Asymptotic Green function for *S* waves in the kiss singularity

Václav Vavryčuk

Geophysical Institute, Academy of Sciences of the Czech Republic, Boční II/1401, 141 31 Praha 4, Czech Republic. E-mail: vv@ig.cas.cz

Abstract

We have derived the far-field asymptotic formula for the *S*-wave elastodynamic Green function in the kiss singularity in anisotropic media. In contrast to standard asymptotics in regular directions the derived formula is more complex and expressed in the form of 1-D integral. This integral is specified for the kiss singularity along the symmetry axis in transversely isotropic media and along the fourfold symmetry axes in the tetragonal and cubic media. The shape of the slowness surface in the singularity is regular in transverse isotropy and the amplitude of the Green function is expressed by means of the Gaussian curvature of this surface in the singularity. However, the shape of the slowness surface of *S* waves is irregular in tetragonal or cubic media. In this case the Gaussian curvature is not defined in the singularity. Therefore, a generalized Gaussian curvature is introduced and the amplitude of the Green function is expressed by means of this new quantity. For regular directions, both the generalized and standard Gaussian curvatures yield the same value.

1. Introduction

Three waves propagate in anisotropic media with, in general, three different phase velocities and polarizations. The phase velocities and polarization vectors are determined as eigenvalues and eigenvectors of the Christoffel tensor. For some directions, however, the Christoffel tensor can degenerate and then two waves (or even all three waves) propagate with the same phase velocity. These directions of degeneracy can cause anomalies in the shape of the slowness surface as well as singularities in the field of polarization vectors (see Alshits, Sarychev & Shuvalov 1985; Shuvalov & Every 1997; Shuvalov 1998). One of important types of degeneracy is the direction in which two *S*-wave slowness surfaces touch tangentially. If this direction is isolated and the polarization vectors are singular, we speak of the *kiss singularity* (Crampin & Yedlin 1981). The simplest example of the kiss singularity is the symmetry axis in transversely isotropic media. This singularity displays a simple pattern of polarization vectors and the shape of the slowness surface is regular in the singularity.

Anomalous behaviour of the polarization vectors near singularities causes that the standard formula for the far-field asymptotics of the Green function (Buchwald 1959; Lighthill 1960; Burridge 1967; Yeatts 1994; Every & Kim 1994; Červený 2001) cannot be applied, but a modified approach is required. The far-field asymptotics in the kiss singularity has been studied by Vavryčuk (1997, 1999) and Gridin (2000) but only for the special case of transversely isotropic media. They showed that the far-field asymptotics in the kiss singularity

really differs from that in the regular direction, hence ignoring this can lead to significant errors. In this paper, we shall study the far-field asymptotics in the kiss singularity in general anisotropy. We assume that the slowness surface is concave or convex in the singularity. The derived formula will be specified for the kiss singularity along a fourfold symmetry axes in the tetragonal and cubic symmetries. We shall show that this singularity is more complex then the kiss singularity in transversely isotropic media because the shape of the slowness surface is not regular in the singularity.

2. Asymptotic formula for the Green function

The exact elastodynamic Green function $G_{kl}(\mathbf{x},t)$ in unbounded, homogeneous, anisotropic and perfectly elastic media can be expressed as follows (Burridge 1967, eq. 4.6; Wang & Achenbach 1994, eq. 13):

$$G_{kl}(\mathbf{x},t) = -\frac{H(t)}{8\pi^2\rho}\sum_{m=1}^{3}\int_{S(\mathbf{n})}\frac{g_k^m g_l^m}{\left(c^m\right)^3}\,\dot{\delta}\left(t-\frac{\mathbf{n}\cdot\mathbf{x}}{c^m}\right)dS(\mathbf{n}) \quad . \tag{1}$$

Superscript m = 1,2,3 denotes the type of wave (P, $S1$ and $S2$), $\mathbf{g} = \mathbf{g}(\mathbf{n})$ is the unit polarization vector, $c = c(\mathbf{n})$ is the phase velocity, ρ is the density of the medium, t is time, $H(t)$ is the Heaviside step function, $\dot{\delta}(t)$ is the time derivative of the Dirac delta function, $\mathbf{x}$ is the position vector of the observation point, and $\mathbf{n}$ is the phase normal. The integration in (1) is over unit sphere $S(\mathbf{n})$. If the integration is performed over the slowness surface the integral takes the following form (Burridge 1967, eq. 5.4):

$$G_{kl}(\mathbf{x},t) = -\frac{H(t)}{8\pi^2\rho}\sum_{m=1}^{3}\int_{S(\mathbf{p})}\frac{g_k^m g_l^m}{v^m}\,\dot{\delta}\left(t-\mathbf{p}\cdot\mathbf{x}\right)dS(\mathbf{p}) \quad , \tag{2}$$

where $v = v(\mathbf{p})$ is the group velocity of the wave, $\mathbf{p}$ is the slowness vector and $S(\mathbf{p})$ is the slowness surface. Both equations (1) and (2) formally represent surface integration, but the Dirac delta function in the integrand causes that the surface integral is actually reduced to a line integral. The integration line is an intersection of the slowness surface with a plane which has the normal parallel to $\mathbf{x}$ and which moves with time from the source toward the observation point.

The evaluation of (1) and (2) is complicated and can be performed analytically only for a few cases of a very simple anisotropy (Payton 1983; Burridge *et al.* 1993; Vavryčuk 2001). For general anisotropy, it should be evaluated either numerically (Wang and Achenbach 1994) or asymptotically (Every & Kim 1994). The asymptotic evaluation of $G_{kl}(\mathbf{x},t)$ in the far field simplifies the problem considerably because the asymptotic $G_{kl}(\mathbf{x},t)$ is significant only in times close to arrival time t_0 for which the integration line collapses into the stationary point $\mathbf{p}_0$ (see Fig.1). The asymptotic Green function $G_{kl}^{far}(\mathbf{x},t)$ for $|\mathbf{x}| \to \infty$ for one particular wave is expressed as follows (Burridge 1967, eq. 6.7):

$$G_{kl}^{far}(\mathbf{x},t) = -\frac{H(t)}{8\pi^2\rho} \int\limits_{S^\varepsilon} \frac{g_k g_l}{v} \dot{\delta}\left(t - \mathbf{p}_0 \cdot \mathbf{x} + \frac{1}{2}|\mathbf{x}|\ \mathbf{s}^T \mathbf{K}\ \mathbf{s} \right) ds_1 ds_2\ , \tag{3}$$

where **K** is the 2x2 matrix called the matrix of curvature at point $\mathbf{p}_0$, s_1 and s_2 are local coordinates with their origin at $\mathbf{p}_0$ and being perpendicular to $\mathbf{p}_0$. The integration is over a small neighbourhood of $\mathbf{p}_0$ of slowness surface $S(\mathbf{p})$. Assuming a concave or convex shape of the slowness surface at $\mathbf{p}_0$, equation (3) can be rewritten as follows:

$$G_{kl}^{far}(\mathbf{x},t) = \frac{I_{kl}(\mathbf{p}_0)}{4\pi^2\rho\, v} \frac{\delta(t - \mathbf{p}_0 \cdot \mathbf{x})}{|\mathbf{x}|}, \tag{4}$$

$$I_{kl}(\mathbf{p}_0) = \int\limits_{S^\varepsilon} g_k g_l\, \delta(\mathbf{s}^T \mathbf{K} \mathbf{s}) ds_1 ds_2\ , \tag{5}$$

where $\mathbf{s} = (s_1, s_2)^T$. Hence the problem of determining $G_{kl}^{far}(\mathbf{x},t)$is reduced now to a problem of evaluating amplitude $I_{kl}(\mathbf{p}_0)$ of the Green function. Amplitude $I_{kl}(\mathbf{p}_0)$ is controlled by: (1) the form of the integration line which is determined from the shape of the slowness surface in the vicinity of the stationary point $\mathbf{p}_0$, and (2) the behaviour of polarization vectors near $\mathbf{p}_0$. For an example of the integration line around the singularity in tetragonal symmetry, see Fig. 1. Examples of behaviour of polarization vectors near a singularity will be shown later.

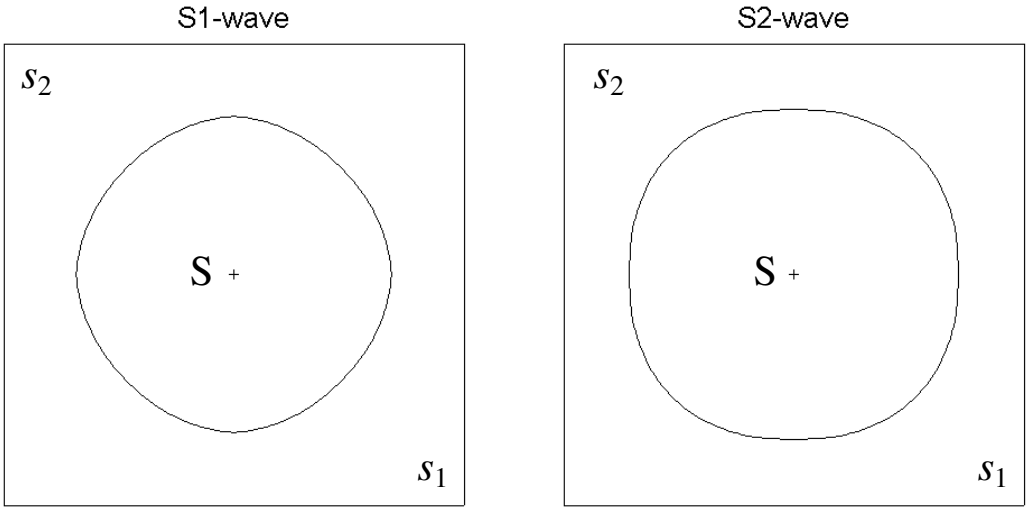

Figure 1. The shape of the integration line for the kiss singularity in tetragonal symmetry. Point *S* denotes the singularity in the direction $\mathbf{n} = (0,0,1)^T$. The elastic parameters are: $a_{11} = a_{22} = 6.25$, $a_{33} = 9.38$, $a_{13} = a_{23} = 2.35$, $a_{12} = 2.71$, $a_{66} = 2.08$, $a_{44} = a_{55} = 2.92$, $a_{16} = 0$.

3. Regular direction

If the behaviour of the slowness surface and polarization vectors is regular near $\mathbf{p}_0$ integral (5) can be evaluated analytically. Since polarization vector $\mathbf{g} = \mathbf{g}(s_1, s_2)$ in (5) is a continuous function at $\mathbf{p}_0$, we can utilize properties of the δ-function and move $\mathbf{g}$ out of the integral

$$I_{kl}(\mathbf{p}_0) = g_k(\mathbf{p}_0) g_l(\mathbf{p}_0) \int_{S^\varepsilon} \delta\left(\mathbf{s}^T \mathbf{K} \mathbf{s}\right) ds_1 ds_2 \ . \tag{6}$$

The integration line in (6) is an ellipse, hence we assume that coordinates s_1 and s_2 are parallel to directions of the principal curvatures K_{11} and K_{22} at $\mathbf{p}_0$

$$\mathbf{s}^T \mathbf{K} \mathbf{s} = K_{11} s_1^2 + K_{22} s_2^2 \,, \tag{7}$$

and substitute coordinates s_1 and s_2 by coordinates

$$u = K_{11} s_1^2 + K_{22} s_2^2 \ , \ \sigma = \operatorname{atan} \frac{\sqrt{K_{22}}\, s_2}{\sqrt{K_{11}}\, s_1} \ , \ ds_1 ds_2 = \frac{1}{2} \frac{1}{\sqrt{K_{11} K_{22}}} du d\sigma \ . \tag{8}$$

We obtain

$$I_{kl}(\mathbf{p}_0) = \frac{1}{2} \frac{g_k(\mathbf{p}_0) g_l(\mathbf{p}_0)}{\sqrt{K_{11} K_{22}}} \int_{S^\varepsilon} \delta(u)\, du d\sigma = \frac{1}{2} \frac{g_k(\mathbf{p}_0) g_l(\mathbf{p}_0)}{\sqrt{K_{11} K_{22}}} \int_0^{2\pi} d\sigma = \pi \frac{g_k(\mathbf{p}_0) g_l(\mathbf{p}_0)}{\sqrt{K_{11} K_{22}}} \ . \tag{9}$$

Finally, the asymptotic Green function reads

$$G_{kl}^{far}(\mathbf{x}, t) = \frac{1}{4\pi\rho} \frac{g_k g_l}{v\sqrt{K}} \frac{\delta\left(t - \mathbf{p}_0 \cdot \mathbf{x}\right)}{|\mathbf{x}|} \,, \tag{10}$$

where $K = K_{11} K_{22}$ is the Gaussian curvature of the slowness surface at $\mathbf{p}_0$. Values of polarization vector $\mathbf{g}$ and group velocity v are also calculated at $\mathbf{p}_0$.

4. Kiss singularity in transverse isotropy

Let us consider the kiss singularity parallel to the p_3-axis. Since the kiss singularity coincides with the symmetry axis, the matrix of curvature $\mathbf{K}$ is diagonal and the principal curvatures K_{11} and K_{22} are equal. Hence eq. (5) takes the following form:

$$I_{kl}(\mathbf{p}_0) = \int_{S^\varepsilon} g_k g_l \, \delta\left[\sqrt{K}\left(s_1^2 + s_2^2\right)\right] ds_1 ds_2 = \frac{1}{\sqrt{K}} \int_{S^\varepsilon} g_k g_l \, \delta\left(s_1^2 + s_2^2\right) ds_1 ds_2 \,, \tag{11}$$

where K is the Gaussian curvature at the singularity. Obviously, the integration line in (11) is a circle, hence we substitute coordinates s_1 and s_2 by coordinates u, φ,

$$u = s_1^2 + s_2^2 \,, \quad \varphi = \operatorname{atan} \frac{s_2}{s_1}, \quad ds_1 ds_2 = \frac{1}{2} du d\varphi \,,$$

and integral (11) takes the form:

$$I_{kl}(\mathbf{p}_0) = \frac{1}{2} \frac{1}{\sqrt{K}} \int_{S^\varepsilon} g_k g_l \, \delta(u) \, du d\varphi \,. \tag{12}$$

The field of polarization vectors is singular for both *S* waves in the kiss singularity (see Fig. 2), which is characterized by topological charge $n_d = 1$ (see Darinskii 1994). Because of the singularity we cannot move term $g_k g_l$ in (13) out of the integral as done in the case of the regular direction, but we have to limit this term for $u \to 0$ and then to integrate over φ. We obtain

$$I_{kl}(\mathbf{p}_0) = \frac{1}{2} \frac{1}{\sqrt{K}} \int_0^{2\pi} \overline{g}_k \overline{g}_l \, d\varphi \,, \tag{13}$$

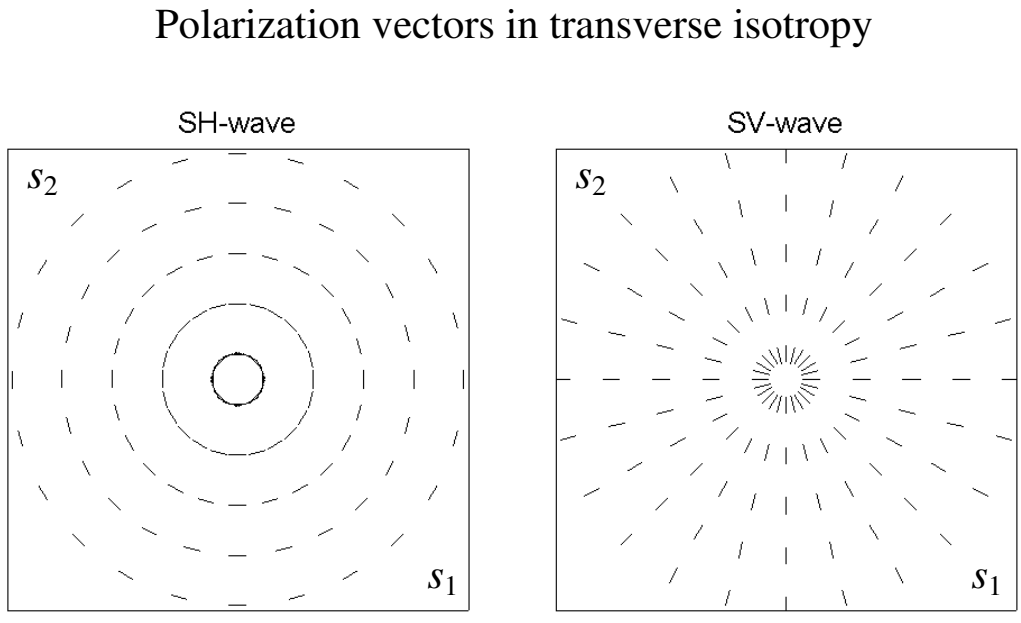

Figure 2. Singular behaviour of polarization vectors near the kiss singularity in transverse isotropy. The singularity has a topological charge $n_d = +1$.

where

$$\overline{g}_k \overline{g}_l = \lim_{u \to 0} g_k g_l \ . \tag{14}$$

Specifying (14) for the *SV* and *SH* waves

$$\overline{g}_k^{SV} \overline{g}_l^{SV} = \begin{bmatrix} \cos^2\varphi & \sin\varphi\cos\varphi & 0 \\ \sin\varphi\cos\varphi & \sin^2\varphi & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ \overline{g}_k^{SH} \overline{g}_l^{SH} = \begin{bmatrix} \sin^2\varphi & -\sin\varphi\cos\varphi & 0 \\ -\sin\varphi\cos\varphi & \cos^2\varphi & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{15}$$

and inserting into (13) we get

$$I_{kl}^{SV}(\mathbf{p}_0) = \frac{1}{2}\frac{\pi}{\sqrt{K^{SV}}}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ I_{kl}^{SH}(\mathbf{p}_0) = \frac{1}{2}\frac{\pi}{\sqrt{K^{SH}}}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{16}$$

Finally, the asymptotic Green function for *S* waves in the kiss singularity reads

$$G_{kl}^{S\,far}(\mathbf{x},t) = \frac{1}{8\pi\rho}\frac{1}{v^S}\left(\frac{1}{\sqrt{K^{SV}}} + \frac{1}{\sqrt{K^{SH}}}\right)\frac{\delta_{kl} - \delta_{k3}\delta_{l3}}{|\mathbf{x}|}\,\delta\left(t - \mathbf{p}_0 \cdot \mathbf{x}\right), \tag{17}$$

where K^{SV} and K^{SH} are Gaussian curvatures of the SV and SH slowness surfaces at the singularity, v^S is the group velocity at the singularity and δ_{kl} is the Kronecker delta. Equation (17) is connected to the local coordinate system with the x3-axis along the singularity. In a general coordinate system, the eq. (17) yields (Vavryčuk 1999, eq. A6)

$$G_{kl}^{S\,far}(\mathbf{x},t) = \frac{1}{8\pi\rho v^S}\left(\frac{1}{\sqrt{K^{SV}}} + \frac{1}{\sqrt{K^{SH}}}\right)\frac{\delta_{kl} - n_k n_l}{|\mathbf{x}|}\,\delta\left(t - \frac{|\mathbf{x}|}{v^S}\right), \tag{18}$$

where **n** denotes the direction of the phase or group velocities in the singularity.

5. Kiss singularity in general anisotropy

If we substitute coordinates s_1 and s_2 by coordinates u and φ defined as:

$$u = \mathbf{s}^T \mathbf{K}\mathbf{s} = K_{11}s_1^2 + 2K_{12}s_1s_2 + K_{12}s_2^2 \ , \ \varphi = \operatorname{atan}\frac{s_2}{s_1}, \tag{19}$$

we obtain for the Jacobian of the transformation

$$ds_1 ds_2 = \frac{1}{2}\frac{dud\varphi}{K_{11}\cos^2\varphi + 2K_{12}\sin\varphi\cos\varphi + K_{22}\sin^2\varphi} = \frac{1}{2}\frac{dud\varphi}{K_{IJ}e_I e_J}, \tag{20}$$

where $\mathbf{e} = (\cos\varphi, \sin\varphi)^T$. Then equation (5) can be rearranged as follows

$$I_{kl}(\mathbf{p}_0) = \frac{1}{2}\int\limits_{S^\varepsilon} \frac{g_k g_l\,\delta(u)}{K_{IJ}e_I e_J} dud\varphi = \frac{1}{2}\int_0^{2\pi} \frac{\overline{g}_k \overline{g}_l}{K_{IJ}e_I e_J} d\varphi \ , \tag{21}$$

where summation over capital indices runs from 1 to 2, and $\overline{\mathbf{g}}$ denotes the limit of the polarization vector $\mathbf{g}$ when $\mathbf{p}$ approaches $\mathbf{p}_0$ for a constant φ,

$$\overline{\mathbf{g}} = \overline{\mathbf{g}}(\varphi) = \lim_{\mathbf{p}\to\mathbf{p}_0} \mathbf{g} \ . \tag{22}$$

Matrix $\mathbf{K}$ in (21) means the curvature matrix in the singularity provided the slowness surface is regular in the singularity. The case of a singular shape of the slowness surface will be discussed in Sec. 6.

To evaluate integral (21) we should express $\overline{\mathbf{g}}$ as a function of angle φ for the $S1$ and $S2$ waves. Let us introduce angle Φ defining the orientation of polarization vectors $\overline{\mathbf{g}}^{S1}$ and $\overline{\mathbf{g}}^{S2}$ of the $S1$ and $S2$ waves in the close vicinity of the singularity:

$$\overline{\mathbf{g}}^{S1} = \mathbf{e}^{S1}\cos\Phi + \mathbf{e}^{S2}\sin\Phi \ , \ \overline{\mathbf{g}}^{S2} = -\mathbf{e}^{S1}\sin\Phi + \mathbf{e}^{S2}\cos\Phi \, , \tag{23}$$

where vectors $\mathbf{e}^{S1}$ and $\mathbf{e}^{S2}$ are arbitrarily chosen orthonormalized vectors in the plane perpendicular to the P-wave polarization vector $\mathbf{g}^P$. The angle $\Phi(\varphi)$ is expressed by the equation (Shuvalov 1998, eq. 2.41)

$$\cot 2\Phi(\varphi) = \frac{F_{IJ}e_I e_J}{G_{KL}e_K e_L}, \tag{24}$$

where matrices $\mathbf{F}$ and $\mathbf{G}$ are defined as (Shuvalov 1998, eq. 2.16)

$$F_{jk} = a_{ijkl}\left(e_i^{S1}e_l^{S1} - e_i^{S2}e_l^{S2}\right) + \frac{1}{\left(v^S\right)^2 - \left(v^P\right)^2}\left(q_j^{S1}q_k^{S1} - q_j^{S2}q_k^{S2}\right),$$

$$G_{jk} = a_{ijkl}\left(e_i^{S1}e_l^{S1} + e_i^{S2}e_l^{S2}\right) + \frac{1}{\left(v^S\right)^2 - \left(v^P\right)^2}\left(q_j^{S1}q_k^{S1} + q_j^{S2}q_k^{S2}\right), \tag{25}$$

$$q_j^{S1} = a_{ijkl}\left(e_i^{S1}n_l + e_l^{S1}n_i\right)n_k \ , \ q_j^{S2} = a_{ijkl}\left(e_i^{S2}n_l + e_l^{S2}n_i\right)n_k \, ,$$

where **n** denotes the direction of the slowness vector in the singularity. Note that matrices **F** and **G** also control the topological charge of the singularity n_d which can attain values 0, ± 1 (see Shuvalov 1998, Table 1).

Substituting eqs (23)-(25) into (21) we obtain a final expression for the amplitude $I_{kl}(\mathbf{p}_0)$ of the asymptotic Green function. Complexity of the obtained integral prevents analytic evaluation for a general anisotropy, hence the integral should be evaluated numerically. Obviously, under higher anisotropy symmetries the integral simplifies and its evaluation becomes more straightforward. Therefore, we shall specify the general formulae for tetragonal and cubic symmetries in the next section.

6. Kiss singularity in tetragonal and cubic symmetries

We shall concern the kiss singularity along the fourfold symmetry axes in tetragonal and cubic media. Let us consider the tetragonal symmetry with the following non-zero elastic parameters (see Musgrave 1970)

$$a_{11} = a_{22},\ a_{33}, a_{13} = a_{23},\ a_{12}, a_{44} = a_{55}, a_{66}, \tag{26}$$

and the cubic symmetry with the following non-zero elastic parameters

$$a_{11} = a_{22} = a_{33},\ a_{12} = a_{13} = a_{23},\ a_{44} = a_{55} = a_{66}. \tag{27}$$

The kiss singularity studied is in both symmetries along the p_3-axis. If we assume vectors $\mathbf{e}^{S1}$ and $\mathbf{e}^{S2}$ parallel to the p_1- and p_2-axis,

$$\mathbf{n} = (0,0,1)^T,\ \mathbf{e}^{S1} = (1,0,0)^T \text{ and } \mathbf{e}^{S2} = (0,1,0)^T, \tag{28}$$

we can express dyadics of polarization vectors $\overline{\mathbf{g}}^{S1}\overline{\mathbf{g}}^{S1}$ and $\overline{\mathbf{g}}^{S2}\overline{\mathbf{g}}^{S2}$ in (21) as follows:

$$\overline{\mathbf{g}}^{S1}\overline{\mathbf{g}}^{S1} = \begin{bmatrix} \cos^2\Phi & \sin\Phi\cos\Phi & 0 \\ \sin\Phi\cos\Phi & \sin^2\Phi & 0 \\ 0 & 0 & 0 \end{bmatrix},\ \overline{\mathbf{g}}^{S2}\overline{\mathbf{g}}^{S2} = \begin{bmatrix} \sin^2\Phi & -\sin\Phi\cos\Phi & 0 \\ -\sin\Phi\cos\Phi & \cos^2\Phi & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{29}$$

where

$$\cos^2\Phi = \frac{1}{2}\left[1 + \frac{F_{IJ}e_I e_J}{\sqrt{(F_{IJ}e_I e_J)^2 + (G_{IJ}e_I e_J)^2}}\right], \tag{30a}$$

$$\sin^2 \Phi = \frac{1}{2}\left[1 - \frac{F_{IJ} e_I e_J}{\sqrt{\left(F_{IJ} e_I e_J\right)^2 + \left(G_{IJ} e_I e_J\right)^2}}\right], \tag{30b}$$

$$\sin\Phi\cos\Phi = \frac{1}{2}\frac{G_{IJ} e_I e_J}{\sqrt{\left(F_{IJ} e_I e_J\right)^2 + \left(G_{IJ} e_I e_J\right)^2}}\ . \tag{30c}$$

Matrices **F** and **G** read

$$F_{11} = (a_{11}a_{33} - a_{11}a_{44} - a_{13}^2 - 2a_{13}a_{44} - a_{33}a_{66} - a_{44}^2 + a_{44}a_{66})/(a_{33} - a_{44})\,,$$
$$F_{12} = 0\,,\quad F_{22} = -F_{11}\,, \tag{31}$$

$$G_{11} = 0\,,\quad G_{12} = (a_{12}a_{33} - a_{12}a_{44} - a_{13}^2 - 2a_{13}a_{44} + a_{33}a_{66} - a_{44}^2 - a_{44}a_{66})/(a_{33} - a_{44})\,,$$
$$G_{22} = 0\,, \tag{32}$$

for tetragonal symmetry, and

$$F_{11} = (a_{11}^2 - 2a_{11}a_{44} - a_{12}^2 - 2a_{12}a_{44})/(a_{11} - a_{44})\,,\quad F_{12} = 0\,,\quad F_{22} = -F_{11}\,, \tag{33}$$

$$G_{11} = 0\,,\quad G_{12} = (a_{11}a_{12} + a_{11}a_{44} - a_{12}^2 - 3a_{12}a_{44} - 2a_{44}^2)/(a_{11} - a_{44})\,,\quad G_{22} = 0\,, \tag{34}$$

for cubic symmetry. Topological charge of the singularity is determined as

$$n_d = \mathrm{sign}\left(F_{11}G_{12}\right),$$

hence it can only attain values of +1 or –1. Behaviour of polarization vectors near a singularity with $n_d = 1$ in tetragonal or cubic symmetries is very similar to that in transverse isotropy (see Fig. 2). For behaviour of polarization vectors near a singularity with $n_d = -1$, see Fig. 3.

Using (31)-(34), equations (30a)-(30c) can be further simplified

$$\cos^2 \Phi = \frac{1}{2}\left[1 + \frac{F_{11}\cos 2\varphi}{\sqrt{F_{11}^2\cos^2 2\varphi + G_{12}^2 \sin^2 2\varphi}}\right],\ \sin^2 \Phi = \frac{1}{2}\left[1 - \frac{F_{11}\cos 2\varphi}{\sqrt{F_{11}^2\cos^2 2\varphi + G_{12}^2 \sin^2 2\varphi}}\right],$$

$$\sin\Phi\cos\Phi = \frac{1}{2}\frac{G_{12}\cos 2\varphi}{\sqrt{F_{11}^2\cos^2 2\varphi + G_{12}^2 \sin^2 2\varphi}}. \tag{35}$$

To evaluate integral (21) we have to specify matrix of curvature in the singularity. Interestingly, in contrary to the singularity in transverse isotropy, the shape of the slowness surface is not regular in the singularity for the tetragonal or cubic symmetries. It means that

the curvature of the slowness surface does not have a unique value in the singularity but instead the curvature depends on the direction of approaching the singularity (see Fig. 4). It implies that the *Gaussian curvature is not defined in the singularity*. We can write (see Shuvalov & Every 1996, eq. 45)

$$k^{S1,S2}(\varphi) = \overline{K}_{IJ}^{S1,S2} e_I e_J = \frac{1}{2}\frac{1}{\sqrt{a_{44}}}\left(f \pm \sqrt{g^2 \cos^2 2\varphi + h^2 \sin^2 2\varphi}\right), \tag{36}$$

where $\overline{K}_{IJ} = \overline{K}_{IJ}(\varphi)$ means the limit of the curvature matrix for $\mathbf{p} \to \mathbf{p}_0$ for a fixed value of φ, and f, g and h are defined for tetragonal symmetry as

$$f = (a_{11}a_{33} - a_{11}a_{44} - a_{23}^2 - 2a_{23}a_{44} + a_{33}a_{66} - a_{44}^2 - a_{44}a_{66})/(a_{33} - a_{44}), \tag{37}$$
$$g = F_{11}\ ,\ h = G_{12}\ ,$$

and for cubic symmetry as

$$f = (a_{11}^2 - a_{12}^2 - 2a_{12}a_{44} - 2a_{44}^2)/(a_{11} - a_{44}), \tag{38}$$
$$g = F_{11}\ ,\ h = G_{12}\ .$$

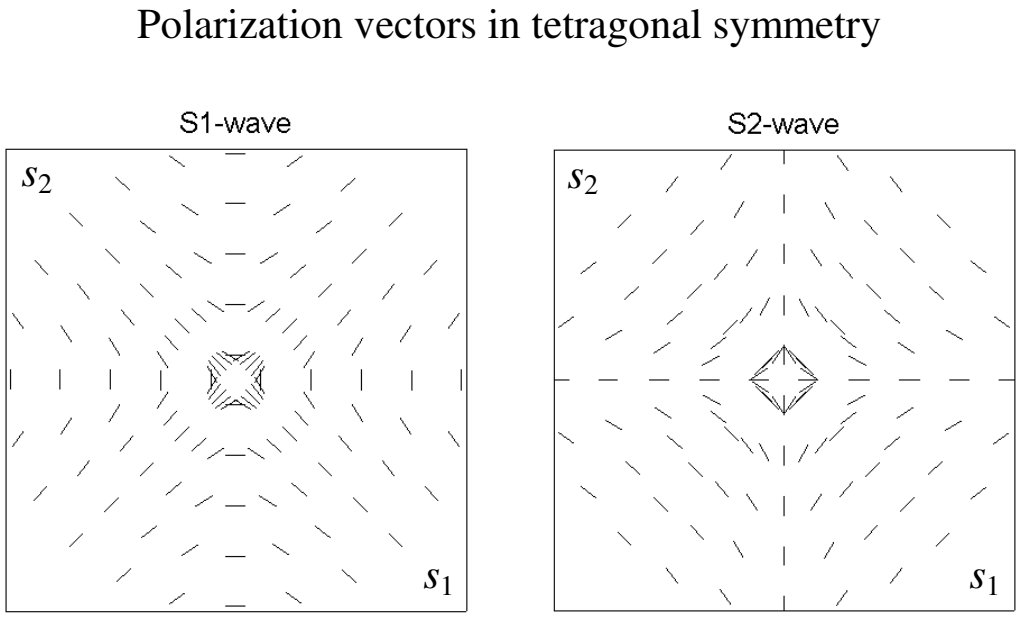

Figure 3. Singular behaviour of polarization vectors near the fourfold symmetry axis in tetragonal anisotropy. The singularity has a topological charge $n_d = -1$. For elastic parameters of the anisotropy, see the caption of Fig. 1.

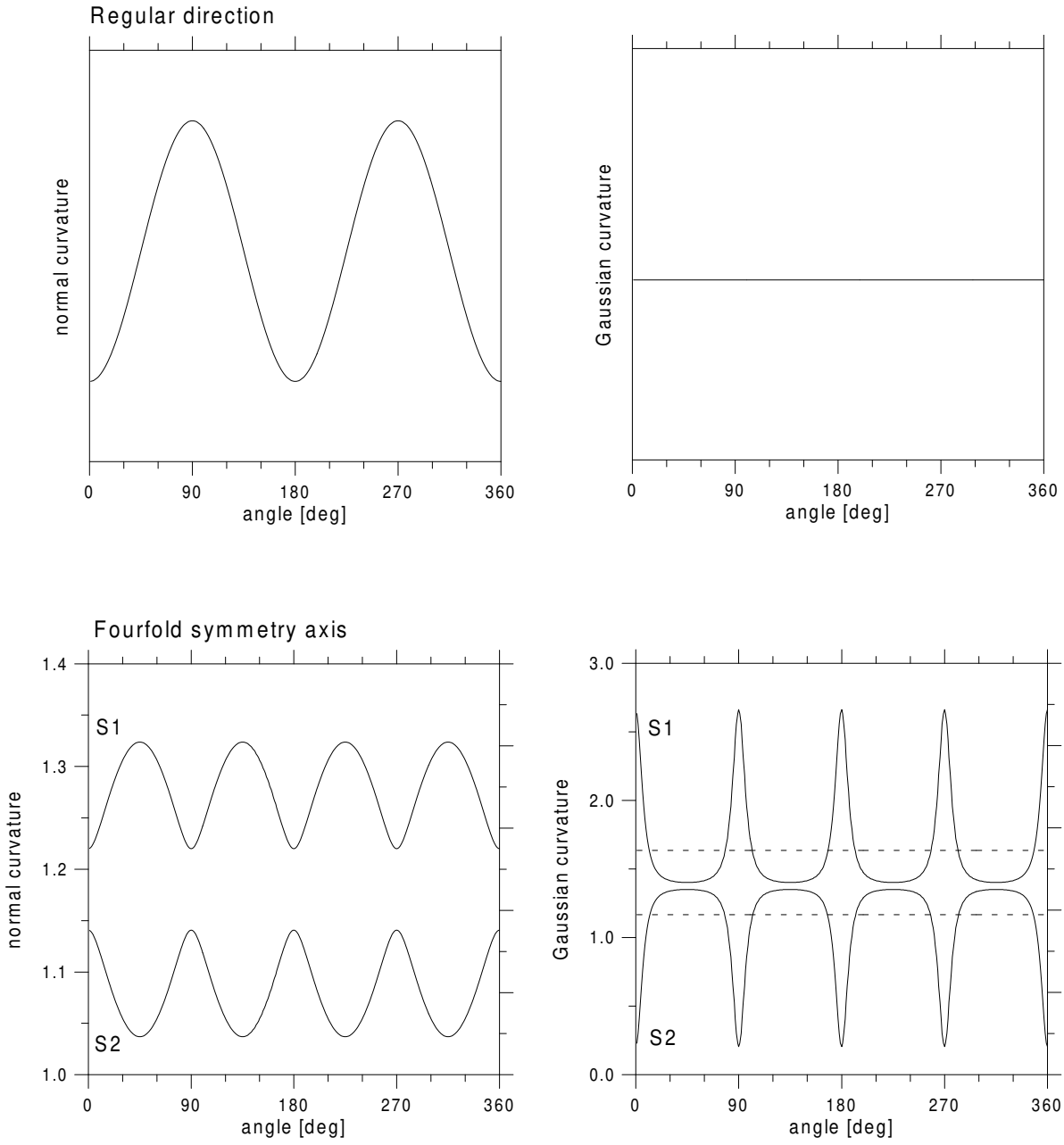

Figure 4. Behaviour of normal (left-hand plots) and Gaussian (right-hand plots) curvatures for a regular direction (upper plots) and for the fourfold symmetry axis in tetragonal anisotropy (lower plots). For elastic parameters of the tetragonal anisotropy, see the caption of Fig. 1. The dashed line shows a value of the generalized Gaussian curvature.

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$$\int_0^{2\pi} \frac{\cos 2\varphi}{\sqrt{F_{11}^2 \cos^2 2\varphi + G_{12}^2 \sin^2 2\varphi}} \frac{1}{f \pm \sqrt{F_{11}^2 \cos^2 2\varphi + G_{12}^2 \sin^2 2\varphi}} d\varphi = 0 \ , \tag{39}$$

we can simplify integral (21) and obtain

$$I_{kl}^{S1,S2}(\mathbf{p}_0) = \frac{\pi}{2\sqrt{\overline{K}^{S1,S2}}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \ , \tag{40}$$

where $\overline{K}^{S1,S2}$ is defined as

$$\frac{1}{\sqrt{\overline{K}^{S1,S2}}} = \frac{1}{2\pi} \int_0^{2\pi} \frac{d\varphi}{k^{S1,S2}(\varphi)} . \tag{41}$$

Quantity $k^{S1,S2}(\varphi)$ is the normal curvature in the singularity defined in (36), and $\overline{K}^{S1,S2}$ will be called the *generalized Gaussian curvature* (see Fig. 4). Note that for regular points of the slowness surface this definition yields the value of the standard Gaussian cuvature. The generalized Gaussian curvature (41) can be evaluated by means of elliptic integrals or numerically.

Finally we can write for the asymptotic Green function in the singularity

$$G_{kl}^{S\ far}(\mathbf{x},t) = \frac{1}{8\pi\rho v^S} \left(\frac{1}{\sqrt{\overline{K}^{S1}}} + \frac{1}{\sqrt{\overline{K}^{S2}}} \right) \frac{\delta_{kl} - n_k n_l}{|\mathbf{x}|} \delta\left(t - \frac{|\mathbf{x}|}{v^S} \right) . \tag{42}$$

Formula for the Green function in the kiss singularity in tetragonal or cubic symmetries is very similar to that in transverse isotropy (18). The only difference is that instead of Gaussian curvatures of the *S* waves in the singularity we use generalized Gaussian curvatures. The reason is that the shape of the slowness surface is not regular in the singularity in tetragonal or cubic symmetries hence the standard definition of the Gaussian curvature fails.

7. Conclusion

The far-field asymptotic formula for the Green function in the kiss singularity is more complex than the analogous formula valid in regular directions. The complications arise due to singularity in the field of polarization vectors and in the shape of the slowness surface. In

general anisotropy, the amplitude of the Green function is expressed in the form of 1-D integral. This integral can easily be evaluated for the kiss singularity along the symmetry axis in transversely isotropic media because behaviour of polarization vectors is simple and the shape of the slowness surface is regular. The amplitude of the Green function is then expressed by means of the Gaussian curvature of the slowness surface in the singularity. The kiss singularity along the fourfold symmetry axes in the tetragonal and cubic media is, however, more complicated. The shape of the slowness surface is not regular and the Gaussian curvature is not defined in this singularity. In this case the amplitude of the Green function is expressed by means of the generalized Gaussian curvature.

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Appendix A: Matrices F and G for general anisotropy

We assume the kiss singularity along the p_3-axis and vectors $\mathbf{e}^{S1}$ and $\mathbf{e}^{S2}$ parallel to the p_1- and p_2-axis:

$$\mathbf{n} = (0,0,1)^T,\ \mathbf{e}^{S1} = (1,0,0)^T \text{ and } \mathbf{e}^{S2} = (0,1,0)^T. \tag{A1}$$

Inserting (A1) into (25) we obtain

$$F_{11} = (a_{11}a_{33} - a_{11}a_{44} - a_{13}^2 - 2a_{13}a_{55} - a_{33}a_{66} + a_{36}^2 + 2a_{36}a_{45} + a_{44}a_{66} + a_{45}^2 - a_{55}^2)/(a_{33} - a_{44}), \tag{A2}$$

$$F_{12} = (-a_{13}a_{36} - a_{13}a_{45} + a_{16}a_{33} - a_{16}a_{44} + a_{23}a_{36} + a_{23}a_{45} - a_{26}a_{33} + a_{26}a_{44} + a_{36}a_{44} - a_{36}a_{55} + a_{44}a_{45} - a_{45}a_{55})/(a_{33} - a_{44}), \tag{A3}$$

$$F_{22} = (-a_{22}a_{33} + a_{22}a_{44} + a_{23}^2 + 2a_{23}a_{44} + a_{33}a_{66} - a_{36}^2 - 2a_{36}a_{45} + a_{44}^2 - a_{44}a_{66} - a_{45}^2)/(a_{33} - a_{44}), \tag{A4}$$

$$G_{11} = 2(-a_{13}a_{36} - a_{13}a_{45} + a_{16}a_{33} - a_{16}a_{44} - a_{36}a_{55} - a_{45}a_{55})/(a_{33} - a_{44}), \tag{A5}$$

$$G_{12} = (a_{12}a_{33} - a_{12}a_{44} - a_{13}a_{23} - a_{13}a_{44} - a_{23}a_{55} + a_{33}a_{66} - a_{36}^2 - 2a_{36}a_{45} - a_{44}a_{55} - a_{44}a_{66} - a_{45}^2)/(a_{33} - a_{44}), \tag{A6}$$

$$G_{22} = 2(-a_{23}a_{36} - a_{23}a_{45} + a_{26}a_{33} - a_{26}a_{44} - a_{36}a_{44} - a_{44}a_{45})/(a_{33} - a_{44}), \tag{A7}$$

We do not specify the other elements of **F** and **G** because they are not used in our calculations.