Acoustical axes in weakly anisotropic media in the first-order approximation [1]

Véronique Farra and **Ivan Pšenčík**

Département de Sismologie, Institut de Physique du Globe de Paris, 4 Place Jussieu, 75252 Paris Cedex 05, France; e-mail: farra@ipgp.jussieu.fr

Geophysical Institute, Acad.Sci.of Czech Republic, Boční II, Praha 4, Czech Republic; e-mail: ip@ig.cas.cz

Summary

Perturbation theory is a useful tool for approximate study of wave properties in weakly anisotropic media. Especially its first-order approximation yields simple and transparent formulae only slightly more complicated than those for isotropic media. In this contribution we study effects of the first-order approximation on the position of acoustical axes in weakly anisotropic media. We show that longitudinal directions in the first-order approximation are identical with actual ones. In singular directions, however, the first-order formulae provide singular directions, which may deviate from the exact ones or which may even provide false singular directions.

Key words: First- and higher-order perturbations, acoustical axes, weak anisotropy

1 Introduction

Acoustical axes are a characteristic feature of anistropic media. We can distinguish two kinds of acoustical axes (Schoenberg and Helbig, 1997). The first are the well-known *singular directions* in which the two qS waves propagating in anisotropic media propagate with the same phase velocity. These directions correspond to the places on the slowness surfaces at which the two qS-wave sheets are in a contact. Characteristic feature of singular directions and their vicinity is strong variation of qS-wave polarization vectors. The second kind of acoustical axes are *longitudinal directions*. Characteristic feature of longitudinal directions is propagation of purely longitudinal and purely transverse waves along them, i.e. propagation of qP waves with polarization parallel to the slowness vector (specifying the longitudinal direction) and propagation of qS waves with polarization perpendicular to the slowness vector. Helbig (1993) and Schoenberg and Helbig (1997) propose to use them

[1]Seismic Waves in Complex 3-D Structures, Report 12 (Department of Geophysics, Charles University, Prague 2002), pp.151-169

in inverting observed data into elastic parameters specifying anisotropy of the medium. It thus seems that their importance might increase in near future. In this contribution, we study effects of frequently used approximations on the accuracy with which the acoustical axes can be determined.

In Sec.2, we briefly review the basic results of Farra (2001). An alternative specification of two unit vectors specifying the qS-wave polarization, which leads to simplified polarization formulae is proposed. In Sec.3, the conditions for the determination of acoustical axes in the first-order approximation of the perturbation theory are specified and discussed. The accuracy of the determination of acoustical axes determined by the perturbation formulae is studied. In Sec.4, accuracy of perturbation formulae is illustrated on several numerical examples in media used in literature.

In the following, component notation is used. All the Roman lowercase indices range over the values 1, 2 and 3 and the uppercase indices over the values 1 and 2. Einstein summation convention is used for the repeated subscripts. Voigt notation $A_{\alpha\beta}$ for density normalized elastic parameters, with α, β running from 1 to 6, is used in parallel with the tensor notation a_{ijkl}.

2 Higher-order perturbation formulae

We briefly review the formulae for the calculation of higher-order terms for the phase velocity and polarization of qP and qS waves derived by Farra (2001).

Let us start from the Christoffel matrix

$$\Gamma_{jk} = a_{ijkl} n_i n_l. \tag{1}$$

The symbol a_{ijkl} denotes a tensor of density-normalized elastic parameters and n_i is a unit wave vector. The Christoffel matrix Γ_{jk} is symmetric and positive definite. Thus it has three positive and real eigenvalues $G_m = G_m(n_j)$ and three corresponding eigenvectors $g_i^m(n_j)$. They satisfy the following system of equations

$$(\Gamma_{jk} - G_m \delta_{jk}) g_k^m = 0. \tag{2}$$

Two eigenvalues G_1 and G_2 with corresponding eigenvectors g_i^1 and g_i^2 correspond to the two qS waves, remaining eigenvalue G_3 with the eigenvector g_i^3 belongs to the qP wave. The eigenvalues G_1 and G_2 are, as a rule, smaller than G_3. Since the relation between G_1 and G_2 can be chosen arbitrarily, we choose $G_2 \leq G_1 < G_3$. The eigenvalues are related to squares of corresponding phase velocities $c_m(n_j)$,

$$G_m(n_j) = c_m^2(n_j). \tag{3}$$

The eigenvectors g_i^m specify polarization vectors of the corresponding waves.

The tensor a_{ijkl} can be expressed as follows:

$$a_{ijkl} = a_{ijkl}^0 + \Delta a_{ijkl}. \tag{4}$$

Here a^0_{ijkl} is a tensor of density normalized elastic parameters in a reference isotropic medium:

$$a^0_{ijkl} = (\alpha^2 - 2\beta^2)\delta_{ij}\delta_{kl} + \beta^2(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \tag{5}$$

and Δa_{ijkl} is its perturbation. In (5), α and β denote P- and S-wave velocities of the reference isotropic medium. In the reference isotropic medium, we introduce three mutually perpendicular unit vectors $g_i^{(0)1}$, $g_i^{(0)2}$ and $g_i^{(0)3}$ so that $g_i^{(0)3} = n_i$. The vectors $g_i^{(0)1}$ and $g_i^{(0)2}$, situated in the plane perpendicular to $g_i^{(0)3}$, can be chosen arbitrarily. An option is the choice (A1) given in the Appendix A.

The vectors $g_i^{(0)1}$ and $g_i^{(0)2}$ chosen in (A1) are used in the Appendix A to define a matrix $\bar{B}_{mn}$

$$\bar{B}_{mn} = \Gamma_{jk} g_j^{(0)m} g_k^{(0)n}. \tag{6}$$

The matrix $\bar{B}_{mn}$ is related to the *weak anisotropy* (WA) matrix B_{mn} used, e.g., by Pšenčík and Gajewski (1998), Eq.(11). From that equation we get

$$\bar{B}_{mn} = B_{mn} + c_0^2\delta_{mn}. \tag{6a}$$

The symbol c_0 in Eq.(6a) stands for the phase velocity. For $m = n = 1$ or 2 $c_0 = \beta$, for $m = n = 3$ $c_0 = \alpha$. For explicit expressions of $\bar{B}_{mn}$ see the Appendix A.

In the following we also frequently use a matrix $\hat{B}_{mn}$ defined by Eq.(6) with vectors $g_i^{(0)1}$ and $g_i^{(0)2}$ substituted by vectors $\hat{g}_i^{(0)1}$ and $\hat{g}_i^{(0)2}$ chosen so that

$$\hat{B}_{12} = 0, \qquad \hat{B}_{11} > \hat{B}_{22}. \tag{7}$$

An important property of the matrices $\bar{B}_{mn}$ and $\hat{B}_{mn}$ is their independency of the phase velocities α and β of the reference isotropic medium.

2.1 qP wave

We follow Farra (2001) and express the polarization vector g_i^3, related to the qP wave, in the vector basis $(\hat{g}_i^{(0)1}, \hat{g}_i^{(0)2}, g_i^{(0)3})$. Inserting g_i^3 specified in this way into (2) and multiplying the resulting equation successively by $\hat{g}_i^{(0)1}$, $\hat{g}_i^{(0)2}$ and $g_i^{(0)3}$, we get a set of three equations from which we can determine a vector parallel to g_i^3:

$$\frac{\hat{B}_{13}}{G_3 - G_1^{(1)}}\hat{g}_i^{(0)1} + \frac{\hat{B}_{23}}{G_3 - G_2^{(1)}}\hat{g}_i^{(0)2} + g_i^{(0)3}. \tag{8}$$

In Eq.(8), $G_1^{(1)}$ and $G_2^{(1)}$ are the first-order approximations of the eigenvalues G_1 and G_2, respectively. Eq.(8) is derived under assumption that the expressions in the denominators are non-zero. For qP waves, this is practically always guaranteed.

The vector (8) specifies the exact direction of polarization vector of the qP wave. Note that the vector (8) is not unit. In order to identify it with g_i^3, normalization of the vector (8) is necessary. Since the determination of the vectors $\hat{g}_j^{(0)K}$ and $g_j^{(0)3}$, of the elements of the matrix $\hat{B}_{mn}$ and of the first-order approximations of the eigenvalues is straightforward, only knowledge of the exact eigenvalue G_3 is necessary for the determination of the direction of the polarization vector g_i^3 from Eq.(8). Eq.(8) can be also used for the determination of the direction of the polarization vector of different order of approximation, see Farra (2001). Specifically, in the first-order approximation we get

$$\frac{\hat{B}_{13}}{G_3^{(0)} - G_1^{(0)}} \hat{g}_i^{(0)1} + \frac{\hat{B}_{23}}{G_3^{(0)} - G_2^{(0)}} \hat{g}_i^{(0)2} + g_i^{(0)3}. \tag{9}$$

It is easy to show that in contrast to the general Eq.(8), Eq.(9) holds for arbitrarily chosen vectors $g_i^{(0)K}$, not only for $\hat{g}_i^{(0)K}$. It can thus be rewritten in the form:

$$\frac{\bar{B}_{13}}{G_3^{(0)} - G_1^{(0)}} g_i^{(0)1} + \frac{\bar{B}_{23}}{G_3^{(0)} - G_2^{(0)}} g_i^{(0)2} + g_i^{(0)3}. \tag{9a}$$

Eq.(9a) corresponds to the expression for the first-order perturbation of the qP-wave polarization vector of Jech and Pšenčík (1989).

If we substitute g_k^m in the Christoffel equation (2) by the vector (8), and multiply the resulting equation by $g_j^{(0)3}$, we get

$$G_3 = G_3^{(1)} + \frac{(\hat{B}_{13})^2}{G_3 - G_1^{(1)}} + \frac{(\hat{B}_{23})^2}{G_3 - G_2^{(1)}}. \tag{10}$$

This is a basic equation for the iterative determination of the higher-order approximations of the eigenvalue G_3, see Eq.(30) of Farra (2001). We can proceed in the same way as Farra (2001) to obtain approximations of G_3 of different order. Specifically, in the first-order approximation we obtain

$$G_3^{(1)} = \hat{B}_{33} = \bar{B}_{33} = \Gamma_{jk} g_j^{(0)3} g_k^{(0)3}. \tag{11}$$

The odd-order approximations of the eigenvalue G_3 are independent of the choice of the reference medium while the even-order perturbations depend on its choice. The first-order approximation $G_3^{(1)}$ in (11) is thus independent of α.

It is shown in the Appendix A that the elements of the matrix $\bar{B}_{mn}$ appearing in Eqs.(9a) and (11) are controlled by 15 WA parameters. This confirms well-known fact that the first-order aproximations of the direction of the polarization vector (9a) and of the eigenvalue (11) are controlled by those 15 parameters. The polarization vector is, in addition, controlled by the difference of squarred phase velocities, $\alpha^{-}\beta^2$.

2.2 qS wave

We cannot usually guarantee non-zero denominators in expressions like Eq.(8) for qS waves. For example, the denominators turn zero if the medium is isotropic. The denominators are also very small or zero in singular regions. If this is not the case, we can use the same procedure as in the case of qP wave for the development of higher-order approximation formulae. Otherwise, we must use a procedure designed for the degenerate cases. Let us mention that this procedure removes problems in formulae for eigenvalues. Formulae for polarization yield indetermined expressions for singular directions. In this section, we exclude therefore singular directions from our considerations. We return to them in Sec.3.

We again follow Farra (2001) and express the polarization vector of the K-th ($K=1,2$) qS wave, g_i^K in the basis $(g_i^{(0)1}, g_i^{(0)2}, g_i^{(0)3})$. Note that arbitrarily chosen vectors $g_i^{(0)K}$ are considered. Inserting g_i^K specified in this way into (2) and multiplying the resulting equation successively by $g_i^{(0)1}$, $g_i^{(0)2}$ and $g_i^{(0)3}$, we get, after some manipulation, a set of three equations

$$(M_{11}-G_K)g_m^K g_m^{(0)1} + M_{12} g_m^K g_m^{(0)2} = 0, \tag{12a}$$

$$M_{12} g_m^K g_m^{(0)1} + (M_{22}-G_K) g_m^K g_m^{(0)2} = 0, \tag{12b}$$

$$\bar{B}_{13} g_m^K g_m^{(0)1} + \bar{B}_{23} g_m^K g_m^{(0)2} + (G_3^{(1)} - G_K) g_m^K g_m^{(0)3} = 0. \tag{12c}$$

Here we use the notation of Farra (2001):

$$M_{ij} = \bar{B}_{ij} + \frac{\bar{B}_{i3}\bar{B}_{j3}}{G_K - G_3^{(1)}}. \tag{13}$$

From the condition of solvability of Eqs.(12a, b), we get expressions for the eigenvalues G_K of the two qS waves:

$$G_K = \frac{1}{2}[M_{11} + M_{22} \pm \sqrt{D}], \tag{14}$$

where

$$D = (M_{11} - M_{22})^2 + 4M_{12}^2. \tag{15}$$

The positive sign in (14) corresponds to the faster $qS1$ wave, the negative sign to the slower $qS2$ wave.

From Eqs.(12a, b) and (13), we then get expressions for the projections of the polarization vectors $g_i^K (K=1,2)$ into the vectors $g_i^{(0)1}$ and $g_i^{(0)2}$:

$$g_m^1 g_m^{(0)1} = g_m^2 g_m^{(0)2} = \sqrt{\frac{1}{2}(1 + \frac{M_{11}-M_{22}}{\sqrt{D}})},$$

$$g_m^1 g_m^{(0)2} = -g_m^2 g_m^{(0)1} = sgn(M_{12})\sqrt{\frac{1}{2}(1 - \frac{M_{11}-M_{22}}{\sqrt{D}})}. \tag{16a}$$

From Eq.(12c), we get expression for the projection of the polarization vector g_m^K into the vector $g_m^{(0)3}$:

$$g_m^K g_m^{(0)3} = \frac{\bar{B}_{13} g_m^K g_m^{(0)1} + \bar{B}_{23} g_m^K g_m^{(0)2}}{G_K - G_3^{(1)}}. \tag{16b}$$

Eqs.(14) and (16), which hold for arbitrarily chosen vectors $g_i^{(0)1}$, $g_i^{(0)2}$, are basic equations for the iterative determination of the higher-order approximations of the eigenvalues and eigenvectors of the two qS waves propagating in anisotropic media, see Farra (2001). It is only necessary to find higher-order approximations $M_{ij}^{(n)}$ of the matrix M_{ij} and insert them into (14) and (16). As in the case of qP wave, the odd-order perturbations of the eigenvalues (14) are independent of the choice of the reference medium while the even-order perturbations depend on it.

Let us consider vectors $\hat{g}_i^{(n)1}$, $\hat{g}_i^{(n)2}$ chosen in a similar way as the vectors $\hat{g}_i^{(0)1}$, $\hat{g}_i^{(0)2}$ in Eq.(7), specifically so that the $(n+1)$-th order approximation $\hat{M}_{ij}^{(n+1)}$ of the matrix M_{ij} satisfies

$$\hat{M}_{12}^{(n+1)} = 0, \qquad \hat{M}_{11}^{(n+1)} > \hat{M}_{22}^{(n+1)}. \tag{17}$$

From Eq.(13) we can see that Eq.(7) is a special case of Eq.(17) corresponding to $n = 0$. Using vectors $\hat{g}_i^{(n)1}$, $\hat{g}_i^{(n)2}$ chosen in the above way, we get from (14) and (15) expressions for the $(n+1)$-th approximation $G_K^{(n+1)}$ of the eigenvalue G_K:

$$G_1^{(n+1)} = \hat{M}_{11}^{(n+1)}, \qquad G_2^{(n+1)} = \hat{M}_{22}^{(n+1)}. \tag{18}$$

Specifically, for the first-order approximation we get

$$G_1^{(1)} = \hat{M}_{11}^{(1)} = \hat{B}_{11}, \qquad G_2^{(1)} = \hat{M}_{22}^{(1)} = \hat{B}_{22}. \tag{18a}$$

The approximations of eigenvalues in Eq.(18a) are independent of the choice of the reference S-wave velocity β.

As the polarization vectors in the zero-order approximation ($n = 0$) we get the vectors $\hat{g}_i^{(0)K}$ specified by the conditions (7). Specifying $n = 1$ in Eq.(17), we get from (16) for the directions of the corresponding polarization vectors in the first-order approximation:

$$\hat{g}_i^{(1)1} + \frac{\hat{B}_{13}}{G_1^{(0)} - G_3^{(0)}} g_i^{(0)3}, \qquad \hat{g}_i^{(1)2} + \frac{\hat{B}_{23}}{G_2^{(0)} - G_3^{(0)}} g_i^{(0)3}. \tag{19}$$

Eq.(19) has useful form for being used in an inverse problem, the determination of the anisotropic parameters from the observed polarizations, see Pšenčík and Zheng (2002). Let us note that the elements $\hat{B}_{K3}$ of the matrix $\hat{B}_{mn}$ are specified for the vectors $\hat{g}_i^{(1)K}$. Since the choice of the vectors $\hat{g}_i^{(1)1}$, $\hat{g}_i^{(1)2}$ used in (19) differs from $\hat{g}_i^{(0)1}$ and $\hat{g}_i^{(0)2}$, Eq.(19) differs from a similar equation for the first-order perturbation of the qS-wave polarization vector given by Jech and Pšenčík (1989).

Let us mention that the procedure of selection vectors $g_i^{(0)K}$ so that Eq.(17) is satisfied can be extended to the limiting case. We can choose vectors $\hat{g}_i^K$ so that the elements of the matrix $\hat{M}_{ij}$ (the matrix $\hat{M}_{ij}$ is the exact matrix M_{ij}, see (13), specified for the vectors $\hat{g}_i^K$) satisfy similar relations to those in Eq.(17):

$$\hat{M}_{12} = 0, \qquad \hat{M}_{11} > \hat{M}_{22}. \tag{17a}$$

For such a specification we get immediately from (14) and (15)

$$G_1 = \hat{M}_{11}, \qquad G_2 = \hat{M}_{22}, \tag{18b}$$

and for the directions of the corresponding exact polarization vectors:

$$\hat{g}_i^1 + \frac{\hat{B}_{13}}{G_1 - G_3^{(1)}} g_i^{(0)3}, \qquad \hat{g}_i^2 + \frac{\hat{B}_{23}}{G_2 - G_3^{(1)}} g_i^{(0)3}. \tag{19a}$$

The elements $\hat{B}_{K3}$ of the matrix $\hat{B}_{mn}$ are specified for the vectors $\hat{g}_i^K$.

The first-order approximation of the qS-wave polarization vectors (19) contains the same elements of the matrix $\bar{B}_{mn}$ ($\hat{B}_{13}$ and $\hat{B}_{23}$ are dependent on $\bar{B}_{13}$ and $\bar{B}_{23}$ only) as the first-order approximation of the qP-wave polarization vector (9a). This means that the polarization vectors of qP and qS waves are controlled by the same set of 15 WA parameters (see the Appendix A) and by the difference of squarred velocities $\alpha^2 - \beta^2$. The first-order approximation of eigenvalues (18a) depends on elements $\bar{B}_{11}$, $\bar{B}_{12}$ and $\bar{B}_{22}$. It is shown in the Appendix A that these elements depend on a different set of 15 WA parameters and their combinations. This set is complementary to the above-mentioned one so that the first-order aproximations of the eigenvalues (18a) and of the directions of the polarization vectors (19) are altogether controlled by the complete set of 21 WA parameters (A3).

3 Acoustical axes

Characteristic feature of any anisotropic medium are so-called *acoustical axes*. According to Schoenberg and Helbig (1997) the acoustical axes specify two kinds of directions: longitudinal and singular. A longitudinal direction is a direction in which the qP wave is purely longitudinal and the qS waves are purely transverse. The polarization vector g_i^3 of the qP wave is parallel to its wave normal n_i, the polarization vectors g_i^K of the qS waves are perpendicular to n_i. By inspecting Eqs.(9) and (19), we can see that the indication of the longitudinal direction in the first-order perturbation formulae is the identity

$$\hat{B}_{13} = \hat{B}_{23} = 0. \tag{20}$$

Since the elements $\bar{B}_{13}$ and $\bar{B}_{23}$ of the matrix $\bar{B}_{mn}$ can be expressed in terms of only the elements $\hat{B}_{13}$ and $\hat{B}_{23}$, we can see that Eq.(20) holds for arbitrary choice of vectors $g_i^{(0)K}$. For the specification (20), the first-order approximation of the qP-wave polarization vector is parallel to the wave vector n_i, and the first-order approximation of the qS-wave polarization vectors are perpendicular to the wave vector n_i. It is interesting to note that the elements $\bar{B}_{13}$ and $\bar{B}_{23}$ appearing in Eq.(20) depend on only 15 weak anisotropy parameters which control qP-wave propagation, see the Appendix A. It is also interesting to check Eqs.(8) and (16) specifying general expressions for the polarization vectors. Under the condition (20), these equations indicate that the longitudinal directions determined from the first-order perturbation formulae are identical with the exact ones.

From Eq.(10) we can derive another interesting property of the first-order perturbation of the qP-wave eigenvalue. Because $G_3 > G_1^{(1)}$ and $G_3 > G_2^{(1)}$, Eq.(10) indicates that $G_3 \geq G_3^{(1)}$, i.e., the qP-wave first-order formula yields value which is less or at most equal to exact eigenvalue. An alternative proof of this inequality can be found in the Appendix of Pšenčík and Gajewski (1998). We can see from Eq.(10) that the equality $G_3 = G_3^{(1)}$ occurs when (20) holds, i.e., in longitudinal directions.

Similar property as above holds also for the qS waves. The proof is only slightly more complicated. Let us consider vectors $g_i^{(0)K}$ satisfying conditions (7) and let us rewrite

Eq.(14) for this specification. We get

$$G_1 = \frac{1}{2}\Big(\hat{B}_{11} + \hat{B}_{22} + \frac{(\hat{B}_{13})^2 + (\hat{B}_{23})^2}{G_1 - G_3^{(1)}} + \sqrt{D}\Big)$$

$$G_2 = \frac{1}{2}\Big(\hat{B}_{11} + \hat{B}_{22} + \frac{(\hat{B}_{13})^2 + (\hat{B}_{23})^2}{G_2 - G_3^{(1)}} - \sqrt{D}\Big). \tag{21}$$

Let us also rewrite Eq.(15) for the same specification:

$$D = (\hat{B}_{11} - \hat{B}_{22})^2 + \Big(\frac{(\hat{B}_{13})^2 + (\hat{B}_{23})^2}{G_K - G_3^{(1)}}\Big)^2 + 2(\hat{B}_{11} - \hat{B}_{22})\frac{(\hat{B}_{13})^2 - (\hat{B}_{23})^2}{G_K - G_3^{(1)}}. \tag{22}$$

The quantity D satisfies the following inequalities

$$\Big(\hat{B}_{11} - \hat{B}_{22} + \frac{(\hat{B}_{13})^2 + (\hat{B}_{23})^2}{G_K - G_3^{(1)}}\Big)^2 \leq D \leq \Big(\hat{B}_{11} - \hat{B}_{22} - \frac{(\hat{B}_{13})^2 + (\hat{B}_{23})^2}{G_K - G_3^{(1)}}\Big)^2, \tag{23}$$

which follows from

$$-[(\hat{B}_{13})^2 + (\hat{B}_{23})^2] \leq (\hat{B}_{13})^2 - (\hat{B}_{23})^2 \leq (\hat{B}_{13})^2 + (\hat{B}_{23})^2 \tag{24}$$

and from the obvious fact that $G_K < G_3^{(1)}$. Applying the right-hand inequality (23) to the expression for G_1 and the left-hand inequality (23) to G_2, we get two important results

$$G_1 \leq \hat{B}_{11} = G_1^{(1)}, \qquad G_2 \leq \hat{B}_{22} = G_2^{(1)}. \tag{25}$$

We can see that the first-order formulae yield values which are greater or at least equal to exact eigenvalues of the qS waves. As in the case of qP wave, we can see from Eqs.(13)-(15) that the equality $G_1 = G_1^{(1)}$ and $G_2 = G_2^{(1)}$ occurs when (20) holds, i.e., in longitudinal directions.

Let us mention that Eqs.(20) are the equations, which could be used in the process of the determination of the WA parameters if the longitudinal directions are known.

The second type of acoustical axes is connected with singular directions. Singular directions exist only for qS waves and occur when the phase velocities (or the eigenvalues G_M) of the two qS waves are equal. By inspecting Eq.(14), we can see that this occurs for any perturbation order if $D = 0$. Because D is given, see (15), by a sum of two quadratic terms, each of these terms must be zero. This yields

$$M_{11} = M_{22}, \quad M_{12} = 0. \tag{26}$$

In the first-order approximation, the singular direction is specified by

$$\bar{B}_{11} = \bar{B}_{22}, \quad \bar{B}_{12} = 0. \tag{27}$$

If a singular direction differs from a longitudinal direction, the equality $G_1 = G_2$ does not imply generally $G_1^{(1)} = G_2^{(1)}$. This means that the singular direction in the first-order approximation may differ from the actual singular direction. This also holds for approximations of higher order. This is illustrated on the following numerical examples. In the limit, the singular direction should converge to the actual one.

4 Numerical examples

We shall now concentrate on study of behaviour of the above derived qS-wave formulae in vicinity of singular directions. In singular directions, the two qS waves have the same phase velocities and thus their eigenvalues G_k coincide. This is so-called degenerate situation in perturbation analysis. In such a situation, it is not possible to specify uniquely the polarization vectors of the qS waves. It is only possible to find the plane in which the polarization vectors are situated, i.e., the plane perpendicular to the third polarization vector. This fact reflects in the indeterminacy of corresponding formulae for the polarization vectors. In applications, the most frequently used formulae are formulae based on the first-order approximation. In the following, we shall compare results of the first- and second-order perturbation formulae with exact results for several types of anisotropic media. This will enable us to estimate effects of proximity of a singular direction on the quality of the first-order formulae. Let us emphasize that the results of this section were obtained by using Eqs.(14) and (16). In order to calculate n-th order approximation of the square of the phase velocity from Eq.(14), n-th order approximation of the matrix M_{ij} was used. For the determination of the n-th order approximation of polarization vectors from (16), $(n+1)$-th order approximation of M_{ij} was used.

For illustration of effects of a singularity on the first-order approximation, let us consider a simple transversely isotropic medium with vertical axis of symmetry. We use the model of thin water-filled cracks of Shearer and Chapman (1989). The model is specified by the density-normalized elastic parameters A_{ij}, in (km/s)^2, with values: $A_{11} = A_{22} = 20.16$, $A_{33} = 19.63$, $A_{12} = 7.40$, $A_{13} = A_{23} = 7.26$, $A_{44} = A_{55} = 3.48$, $A_{66} = 6.38$. Anisotropy of this model is rather strong, about 29% and 30% for qS waves. Due to the axial symmetry of the medium, it is sufficient to investigate just a quadrant of a vertical plane containing the axis of symmetry. Figure 1 shows variations of phase velocities of the two qS waves in such a section. The angle θ specifies the direction of the wave vector, $\theta = 0^0$ corresponds to the direction along the axis of symmetry, $\theta = 90^0$ corresponds to the direction perpendicular to it. The two distinct curves correspond to the SH and qSV waves. Polarization of the SH wave is perpendicular to the plane of the plot and thus to the wave vector (therefore we omit the letter q in front of SH), the qSV wave is polarized in the plane of the plot, its polarization is generally not perpendicular to the wave vector. The first-order phase velocity formula for the SH wave yields exact values of the phase velocity. For the qSV wave the values generally differ, the approximate values being always greater or equal to the exact ones, see the analytical proof above. Results of the approximate formula coincide with exact for $\theta = 0^0$ (kiss singularity), for approximately 45^0 (longitudinal direction) and for 90^0 (another longitudinal direction). For the goals of this paper, the most interesting is the intersection of the SH- and qSV-wave phase velocities at about 60^0, which corresponds to an intersection singularity. We can see that the singularity estimated by the first-order formula is about 2^0 shifted from the direction of the true singularity. The first-order perturbation formula thus yields distorted result. It is of interest that there is no shift in case of the kiss singularity. The kiss singularity direction coincides with the longitudinal direction, in which the first-order formula yields exact results.

Let us now consider a more complicated situation, namely an orthorhombic model used by Farra (2001). The model is specified by the density-normalized elastic parameters

A_{ij}, in (km/s)^2, with values: $A_{11} = 10.8$, $A_{22} = 11.3$, $A_{33} = 8.5$, $A_{12} = 2.2$, $A_{13} = 1.9$, $A_{23} = 1.7$, $A_{44} = 3.6$, $A_{55} = 3.9$ and $A_{66} = 4.3$. Figure 2 shows exact (top), first-order (middle) and second-order (bottom) maps of relative differences (in %) of $qS1$ and $qS2$ phase velocities as functions of the polar angle θ and the azimuth φ specifying the wavevector. The relative differences are calculated as differences of the qS-wave velocities normalized by their average. There is a point singularity at $\varphi = 0^0$ and $\theta = 34.4^0$ and the region of small differences extends to larger values of φ in the map of exact differences. Another region of small differences can be observed on the right-hand side of the exact map for $\varphi = 90^0$ and $\theta \sim 35^0$. Let us now compare the exact map with its first-order approximation in the middle frame. We can see several interesting phenomena. The most significant is the shift of the region of minimum differences to higher values of θ, from 34.4^0 to, approximately, 39^0. Another important feature is appearence of two minima seemingly indicating existence of singularities for $\varphi = 90^0$, $\theta = 33^0$ and $\theta = 45^0$. The bottom of the valley crossing the plots for $\theta \sim 30 - 40^0$ is deeper in the first-order map. For relative differences of $qS1$ and $qS2$ phase velocities greater than, approximately 3%, the differences between the top and middle frames are minimum. From inspection of the frame of the second-order approximation, we can see that it gives effectively the same results as the exact formula. Slight exception is the region of $\varphi = 90^0$ and $\theta \sim 35^0$.

In the following, we show effects of the first-order perturbations on the model of orthorhombic medium proposed and studied in detail by Schoenberg and Helbig (1997). The model has been obtained by combining finely layered (VTI) model with vertical fractures. The model is specified by the density-normalized elastic parameters A_{ij}, in (km/s)^2, with values: $A_{11} = 9.00$, $A_{22} = 9.84$, $A_{33} = 5.94$, $A_{12} = 3.60$, $A_{13} = 2.25$, $A_{23} = 2.40$, $A_{44} = 2.00$, $A_{55} = 1.60$ and $A_{66} = 2.18$. Figure 3 shows again exact (top), first-order (middle) and second-order (bottom) maps of relative differences of $qS1$ and $qS2$ phase velocities. This model has four singularities in the shown map: two singularities (conical points) for $\varphi = 0^0$ and $\theta = 20.1^0$ and $\theta = 59.8^0$, one conical point for $\varphi = 90^0$ and $\theta = 72.5^0$, and another one for $\varphi = 44.89^0$ and $\theta = 46.53^0$, see Schoenberg and Helbig (1997). Comparison of the exact and first-order maps yields several interesting phenomena. We can see that, as in Fig.2, the singularity $\varphi = 0^0$, $\theta = 59.8^0$ is shifted by about 5^0 within the plane of symmetry $\varphi = 0^0$. The other singularity in the same plane, $\varphi = 0^0$, $\theta = 20.1^0$ remains, however, at the same position. This is result of closeness of the longitudinal direction $\varphi = 0^0$ and $\theta = 20.3^0$. As mentioned above, the first-order formulae yield exact results for the longitudinal directions. Thus the differences between the first-order approximations of phase velocities of the qS waves in a vicinity of a longitudinal direction are also close to exact ones. The singularity $\varphi = 90^0$, $\theta = 72.5^0$ is shifted by about 4^0 in the plane $\varphi = 90^0$ to approximately $\theta = 77^0$. Interesting is the shift of the point singularity $\varphi = 44.89^0$, $\theta = 46.53^0$ situated off the symmetry planes. The shift of the singularity in the first-order approximation is in both angles. It is about 2^0 in φ and about 4^0 in the polar angle θ. As in Fig.2, the results of the second-order formula practically coincide with exact results. The only exception is the singularity $\varphi = 90^0$, $\theta = 72.5^0$ which remains shifted by about 2^0. The use of the third-order formula would shift the singularity to the proper position.

The effect of singularities on polarization vectors is much more pronounced. As mentioned above, in singular directions the approximate formulae for polarization vectors are indetermined. In their vicinity, polarization vectors change fast. This can be seen in Fig.4.

Fig.4 shows deviations of the exact polarization vector of the faster $qS1$ wave from the zero-order (top), first-order (middle) and (second-order) approximation of the same polarization vector. From the upper plot we can see that the zero-order approximation yields deviation less than 1^0 in only small regions. Generally, the deviation of the zero-order approximation is large, it reaches values of 90^0! The points, in which the isolines concentrate, represent actual and shifted positions of singularities. The actual positions are related to the exact polarization vector, which varies significantly in their vicinity. The shifted positions are related to the zero-order approximation of the polarization vector, which varies significantly in the vicinity of the shifted positions, see the top plot in Figure 3. There is only very slightly increased variation of deviations in the vicinity of the singularity $\varphi = 0^0$, $\theta = 20.1^0$. The first-order approximation (middle plot) of the polarization vector reduces the deviations significantly. The maximum deviations are again upto 90^0 but they are concentrated to the narrow region connecting three singularities. Except the singularity $\varphi = 90^0$, $\theta = 72.5^0$, the concentrations of isolines are close to actual positions of singularities. This picture further improves when second-order approximation of the polarization vector is used, the bottom plot in Fig.4. The maximum deviation does not exceed 1^0 in most of the studied region, all non-negligible deviations are concentrated in the narrow belt connecting the three actual singularities whose positions are indicated by concentration of isolines.

Fig.5 shows detailed maps of the same deviations as in Fig.4 of the exact polarization vector of the faster $qS1$ wave from the zero-order (top), first-order (middle) and (second-order) approximation of the same polarization vector. We can see in it, in detail, behaviour of the $qS1$ polarization in the vicinity of the singularity singularity in the region of φ and θ between 40^0 and 50^0. The shift of the singularity with varying order of the approximation is also clearly visible.

5 Conclusions

The main reason of this contribution was to show the effect of the first-order perturbation formulae on the determination of acoustical axes. For this purpose, explicit analytical conditions for the determination of the acoustical axis were given. We have found, both analytically and in numerical examples, that the approximations do not affect positions of the longitudinal directions. They may, however, affect quite significantly positions of singularities. The effects can be removed by using higher-order approximations.

The computation of higher-order approximations of the eigenvalues and directions of the polarization vectors of qP and qS waves was based on the formulae of Farra (2001). These formulae were supplemented by an alternative way of determination of auxiliary vectors, which leads to simplification of formulae.

Acknowledgements

This work was supported by the CNRS-AS ČR project, by consortium project "Seismic waves in complex 3-D structures (SW3D)" and by the Grant Agency of the Czech Republic under the contract No. 205/00/1350.

References

Farra,V., 2001. High order expressions of the phase velocity and polarization of qP and qS waves in anisotropic media. *Geophys. J. Int.*, **147**, 93-105.

Helbig,K., 1993. Longitudinal directions in media of arbitrary anisotropy. *Geophysics*, **58**, 680-691.

Jech,J. and Pšenčík,I., 1989. First-order perturbation method for anisotropic media. *Geophys.J.Int.*, **99**, 369-376.

Pšenčík,I. and Gajewski,D., 1998. Polarization, phase velocity and NMO velocity of qP waves in arbitrary weakly anisotropic media, *Geophysics*, **63**, 1754-1766.

Pšenčík,I. and Zheng,X., 2002. Local determination of WA parameters from qS waves. In: Seismic Waves in Complex 3-D Structures, Report 12, pp.19-34, Dept.of Geophysics, Charles University Prague.

Shearer,P.M. and Chapman,C.H., 1989. Ray tracing in azimuthally anisotropic media - I. Results for models of aligned cracks in the upper crust. *Geophys. J. Int.*, **96**, 51-64.

Schoenberg,M. and Helbig,K., 1997. Orthorhombic media: Modeling elastic wave behavior in a vertically fractured earth. *Geophysics*, **62**, 1954-1974.

Appendix A: Expressions for the elements of the matrix $\bar{B}_{mn}$

We introduce three mutually perpendicular unit vectors $g_i^{(0)1}$, $g_i^{(0)2}$ and $g_i^{(0)3}$ so that $g_i^{(0)3} = n_i$, where n_i is a unit wave vector. The vectors $g_i^{(0)1}$ and $g_i^{(0)2}$ are situated in the plane perpendicular to $g_i^{(0)3}$ and they are selected so that

$$\vec{g}^{(0)1} = D^{-1}(n_1 n_3, n_2 n_3, n_3^2 - 1), \quad \vec{g}^{(0)2} = D^{-1}(-n_2, n_1, 0), \quad \vec{g}^{(0)3} = \vec{n} = (n_1, n_2, n_3), \quad (A1)$$

where

$$D = (n_1^2 + n_2^2)^{1/2}, \quad n_1^2 + n_2^2 + n_3^2 = 1. \qquad (A2)$$

In the following, a reference isotropic medium with the P-wave velocity α and S-wave velocity β is considered. We express the symmetric matrix $\bar{B}_{mn}$ given in Eq.(7) in terms of WA parameters

$$\delta_x = \frac{A_{13} + 2A_{55} - \alpha^2}{\alpha^2}, \quad \delta_y = \frac{A_{23} + 2A_{44} - \alpha^2}{\alpha^2}, \quad \delta_z = \frac{A_{12} + 2A_{66} - \alpha^2}{\alpha^2},$$

$$\chi_x = \frac{A_{14} + 2A_{56}}{\alpha^2}, \quad \chi_y = \frac{A_{25} + 2A_{46}}{\alpha^2}, \quad \chi_z = \frac{A_{36} + 2A_{45}}{\alpha^2},$$

$$\epsilon_{15} = \frac{A_{15}}{\alpha^2}, \quad \epsilon_{16} = \frac{A_{16}}{\alpha^2}, \quad \epsilon_{24} = \frac{A_{24}}{\alpha^2}, \quad \epsilon_{26} = \frac{A_{26}}{\alpha^2},$$

$$\epsilon_{34} = \frac{A_{34}}{\alpha^2}, \quad \epsilon_{35} = \frac{A_{35}}{\alpha^2}, \quad \epsilon_{46} = \frac{A_{46}}{\alpha^2}, \quad \epsilon_{56} = \frac{A_{56}}{\alpha^2}, \quad \epsilon_{45} = \frac{A_{45}}{\beta^2},$$

$$\epsilon_x = \frac{A_{11} - \alpha^2}{2\alpha^2}, \quad \epsilon_y = \frac{A_{22} - \alpha^2}{2\alpha^2}, \quad \epsilon_z = \frac{A_{33} - \alpha^2}{2\alpha^2},$$

$$\gamma_x = \frac{A_{55} - \beta^2}{2\beta^2}, \quad \gamma_y = \frac{A_{44} - \beta^2}{2\beta^2}, \quad \gamma_z = \frac{A_{66} - \beta^2}{2\beta^2}. \qquad (A3)$$

Let us repeat again that although the WA parameters depend on velocities α and β of the reference isotropic medium, the matrix $\bar{B}_{mn}$ is *independent* of these velocities. Its elements can be then expressed in the following way:

$$\bar{B}_{11} = \beta^2 + 2\alpha^2 D^{-2}\Big(n_3^3[(\epsilon_{15} - \epsilon_{35})n_1^3 + (\epsilon_{24} - \epsilon_{34})n_2^3 + (\chi_x - \epsilon_{34})n_1^2 n_2 + (\chi_y - \epsilon_{35})n_1 n_2^2]$$
$$+n_3^2[(\epsilon_x - \delta_x + \epsilon_z)n_1^4 + (\epsilon_y - \delta_y + \epsilon_z)n_2^4 + (\delta_z - \delta_x - \delta_y + 2\epsilon_z)n_1^2 n_2^2$$
$$+2(\epsilon_{16} - \chi_z)n_1^3 n_2 + 2(\epsilon_{26} - \chi_z)n_1 n_2^3]$$
$$+n_3[(\epsilon_{35} - \epsilon_{15})n_1^5 + (\epsilon_{34} - \epsilon_{24})n_2^5 + (\epsilon_{34} - \chi_x)n_1^4 n_2 + (\epsilon_{35} - \chi_y)n_1 n_2^4$$
$$+(2\epsilon_{35} - \chi_y - \epsilon_{15})n_1^3 n_2^2 + (2\epsilon_{34} - \chi_x - \epsilon_{24})n_1^2 n_2^3]\Big)$$
$$+2\beta^2 D^{-2}(\gamma_x n_1^2 + \gamma_y n_2^2 + \epsilon_{45} n_1 n_2),$$

$$\bar{B}_{12} = \alpha^2\Big(n_3^2[(\chi_x - \epsilon_{34})n_1^3 - (\chi_y - \epsilon_{35})n_2^3 + (\epsilon_{35} + \chi_y - 2\epsilon_{15})n_1^2 n_2 + (2\epsilon_{24} - \chi_x - \epsilon_{34})n_1 n_2^2]$$
$$+n_3[(\epsilon_{16}-\chi_z)n_1^4-(\epsilon_{26}-\chi_z)n_2^4+(\delta_x-\delta_y-2\epsilon_x+\delta_z)n_1^3n_2+(\delta_x-\delta_y+2\epsilon_y-\delta_z)n_1n_2^3+3(\epsilon_{26}-\epsilon_{16})n_1^2n_2^2]$$
$$+[(\epsilon_{15}-\chi_y)n_1^4n_2+(\chi_x-\epsilon_{24})n_1n_2^4+(\epsilon_{15}-\chi_y)n_1^2n_2^3+(\chi_x-\epsilon_{24})n_1^3n_2^2+\epsilon_{46}n_2^3-\epsilon_{56}n_1^3+\epsilon_{46}n_1^2n_2-\epsilon_{56}n_1n_2^2]\Big)$$
$$+\beta^2 D^{-2} n_3[2(\gamma_y - \gamma_x)n_1 n_2 + \epsilon_{45}n_1^2 - \epsilon_{45}n_2^2],$$

$$\bar{B}_{22} = \beta^2 + 2\beta^2 D^{-2} n_3^2(\gamma_x n_2^2 + \gamma_y n_1^2 - \epsilon_{45}n_1 n_2)$$
$$+2\alpha^2 D^{-2}\Big(n_3[\epsilon_{46}n_1^3 + \epsilon_{56}n_2^3) + (\epsilon_{56} - \chi_x + \epsilon_{24})n_1^2 n_2 + (\epsilon_{46} - \chi_y + \epsilon_{15})n_1 n_2^2]$$
$$+(\epsilon_x + \epsilon_y - \delta_z)n_1^2 n_2^2 + (\epsilon_{26} - \epsilon_{16})n_1^3 n_2 - (\epsilon_{26} - \epsilon_{16})n_1 n_2^3]\Big) + 2\beta^2 D^{-2}(\gamma_z n_1^4 + \gamma_z n_2^4 + 2\gamma_z n_1^2 n_2^2),$$

$$\bar{B}_{13} = \alpha^2 D^{-1}\{2\epsilon_z n_3^5 + n_3^4(\epsilon_{34}n_2 + \epsilon_{35}n_1) + n_3^3(\delta_x n_1^2 + \delta_y n_2^2 + 2\chi_z n_1 n_2 - 2\epsilon_z)$$
$$+n_3^2[(4\chi_x - 3\epsilon_{34})n_1^2 n_2 + (4\chi_y - 3\epsilon_{35})n_1 n_2^2 + (4\epsilon_{15} - 3\epsilon_{35})n_1^3 + (4\epsilon_{24} - 3\epsilon_{34})n_2^3]$$
$$+n_3[(2\delta_z - \delta_x - \delta_y)n_1^2 n_2^2 + 2(2\epsilon_{16} - \chi_z)n_1^3 n_2 + 2(2\epsilon_{26} - \chi_z)n_1 n_2^3 + (2\epsilon_x - \delta_x)n_1^4 + (2\epsilon_y - \delta_y)n_2^4]$$
$$-\chi_x n_1^2 n_2 - \chi_y n_1 n_2^2 - \epsilon_{15}n_1^3 - \epsilon_{24}n_2^3\},$$

$$\bar{B}_{23} = \alpha^2 D^{-1}\{n_3^3(\epsilon_{34}n_1 - \epsilon_{35}n_2) + n_3^2[(\delta_y - \delta_x)n_1 n_2 + \chi_z n_1^2 - \chi_z n_2^2]$$
$$+n_3[(2\chi_y - 3\epsilon_{15})n_1^2 n_2 - (2\chi_x - 3\epsilon_{24})n_1 n_2^2 + \chi_x n_1^3 - \chi_y n_2^3]$$
$$+(\delta_z - 2\epsilon_x)n_1^3 n_2 + (2\epsilon_y - \delta_z)n_1 n_2^3 + 3(\epsilon_{26} - \epsilon_{16})n_1^2 n_2^2 + \epsilon_{16}n_1^4 - \epsilon_{26}n_2^4\},$$

$$\bar{B}_{33} = \alpha^2 + 2\alpha^2[\epsilon_z n_3^4 + 2n_3^3(\epsilon_{34}n_2 + \epsilon_{35}n_1) + n_3^2(\delta_x n_1^2 + \delta_y n_2^2 + 2\chi_z n_1 n_2)$$
$$+2n_3(\chi_x n_1^2 n_2 + \chi_y n_1 n_2^2 + \epsilon_{15}n_1^3 + \epsilon_{24}n_2^3) + \epsilon_x n_1^4 + \delta_z n_1^2 n_2^2 + \epsilon_y n_2^4 + 2\epsilon_{16}n_1^3 n_2 + 2\epsilon_{26}n_1 n_2^3]. \quad (A4)$$

By simple inspection of elements $\bar{B}_{33}$, $\bar{B}_{13}$ and $\bar{B}_{23}$ which are related to qP-wave propagation in the first-order approximation, see Eqs.(9a) and (11), we can find that the elements are controlled by 15 parameters ϵ_x, ϵ_y, ϵ_z, δ_x, δ_y, δ_z, χ_x, χ_y, χ_z, ϵ_{15}, ϵ_{16}, ϵ_{24}, ϵ_{26}, ϵ_{34} and ϵ_{35}. By a similar inspection of elements $\bar{B}_{11}$, $\bar{B}_{12}$ and $\bar{B}_{22}$ which are related to qS-wave propagation in the first-order approximation, we can find that the elements are controlled by different 15 parameters, namely: $\epsilon_{15} - \epsilon_{35}$, $\epsilon_{24} - \epsilon_{34}$, $\chi_x - \epsilon_{34}$, $\chi_y - \epsilon_{35}$, $\epsilon_x - \delta_x + \epsilon_z$, $\epsilon_y - \delta_y + \epsilon_z$, $\delta_z - \delta_x - \delta_y + 2\epsilon_z$, $\chi_z - \epsilon_{16}$, $\chi_z - \epsilon_{26}$, γ_x, γ_y, γ_z, ϵ_{45}, ϵ_{46} and ϵ_{56}.

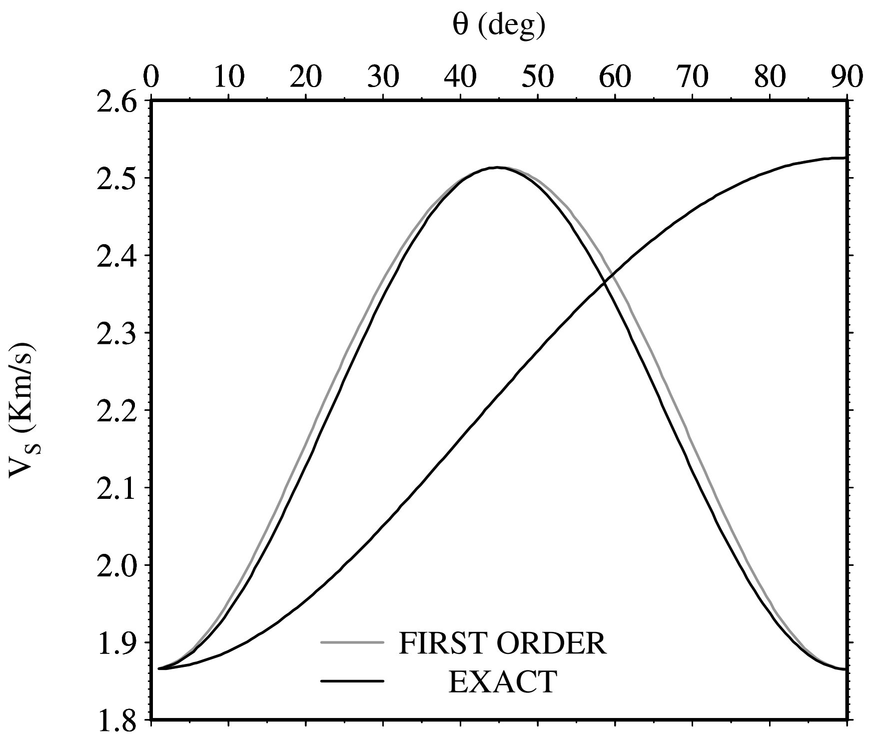
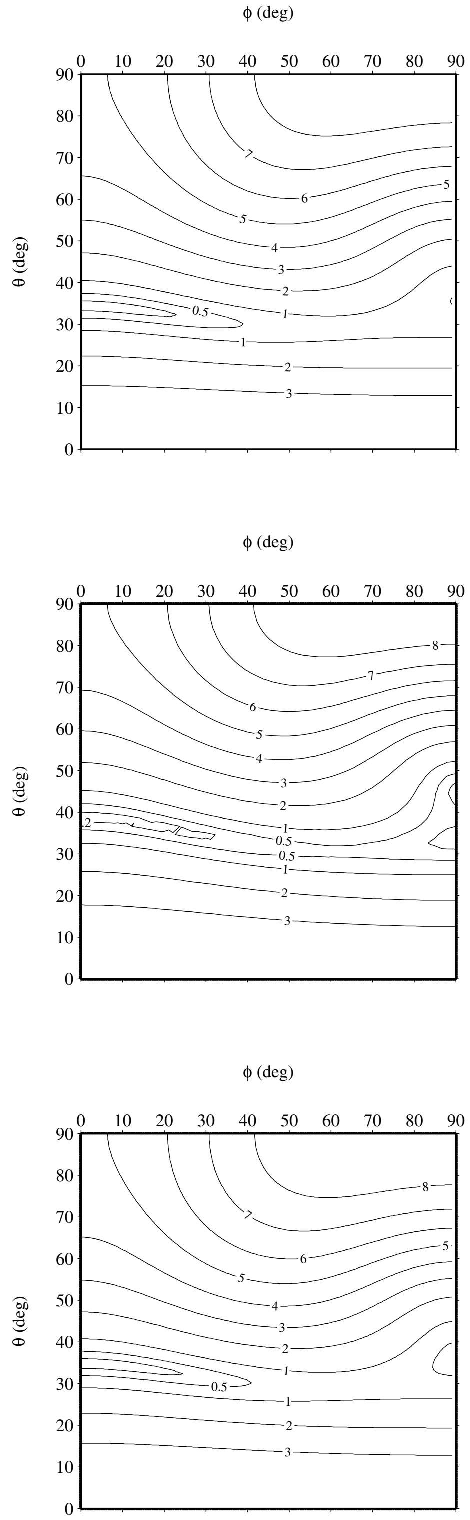

θ (deg)
0
10
20
30
40
50
60
70
80
90
2.6
2.5
2.4
2.3
2.2
2.1
2.0
1.9
1.8
V_S (Km/s)
FIRST ORDER
EXACT

Figure 1

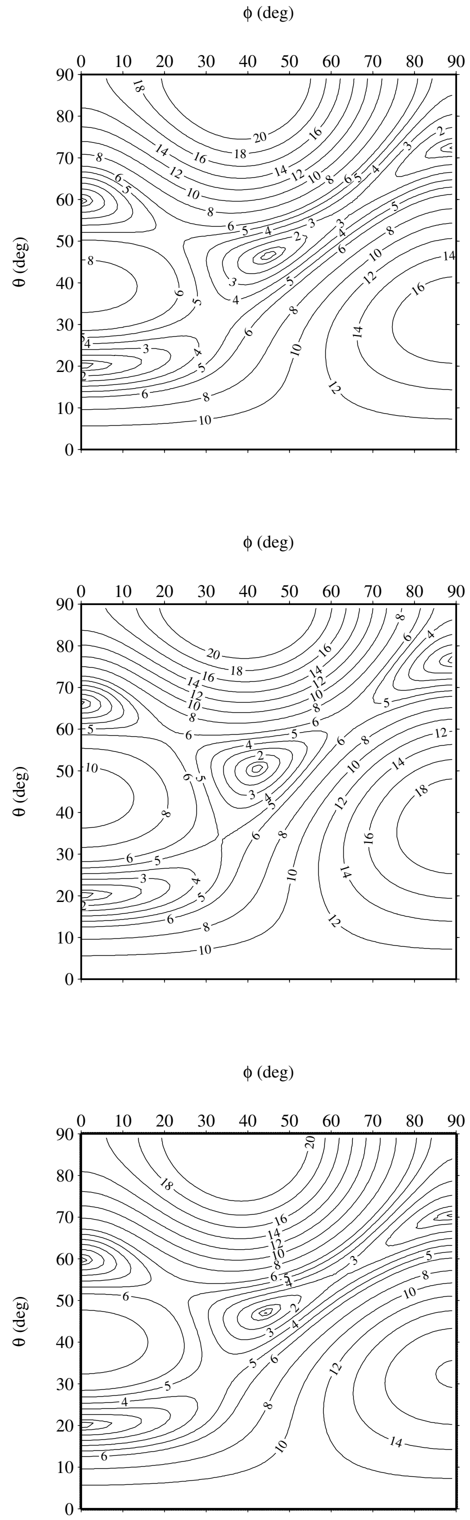


exact

1st order

2nd order

Figure 2

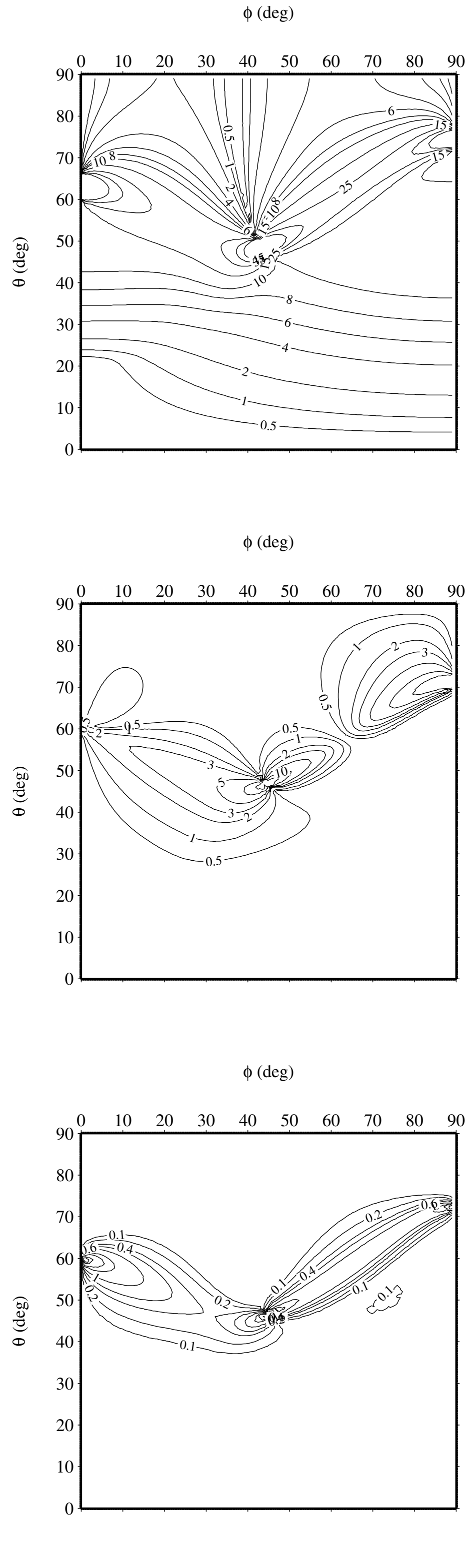


exact

1st order

2nd order

Figure 3

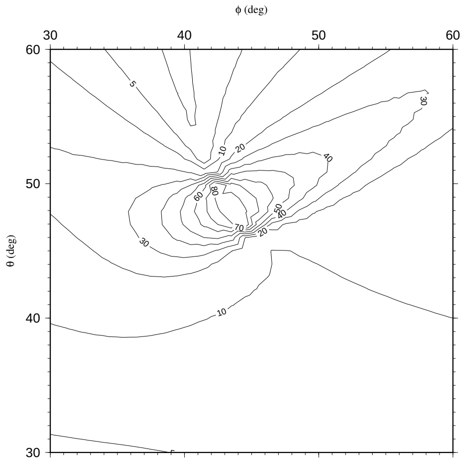


0th order

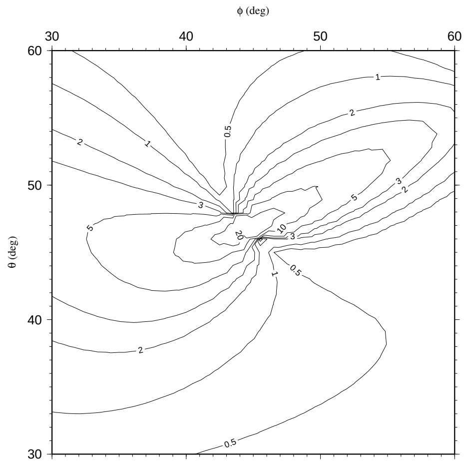
1st order

2nd order

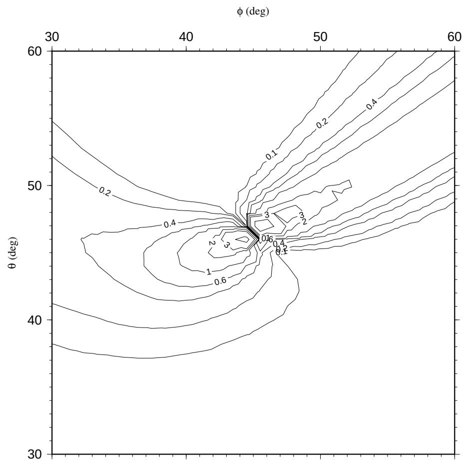
Figure 4



0st order



1st order



2nd order

Figure 5