_i, p_i)$ under consideration. The reference ray Ω^0, with the initial point at S, is situated in the real space and is specified by equations

$$x_i^0 = x_i^0(u) \ , \qquad p_i^0 = p_i^0(u) \ , \qquad u_S \le u \le u_R \ . \tag{1}$$

Here x_i^0 and p_i^0 are real-valued coordinates and components of the slowness vector of the reference ray Ω^0, u_S and u_R denote the values of the ray parameter u at points S and R, respectively. The reference ray Ω^0 can be computed by the standard ray tracing.

Along Ω^0, we can perform dynamic ray tracing, and determine the 6×6 ray propagator matrix $\mathbf{\Pi}^0(u, u_S)$ in Cartesian coordinates. Moreover, we introduce the 3×3 partition matrices $\mathbf{\Pi}_{IJ}(u, u_S)$ $(I, J = 1, 2)$ as follows:

$$\mathbf{\Pi}^0(u, u_S) = \begin{pmatrix} \mathbf{\Pi}_{11}^0(u, u_S) & \mathbf{\Pi}_{12}^0(u, u_S) \\ \mathbf{\Pi}_{21}^0(u, u_S) & \mathbf{\Pi}_{22}^0(u, u_S) \end{pmatrix} . \tag{2}$$

Let us emphasize again that $x_i^0(u)$, $p_i^0(u)$, and $\mathbf{\Pi}_{IJ}^0(u, u_S)$ are real-valued along the reference ray Ω^0.

Consider now a ray Ω of the elementary wave under consideration in the *perturbed, weakly dissipative medium* $\mathcal{M}$, also connecting points S and R, situated in the real space. The ray can be described by the parameteric equations,

$$x_i(u) = x_i^0(u) + \Delta x_i(u) \ , \qquad p_i(u) = p_i^0(u) + \Delta p_i(u) \ . \tag{3}$$

The general expressions for the ray perturbations $\Delta x_i(u)$ and $\Delta p_i(u)$ are derived in Červený (2002). These equations are valid also for $\Delta x_i(u)$ and $\Delta p_i(u)$ complex-valued, which must be considered in weakly-dissipative media. The complex-valued Hamiltonian function $\mathcal{H}(x_i, p_i)$ corresponding to the elementary wave under consideration in the perturbed, weakly dissipative medium $\mathcal{M}$ can be expressed as follows:

$$\mathcal{H}(x_i, p_i) = \mathcal{H}^0(x_i, p_i) + \Delta\mathcal{H}(x_i, p_i) \ . \tag{4}$$

Here $\mathcal{H}^0(x_i, p_i)$ is real-valued, and $\Delta\mathcal{H}(x_i, p_i)$ is presumably imaginary-valued.

It was shown by Červený (2002) that for the rays Ω^0 and Ω, connecting the same points S and R, the perturbation of the slowness vector $\Delta\mathbf{p}^S = \Delta\mathbf{p}(u_S)$ at the initial point S is given by the relation

$$\Delta\mathbf{p}^S = -(\mathbf{\Pi}^0_{12})^{-1}(\mathbf{C}(u_R,u_S) + \boldsymbol{\mathcal{U}}^0(u_S)\Delta u) \ , \tag{5}$$

where Δu is given by the equation,

$$\Delta u = \frac{\Delta\mathcal{H}(u_S) - \boldsymbol{\mathcal{U}}^{0T}(u_S)(\mathbf{\Pi}^0_{12})^{-1}\mathbf{C}(u_R,u_S)}{\boldsymbol{\mathcal{U}}^{0T}(u_S)(\mathbf{\Pi}^0_{12})^{-1}\boldsymbol{\mathcal{U}}^0(u_R)} \ . \tag{6}$$

Here $\mathbf{\Pi}^0_{12} = \mathbf{\Pi}^0_{12}(u_R,u_S)$, and $\boldsymbol{\mathcal{U}}^0$ denotes the vector tangent to the reference ray Ω^0, with Cartesian components $\mathcal{U}^0_i = \partial\mathcal{H}^0/\partial p_i$, T denotes transpose. The physical meaning of Δu is explained in Červený (2002).

The 3×1 column matrix $\mathbf{C}(u,u_S)$, and the closely related 3×1 column matrix $\mathbf{D}(u,u_S)$, are given by relations:

$$\begin{aligned}\mathbf{C}(u,u_S) &= \int_{u_S}^{u}[\mathbf{\Pi}^0_{11}(u,u')\mathbf{E}(u') + \mathbf{\Pi}^0_{12}(u,u')\mathbf{F}(u')]\mathrm{d}u' \ , \\ \mathbf{D}(u,u_S) &= \int_{u_S}^{u}[\mathbf{\Pi}^0_{21}(u,u')\mathbf{E}(u') + \mathbf{\Pi}^0_{22}(u,u')\mathbf{F}(u')]\mathrm{d}u' \ .\end{aligned} \tag{7}$$

Finally, the 3×1 column matrices $\mathbf{E}$ and $\mathbf{F}$ are specified as follows,

$$E_i(u) = \frac{\partial\Delta\mathcal{H}(u)}{\partial p_i} \ , \qquad F_i(u) = -\frac{\partial\Delta\mathcal{H}(u)}{\partial x_i} \ . \tag{8}$$

Equations (5)–(8) are valid even in weakly dissipative perturbed media. In this case, however, $\Delta\mathcal{H}(u_S)$, $\mathbf{E}(u)$, $\mathbf{F}(u)$, $\mathbf{C}(u,u_S)$, $\mathbf{D}(u,u_S)$ and Δu are imaginary-valued. Consequently, $\Delta\mathbf{p}^S$ *is also imaginary-valued*.

The *complex ray* Ω in the weakly dissipative anisotropic inhomogeneous medium $\mathcal{M}$, connecting two points S and R (which are situated in the real space), is then given by the relations (3) with

$$\begin{aligned}\Delta\mathbf{x}(u) &= \mathbf{\Pi}^0_{12}(u,u_S)\Delta\mathbf{p}^S + \mathbf{C}(u,u_S) \ , \\ \Delta\mathbf{p}(u) &= \mathbf{\Pi}^0_{22}(u,u_S)\Delta\mathbf{p}^S + \mathbf{D}(u,u_S) \ ,\end{aligned} \tag{9}$$

see Červený (2002). Both $\Delta\mathbf{x}(u)$ and $\Delta\mathbf{p}(u)$ are imaginary-valued along the whole ray Ω except the points S and R, at which $\Delta\mathbf{x}(u)$ is zero.

In the theory of plane waves propagating in homogeneous dissipative media, the slowness vector $\mathbf{p}$ has been usually expressed as follows

$$\mathbf{p} = \mathbf{P} + \mathrm{i}\mathbf{A} \ , \tag{10}$$

where the real-valued vector $\mathbf{P}$ is called the *propagations vector*, and the real-valued vector $\mathbf{A}$ the *attenuation vector*. Consequently, in weakly dissipative media, $\Delta\mathbf{p}^S$ given by (5) represents the attenuation vector at point S. Clearly, the attenuation vector $\Delta\mathbf{p}^S$ depends not only on the dissipative parameters of the medium, but also on the position of the receiver R. This corresponds fully to the conclusions of Hearn and Krebes (1990a,b), derived for 1-D isotropic dissipative media. Equations (5)–(8), presented here, generalize the results given by Hearn and Krebes to 3-D anisotropic inhomogeneous media. However, they are applicable only to weakly dissipative media, not to strongly dissipative media.

3 Simple example

Now we shall consider the same example as in Červený (2002), in which the velocity $V(x_i)$ of an elementary wave (P or S) in an isotropic medium is specified by a simple relation

$$\frac{1}{V^2(x_i)} = \frac{1}{V_S^2} + b(x_3 - x_3^S) \ . \tag{11}$$

Here x_i are Cartesian coordinates, x_i^S the real-valued Cartesian coordinates of a fixed point S, $V_S = V(x_i^S)$. Thus, (11) represents a one-dimensional isotropic medium, in which the square of slowness varies linearly with the coordinate x_3. The point S is, without loss of generality, considered as an initial point of the rays Ω and Ω^0. The point R is arbitrary.

Considering the same Hamiltonian $\mathcal{H}(x_i, p_i)$ as in Section 4 of Červený (2002), $\mathcal{H}(x_i, p_i) = \frac{1}{2}(p_k p_k - V^{-2}(x_i))$, and using (4), we obtain the reference Hamiltonian $\mathcal{H}^0(x_i, p_i)$ and the Hamiltonian perturbation $\Delta\mathcal{H}(x_i, p_i)$ as follows

$$\mathcal{H}^0(x_i, p_i) = \tfrac{1}{2}(p_k p_k - 1/V_S^2) \ , \qquad \Delta\mathcal{H}(x_i, p_i) = -\tfrac{1}{2}b(x_3 - x_3^S) \ . \tag{12}$$

The reference medium $\mathcal{M}^0$ is homogeneous, thus the reference ray Ω^0 is a straight line connecting points S and R. The perturbed model $\mathcal{M}$ is non-dissipative for b real-valued, and dissipative for b imaginary-valued. (Of course, it would be possible to consider even complex-valued b.) The relevant parameter u along the reference ray equals σ, related to the elementary arclength $\mathrm{d}s$ along the ray by the relation $\mathrm{d}\sigma = V \mathrm{d}s$.

We now consider the weakly dissipative model $\mathcal{M}$ specified by (11), with b imaginary valued and $|b|$ small. Equations (5)–(8) can be then used to compute *the initial value of the attenuation vector* $\Delta\mathbf{p}^S$, see Červený (2002) for details:

$$\Delta\mathbf{p}^S = -\tfrac{1}{4} b l_0 V_S (\mathbf{n} - V_S^2 p_3^0 \mathbf{p}^0) \ . \tag{13}$$

Here $\mathbf{n}$ is the unit vector along the x_3-axis, and l_0 is the distance between S and R. As b is pressumably imaginary-valued, also $\Delta\mathbf{p}^S$ is imaginary-valued.

As $\mathbf{p}^0$ specifies the direction of the reference ray (connecting points S and R), the initial-value attenuation vector $\Delta\mathbf{p}^S$ is situated in the plane specified by the reference ray and by $\mathbf{n}$, which represents the gradient of $\mathrm{Im}(V^{-2}(x_i))$. Using (9), it is possible to write the equations for the complex ray in the medium (11) and to show that $\Delta\mathbf{p}(u)$ remains confined to this plane along the whole ray.

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