Energy flux of time-harmonic waves in anisotropic dissipative media

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Summary

The energy flux of time-harmonic waves propagating in anisotropic dissipative media may be introduced in several ways. It is most common to consider the average energy flux, which is real-valued and time-independent. An extension of this definition is the complex Poynting vector, which is also time-independent, but complex-valued. The real part of it equals the average energy flux. Recently, also an instantaneous energy flux, which is complex-valued and time-harmonic, has found interesting applications. In this article, all these three energy flux quantities are investigated for general anisotropic dissipative media, mainly in relation to the reflection/transmission problem. Energy flux quantities related to the summary wavefield, composed of several waves, are derived. It is shown that, in a dissipative medium, also the interaction energy contributions must be considered in the summary energy flux in addition to the energy fluxes of the individual waves. The interaction energy fluxes in anisotropic dissipative media are derived and discussed in detail. In non-dissipative media, the interaction contributions vanish. It is proved that the interaction contributions, corresponding to the summary vanish even in dissipative media. They, however, play an important role in the summary average energy flux and in the summary complex Poynting vector. Using the results derived for energy fluxes of summary waves, the energy balance at a structural interface between two anisotropic dissipative media is studied. In this case the conservation laws for the complex Poynting vector and for the average energy flux contain interaction terms. In the R/T problem, at an interface between two anisotropic dissipative media, the number of interaction terms may reach eighteen. The conservation laws for the instantaneous energy fluxes, however, do not contain any interaction terms; they contain only the instantaneous energy fluxes of the individual waves (incident, reflected, transmitted). Using the conservation laws, the relevant energy-based R/T coefficients may be introduced. The most important role is played by the instantaneous energy flux R/T coefficients. They include the appropriate phase shifts, needed in the seismic ray theory. The complex Poyinting vector and average energy flux energy R/T coefficients are not as important, as they do not contain any phase information. The instantaneous energy flux R/T coefficients are closely connected with the energy normalized R/T coefficients, used broadly in the seismic ray method. All results derived here for anisotropic dissipative media are specified also for isotropic dissipative media. They remain valid even for non-dissipative media.

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1 Introduction

The energy flux of time-harmonic seismic waves is one of the basic physical concepts in wave propagation problems. The average energy flux has mostly been used in applications. It is real-valued and time-independent. In addition to the average energy flux, also some other energy flux quantities have found applications in seismology, particularly the complex Poynting vector and instantaneous energy flux. For more details refer to Section 2.

For isotropic and anisotropic non-dissipative media, the properties of the average energy flux are well known. The average energy flux has been used to formulate the local radiation conditions for reflected and transmitted waves in the problem of reflection and transmission of a plane wave at a plane structural interface. Using these radiation conditions, it is possible to select the proper "upgoing" and "downgoing" waves of all admissible waves. See Červený (2001, pp. 50–1). As far as the author knows, such radiation conditions have not yet been proposed for dissipative media. Instead of simple local radiation conditions, it has been necessary to use a more cumbersome method based on a global analytical continuation. The author believes that a detailed investigation of various energy fluxes for dissipative media may help to find suitable local radiation conditions for transmitted and reflected waves, and to make the solution of the reflection/transmission problem in dissipative media more efficient.

Close to a structural interface and/or to the boundary of the model, it is necessary to consider the energy fluxes of summary waves. The summary wave is composed of the incident wave and of all reflected waves on one side of the interface, and of all transmitted waves on the other side of the interface. In general, the energy flux of the summary wavefield is not equal to the sum of energy fluxes of the individual waves in dissipative media; the interaction contributions must be introduced. See Section 5 for a detailed derivation, and Section 6 for the discussion of such interaction contributions in anisotropic dissipative media. We shall prove that the interaction contributions vanish if the summary instantaneous energy flux is considered. This makes the computation of the summary instantaneous energy flux considerably simpler than the computation of the summary average energy flux and of the summary complex Poynting vector.

Using the summary energy flux, it is straightforward to find the conservation laws for normal energy fluxes across a structural interface, see Section 7. Again, the energy balance equations are considerably simpler for instantaneous energy fluxes, as interaction contributions are not involved. For the complex Poynting vector and for the average energy flux, the situation is considerably more complicated. In the most general case, the number of interactive terms in the energy balance equations may reach eighteen. In non-dissipative models, all the interaction terms vanish.

Using the energy balance equations, it is possible to introduce various energy-based R/T coefficients see Section 8. The most important are the instantaneous energy flux coefficients. They also include the phase information, which is lost in the average energy flux R/T coefficients and in the complex Poynting vector R/T coefficients. The instantaneous energy flux R/T coefficients are closely related to the energy-normalized R/T coefficients, broadly used in the seismic ray theory. In the seismic ray method, the energy-normalized R/T coefficients have been introduced more or less artificially, to make the resulting equations more symmetric and reciprocal, see Červený (2001). Now we can see what the actual physical background of these coefficients is.

All the derived equations are, of course, applicable even in isotropic dissipative media. The situation is, however, more complicated than in isotropic non-dissipative media. In non-dissipative isotropic media, the system of six linear equations for the R/T coefficients can always be decomposed into two subsystems, corresponding to the P-SV and SH cases. In dissipative media, however, this decomposition is not possible if the propagation and attenuation vectors of the incident wave are not coplanar with the normal to the interface. All P and S waves are then coupled, and eighteen R/T coefficients are obtained. If the propagation and attenuation vectors of the incident wave are coplanar with the normal to the interface, the decomposition is possible, and the P-SV and SH waves may be studied independently. The energy balance equations, corresponding to these subsystems, however, contain interaction terms, if the average energy flux or the complex Poynting vector are considered. For instantaneous energy, the interaction terms again vanish. For explicit expressions see Section 9.

We shall use the Cartesian coordinates x_i, denote the time by t and the circular frequency by ω throughout the whole paper. We also denote the elastic moduli by c_{ijkl}. The elastic moduli are assumed to be complex-valued, as a dissipative medium is being considered. The Cartesian components of the displacement vector and of the particle velocity vector are denoted by $u_i(x_j, t)$ and $\dot{u}_i(x_j, t) = \partial u_i(x_j, t)/\partial t$, respectively. We assume the validity of Hooke's law in the form $\tau_{ij} = c_{ijkl} u_{k,l}$, where τ_{ij} are the Cartesian components of the stress tensor. The Einstein summation convention is used throughout the paper. Finally, a^* denotes the complex-conjugate of a.

2 Energy flux

We shall consider time-harmonic wave fields throughout the whole study. The most important energy flux quantity is the *average energy flux* $\vec{S}$, with its Cartesian components S_i given by the relation, see Auld (1973), Burridge (1976),

$$S_i = -\tfrac{1}{2}\mathrm{Re}(\tau_{ij}\dot{u}_j^*) \ . \tag{1}$$

The average energy flux is a real-valued and time-independent quantity. It has been used successfully and broadly in various seismological applications. The averaging in the definition of S_i is over one cycle in time, $T^{-1}\int_0^T ...\mathrm{d}t$, where $T = 2\pi/\omega$ is the period.

A simple generalization of the average energy flux is the complex vector $\vec{F}$, with Cartesian components F_i given by the relation

$$F_i = -\tfrac{1}{2}\tau_{ij}\dot{u}_j^* \ . \tag{2}$$

The vector $\vec{F}$ has often been called the *complex Poynting vector*, similarly as in electromagnetic waves (see Auld 1973; Ainslie and Burns 1995). It should be noted that the average energy flux is a real part of the complex Poynting vector,

$$S_i = \mathrm{Re}(F_i) \ . \tag{3}$$

Finally, we shall introduce vector $\vec{W}$, with Cartesian components W_i given by the relation

$$W_i = -\tau_{ij}\dot{u}_j \ . \tag{4}$$

We shall refer to $\vec{W}$ as the *instantaneous energy flux* here. The instantaneous energy flux is complex-valued and time-harmonic. Its frequency is twice higher (2ω) than the frequency ω of both its constituents, τ_{ij} and $\dot{u}_j$. The instantaneous energy flux does not represent a physical energy flux, but has, more or less, a formal meaning only. It is also known under different names, like "pseudointensity vector", "complex pseudointensity vector", "Deschamps' vector" and "Deschamps'complex pseudointensity vector", see Ainslie and Burns (1995). Certain of these terms acknowledge the work of M. Deschamps in this field, see Deschamps (1990), for example.

Note that the normal components of all the three energy fluxes, $n_i S_i$, $n_i F_i$ and $n_i W_i$ (where $\vec{n}$ is the normal to the interface) are continuous across the interface. This is obvious from the fact that both $n_i \tau_{ij}$ and $\dot{u}_j$ are continuous across the interface.

3 Energy flux of a plane wave

We shall consider a time-harmonic elastic plane wave propagating in a homogeneous anisotropic dissipative medium,

$$u_i(x_j, t) = U_i \exp[-\mathrm{i}\omega\theta(x_j, t)] \ , \tag{5}$$

where phase $\theta(x_j, t)$ is given by the relation

$$\theta(x_j, t) = t - p_k x_k \ . \tag{6}$$

Here $\vec{p}$ is the slowness vector, which is complex-valued:

$$\vec{p} = \vec{P} + \mathrm{i}\vec{A} \ . \tag{7}$$

$\vec{P}$ is the propagation vector and $\vec{A}$ the attenuation vector. For $\dot{u}_i$ and τ_{ij}, we obtain

$$\begin{aligned} \dot{u}_i &= -\mathrm{i}\omega U_i \exp[-\mathrm{i}\omega\theta(x_j, t)] \ , \\ \tau_{ij} &= \mathrm{i}\omega c_{ijkl} U_k p_l \exp[-\mathrm{i}\omega\theta(x_j, t)] \ . \end{aligned}$$

The energy fluxes for the plane wave are then as follows:

$$\begin{aligned} W_i &= -\omega^2 c_{ijkl} U_j U_k p_l \exp[-2\mathrm{i}\omega\theta(x_j, t)] \ , \\ F_i &= \tfrac{1}{2}\omega^2 Q c_{ijkl} U_j^* U_k p_l \ , \\ S_i &= \tfrac{1}{2}\omega^2 Q \mathrm{Re}[c_{ijkl} U_j^* U_k p_l] \ , \end{aligned} \tag{8}$$

where Q is given by the relation

$$Q = \exp[-2\omega A_k x_k] \ . \tag{9}$$

We can also insert

$$U_i = U g_i \ , \tag{10}$$

where g_i is the appropriate eigenvector (complex-valued). The general relations for the energy fluxes of plane waves are then as follows:

$$\begin{aligned} W_i &= -\omega^2 U^2 c_{ijkl} g_j g_k p_l \exp[-2\mathrm{i}\omega\theta(x_j, t)] \ , \\ F_i &= \tfrac{1}{2}\omega^2 Q (UU^*) c_{ijkl} g_j^* g_k p_l \ , \\ S_i &= \tfrac{1}{2}\omega^2 Q (UU^*) \mathrm{Re}(c_{ijkl} g_j^* g_k p_l) \ . \end{aligned} \tag{11}$$

4 Energy flux of a plane wave in the direction of $\vec{n}$

Assume that $\vec{n}$ is an arbitrary real-valued unit vector. The energy fluxes of a plane wave along $\vec{n}$ are then as follows:

$$\begin{array}{rcl} \overline{W} = W_i n_i &=& -\omega^2 U^2 X_j g_j \exp[-2\mathrm{i}\omega\theta(x_j,t)] \ , \\ \overline{F} = F_i n_i &=& \frac{1}{2}\omega^2 Q(UU^*) X_j g_j^* \ , \\ \overline{S} = S_i n_i &=& \frac{1}{2}\omega^2 Q(UU^*)\mathrm{Re}(X_j g_j^*) \ . \end{array} \tag{12}$$

Here we have used the notation

$$X_j = c_{ijkl} n_i g_k p_l \ . \tag{13}$$

We shall refer to $\overline{W}$ as the normal instantaneous energy flux, to $\overline{F}$ as the normal complex Poynting vector, and to $\overline{S}$ as the normal average energy flux. (The reason is that we shall use the symbol $\vec{n}$ later to denote the *normal* to an interface.)

5 Energy flux of a superposition of several plane waves in the direction of $\vec{n}$

Let us consider a superposition of n plane waves and calculate their summary energy flux. (Example: incident and reflected waves at a plane interface.) Let the k-th wave be described by the relation

$$u_i^{(k)} = U_i^{(k)} \exp[-\mathrm{i}\omega\theta^{(k)}(x_j,t)] \ , \tag{14}$$

where

$$\theta^{(k)}(x_j,t) = t - p_i^{(k)} x_i \ .$$

Then

$$u_i = \sum_{k=1}^{n} u_i^{(k)} = \sum_{k=1}^{n} U_i^{(k)} \exp[-\mathrm{i}\omega\theta^{(k)}(x_j t)] \ .$$

We assume that the phase functions $\theta^{(k)}$ are the same for all k at the selected observation point. The spatial derivatives $\theta_{,i}^{(k)}$, however, are different for different k. We can now calculate $\dot{u}_j^{(k)}$ and $\tau_{ij}^{(k)}$ in the same way as before and find the expressions for the summary energy fluxes. In the final expression, we put

$$\theta^{(1)} = \theta^{(2)} = ... = \theta^{(n)} = \theta \ . \tag{15}$$

The expressions for the normal summary energy fluxes are as follows

$$\begin{array}{rcl} \overline{W} &=& -\omega^2 (\sum_{k=1}^{n} U^{(k)} X_j^{(k)})(\sum_{m=1}^{n} U^{(m)} g_j^{(m)}) \exp[-2\mathrm{i}\omega\theta(x_j,t)] \ , \\ \overline{F} &=& \frac{1}{2}\omega^2 Q(\sum_{k=1}^{n} U^{(k)} X_j^{(k)})(\sum_{m=1}^{n} U^{(m)} g_j^{(m)})^* \ , \\ \overline{S} &=& \frac{1}{2}\omega^2 Q \mathrm{Re}\{(\sum_{k=1}^{n} U^{(k)} X_j^{(k)})(\sum_{m=1}^{n} U^{(m)} g_j^{(m)})^*\} \ . \end{array} \tag{16}$$

These expressions can be modified to read:

$$\begin{aligned}\overline{W} &= \sum_{k=1}^{n} \overline{W}^{(k)} + \overline{W}^{\mathrm{int}} \ , \\ \overline{F} &= \sum_{k=1}^{n} \overline{F}^{(k)} + \overline{F}^{\mathrm{int}} \ , \\ \overline{S} &= \sum_{k=1}^{n} \overline{S}^{(k)} + \overline{S}^{\mathrm{int}} \ .\end{aligned} \tag{17}$$

Here $\overline{W}^{(k)}$, $\overline{F}^{(k)}$ and $\overline{S}^{(k)}$ represent the normal energy fluxes of the individual waves ($k = 1, 2, ..., n$), and $\overline{W}^{int}$, $\overline{F}^{int}$, $\overline{S}^{int}$ are so called *interaction contributions*. The normal energy fluxes of the individual waves are given by standard equations:

$$\begin{aligned}\overline{W}^{(k)} &= -\omega^2 (U^{(k)})^2 X_j^{(k)} g_j^{(k)} \exp[-2\mathrm{i}\omega\theta(x_j, t)] \ , \\ \overline{F}^{(k)} &= \tfrac{1}{2}\omega^2 Q |U^{(k)}|^2 X_j^{(k)} g_j^{(k)*} \ , \\ \overline{S}^{(k)} &= \tfrac{1}{2}\omega^2 Q |U^{(k)}|^2 \mathrm{Re}(X_j^{(k)} g_j^{(k)*}) \ .\end{aligned} \tag{18}$$

The interaction contributions are given by relations:

$$\begin{aligned}\overline{W}^{int} &= \sum_{k=1}^{n} \sum_{\substack{m=1 \\ m\neq k}}^{n} \overline{W}^{(k,m)} \ , \\ \overline{F}^{int} &= \sum_{k=1}^{n} \sum_{\substack{m=1 \\ m\neq k}}^{n} \overline{F}^{(k,m)} \ , \\ \overline{S}^{int} &= \sum_{k=1}^{n} \sum_{\substack{m=1 \\ m\neq k}}^{n} \overline{S}^{(k,m)} \ .\end{aligned} \tag{19}$$

The individual interaction contributions $\overline{W}^{(k,m)}$, $\overline{F}^{(k,m)}$ and $\overline{S}^{(k,m)}$ read

$$\begin{aligned}\overline{W}^{(k,m)} &= -\omega^2 U^{(k)} U^{(m)} X_j^{(k)} g_j^{(m)} \exp[-2\mathrm{i}\omega\theta(x_j, t)] \ , \\ \overline{F}^{(k,m)} &= \tfrac{1}{2}\omega^2 Q U^{(k)} U^{(m)*} X_j^{(k)} g_j^{(m)*} \ , \\ \overline{S}^{(k,m)} &= \tfrac{1}{2}\omega^2 Q \mathrm{Re}(U^{(k)} U^{(m)*} X_j^{(k)} g_j^{(m)*}) \ .\end{aligned} \tag{20}$$

6 Interaction contributions

We shall now discuss the interaction contributions $\overline{W}^{int}$, $\overline{F}^{int}$ and $\overline{S}^{int}$ in greater detail.

An interesting result is obtained for the **interaction contribution to the normal instantaneous energy flux** $\overline{W}^{int}$. We can write

$$\overline{W}^{(k,m)} + \overline{W}^{(m,k)} = -\omega^2 U^{(k)} U^{(m)} (X_j^{(k)} g_j^{(m)} + X_j^{(m)} g_j^{(k)}) \exp[-2\mathrm{i}\omega\theta(x_j, t)] \ . \tag{21}$$

This immediately yields

$$\overline{W}^{(k,m)} + \overline{W}^{(m,k)} = 0 \qquad \text{for} \quad m \neq k \ , \tag{22}$$

because

$$X_j^{(k)} g_j^{(m)} + X_j^{(m)} g_j^{(k)} = 0 \qquad \text{for} \quad m \neq k \; . \tag{23}$$

Relation (**??**) was derived by Červený (2001, Section 5.4.7), exactly in the same form as given here. Analogous equations can also be deduced from relations given earlier by Frazer and Fryer (1989), Chapman (1994), Thomson (1996). Equation (**??**) then yields

$$\overline{W}^{int} = 0 \; . \tag{24}$$

Thus, in the expressions for the normal instantaneous energy flux, the interaction terms vanish.

The **interaction contributions to the normal complex Poynting vector** $\overline{F}^{int}$. They do not vanish. Equation (**??**), however, can be used even in this case to obtain an alternative form of the interaction contribution.

For $\overline{F}^{(k,m)} + \overline{F}^{(m,k)}$, we can write

$$\begin{aligned} \overline{F}^{(k,m)} + \overline{F}^{(m,k)} &= \tfrac{1}{2}\omega^2 Q\{(U^{(k)}X_j^{(k)})(U^{(m)}g_j^{(m)})^* + (U^{(m)}X_j^{(m)})(U^{(k)}g_j^{(k)})^*\} \\ &= \tfrac{1}{2}\omega^2 Q[(U^{(k)}X_j^{(k)})(U^{(m)}g_j^{(m)}) + (U^{(m)}X_j^{(m)})(U^{(k)}g_j^{(k)})] \\ &- 2\mathrm{i}(U^{(k)}X_j^{(k)})\mathrm{Im}(U^{(m)}g_j^{(m)}) - 2\mathrm{i}(U^{(m)}X_j^{(m)})\mathrm{Im}(U^{(k)}g_j^{(k)}) \; . \end{aligned}$$

Since the expression in square brackets vanishes (due to (**??**)), the final result is as follows:

$$\overline{F}^{(k,m)} + \overline{F}^{(m,k)} = -\mathrm{i}\omega^2 Q[(U^{(k)}X_j^{(k)})\mathrm{Im}(U^{(m)}g_j^{(m)}) + (U^{(m)}X_j^{(m)})\mathrm{Im}(U^{(k)}g_j^{(k)})] \; . \tag{25}$$

The interaction contribution $\overline{F}^{int}$ is then given by the relation,

$$\overline{F}^{int} = \sum_{k=1}^{n} \sum_{m=k+1}^{n} (\overline{F}^{(k,m)} + \overline{F}^{(m,k)}) \; . \tag{26}$$

All the interaction contributions $\overline{F}^{(k,m)} + \overline{F}^{(m,k)}$ are complex-valued. It is simple to see that $\overline{F}^{int}$ contains 6 such contributions for $n = 4$, 3 contributions for $n = 3$, and 1 contribution for $n = 2$.

The **interaction contribution to the normal average energy flux** $\overline{S}^{int}$ is analogous to $\overline{F}^{int}$, only we take the real parts:

$$\overline{S}^{int} = \sum_{k=1}^{n} \sum_{m=k+1}^{n} (\overline{S}^{(k,m)} + \overline{S}^{(m,k)}) \; , \tag{27}$$

where

$$\begin{aligned} \overline{S}^{(k,m)} + \overline{S}^{(m,k)} &= \omega^2 Q[\mathrm{Im}(U^{(k)}X_j^{(k)})\mathrm{Im}(U^{(m)}g_j^{(m)}) + \mathrm{Im}(U^{(m)}X_j^{(m)})\mathrm{Im}(U^{(k)}g_j^{(k)})] \\ &= \tfrac{1}{2}\omega^2 Q \; \mathrm{Re}[(U^{(k)}X_j^{(k)})(U^{(m)}g_j^{(m)})^* + (U^{(m)}X_j^{(m)})(U^{(k)}g_j^{(k)})^*] \; . \end{aligned} \tag{28}$$

7 Conservation of normal energy fluxes across a structural interface

We shall now consider a structural interface with an arbitrarily oriented unit normal $\vec{n}$. The quantities u_i and $\tau_{ij}n_j$ are continuous across the interface. This implies that also the nonlinear quantities $\tau_{ij}n_ju_i$, $\tau_{ij}n_ju_i^*$ and $\mathrm{Re}(\tau_{ij}n_ju_i^*)$ are continuous across the interface. This, of course, applies to the summary quantities, not to individual waves. In general, interaction contributions must also be considered.

In the most general case, we have one incident wave, three reflected waves and three transmitted waves. In the following conservation equations, we shall use the superscript *inc* for the incident wave, the superscripts rP, $rS1$, $rS2$ for reflected waves, and the superscripts tP, $tS1$, $tS2$ for the transmitted waves.

a) **Conservation law for the normal instantaneous energy flux**.
In this case, the conservation law is simple, as the interaction contributions vanish. The conservation law reads

$$\overline{W}^{inc} + \overline{W}^{rP} + \overline{W}^{rS1} + \overline{W}^{rS2} = \overline{W}^{tP} + \overline{W}^{tS1} + \overline{W}^{tS2} . \tag{29}$$

Note that all terms in (**??**) contain the same factor $-\omega^2 \exp[-2\mathrm{i}\omega\theta]$ so that this factor may be excluded. Thus, the individual terms can be expressed in a simpler form:

$$\overline{W}^{(k)} = (U^{(k)})^2 X_j^{(k)} g_j^{(k)} . \tag{30}$$

Expressions $\overline{W}^{(k)}$ are complex-valued, so that (**??**) represents two conservation laws, for the real and imaginary parts.

b) **Conservation law for the normal complex Poynting vector**
In this case, the interaction contributions do not vanish, so that we obtain

$$\overline{F}^{inc} + \overline{F}^{rP} + \overline{F}^{rS1} + \overline{F}^{rS2} + \overline{F}_1^{int} = \overline{F}^{tP} + \overline{F}^{tS1} + \overline{F}^{tS2} + \overline{F}_2^{int} . \tag{31}$$

Here all $\overline{F}^{(k)}$ contain the same factors $\frac{1}{2}\omega^2 Q$, which may be excluded. Thus, we obtain

$$\overline{F}^{(k)} = |U^{(k)}|^2 X_j^{(k)} g_j^{(k)*} . \tag{32}$$

The interaction contribution $\overline{F}_1^{int}$ corresponds to the medium with the incident wave, $\overline{F}_2^{int}$ to the second medium. They are given by relations

$$\overline{F}_1^{int} = \sum_{k=1}^{4} \sum_{m=k+1}^{4} (\overline{F}^{(k,m)} + \overline{F}^{(m,k)}) , \qquad \overline{F}_2^{int} = \sum_{k=1}^{3} \sum_{m=k+1}^{3} (\overline{F}^{(k,m)} + \overline{F}^{(m,k)}) , \tag{33}$$

where $(\overline{F}^{(k,m)} + \overline{F}^{(m,k)})$ are given by two alternative expressions:

$$\overline{F}^{(k,m)} + \overline{F}^{(m,k)} = (U^{(k)} X_j^{(k)})(U^{(m)} g_j^{(m)})^* + (U^{(m)} X_j^{(m)})(U^{(k)} g_j^{(k)})^* \tag{34}$$

or

$$\overline{F}^{(k,m)} + \overline{F}^{(m,k)} = -2\mathrm{i}[(U^{(k)} X_j^{(k)})\mathrm{Im}(U^{(m)} g_j^{(m)}) + (U^{(m)} X_j^{(m)})\mathrm{Im}(U^{(k)} g_j^{(k)})] . \tag{35}$$

In the expression for $\overline{F}_1^{int}$, four waves are considered (incident, reflected P, S1 and S2), so that $n = 4$. The incident wave may be of any type (P, S1, S2). Similarly, in the expression for $\overline{F}_2^{int}$, three waves are considered (transmitted P, S1 and S2), so that $n = 3$.

Since $\overline{F}$ in Equation (**??**) are complex-valued, (**??**) represents two conservation laws, for the real and imaginary parts.

c) **Conservation law for the normal average energy flux**
As $\overline{S} = \mathrm{Re}\overline{F}$, the conservation law for the normal average energy flux across an interface is equivalent to the real part of (**??**):

$$\overline{S}^{inc} + \overline{S}^{rP} + \overline{S}^{rS1} + \overline{S}^{rS2} + \overline{S}_1^{int} = \overline{S}^{tP} + \overline{S}^{tS1} + \overline{S}^{tS2} + \overline{S}_2^{int} \ . \tag{36}$$

If we again exclude factors $\frac{1}{2}\omega^2 Q$ from all terms, we obtain

$$\overline{S}^{(k)} = |U^{(k)}|^2 \mathrm{Re}(X_j^{(k)} g_j^{(k)*}) \ . \tag{37}$$

$$\overline{S}_1^{int} = \sum_{k=1}^{4} \sum_{m=k+1}^{4} (\overline{S}^{(k,m)} + \overline{S}^{(m,k)}) \ , \qquad \overline{S}_2^{int} = \sum_{k=1}^{3} \sum_{m=k+1}^{3} (\overline{S}^{(k,m)} + \overline{S}^{(m,k)}) \ , \tag{38}$$

$$\begin{aligned} \overline{S}^{(k,m)} + \overline{S}^{(m,k)} &= \mathrm{Re}\{(U^{(k)} X_j^{(k)})(U^{(m)} g_j^{(m)})^* + (U^{(m)} X_j^{(m)})(U^{(k)} g_j^{(k)})^*\} \\ &= 2\{\mathrm{Im}(U^{(k)} X_j^{(k)})\mathrm{Im}(U^{(m)} g_j^{(m)}) + \mathrm{Im}(U^{(m)} X_j^{(m)})\mathrm{Im}(U^{(k)} g_j^{(k)})\} \ . \end{aligned} \tag{39}$$

d) **Alternative equations**

Using relations (**??**), we can also obtain other equations, alternative to the conservation laws. These equations, however, do not use explicitly the normal energy flux quantities as the conservation laws. They read,

$$\begin{aligned} (\sum_{k=1}^{4} U^{(k)} X_j^{(k)})(\sum_{m=1}^{4} U^{(m)} g_j^{(m)}) &= (\sum_{k=5}^{7} U^{(k)} X_j^{(k)})(\sum_{m=5}^{7} U^{(m)} g_j^{(m)}) \ , \\ (\sum_{k=1}^{4} U^{(k)} X_j^{(k)})(\sum_{m=1}^{4} U^{(m)} g_j^{(m)})^* &= (\sum_{k=5}^{7} U^{(k)} X_j^{(k)})(\sum_{m=5}^{7} U^{(m)} g_j^{(m)})^* \ , \\ \mathrm{Re}[(\sum_{k=1}^{4} U^{(k)} X_j^{(k)})(\sum_{m=1}^{4} U^{(m)} g_j^{(m)})^*] &= \mathrm{Re}[(\sum_{k=5}^{7} U^{(k)} X_j^{(k)})(\sum_{m=5}^{7} U^{(m)} g_j^{(m)})^*] \ . \end{aligned} \tag{40}$$

Here the numbers 1,2,3,4 correspond to one incident wave and three reflected waves, and the numbers 5,6,7 to three transmitted waves.

e) **Conservation laws: discussion**

Thus, we have obtained two complex-valued conservation laws, (**??**) and (**??**). The real and imaginary parts of (**??**) and (**??**) represent four independent conservation laws. The remaining conservation law (**??**) for the normal average energy flux is not independent; it corresponds to the real part of the Poynting vector (**??**).

8 Energy-based R/T coefficients

In all conservation laws, we can use $U^{inc} = 1$. The relevant U^{rP}, U^{rS1}, U^{rS2}, U^{tP}, U^{tS1} and U^{tS2} then have the meaning of the appropriate displacement R/T coefficients.

If we put $U^{inc} = 1$ in relations (**??**), we obtain several useful relations between the displacement R/T coefficients.

In principle, it is also possible to introduce various R/T coefficients for normal energy fluxes. Let us consider independently the normal instantaneous energy flux $\overline{W}$, the normal complex Poynting vector $\overline{F}$, the normal average energy flux $\overline{S}$, and the relevant R/T coefficients.

a) **Normal instantaneous energy flux R/T coefficients** R^W

If we divide Eq. (**??**) by $X_j^{inc} g_j^{inc}$, the first term in (**??**), corresponding to the incident wave, equals 1. The other terms then have the meaning of the normal instantaneous energy flux R/T coefficients R^W, which are given by the relations,

$$(R^W)^{(k)} = (R^{(k)})^2 (X_j^{(k)} g_j^{(k)} / X_l^{inc} g_l^{inc}) \ . \tag{41}$$

We can also use the notation

$$\mathcal{U}_i = \rho^{-1} c_{ijkl} g_j g_k p_l \ . \tag{42}$$

For homogeneous waves in nondissipative media, $\mathcal{U}_i$ represents the i-th Cartesian component of the group velocity vector. In dissipative media, however, $\mathcal{U}_i$ are complex-valued. Using notation (**??**), we obtain

$$X_i g_i = \rho(\mathcal{U}_i n_i) = \rho \overline{\mathcal{U}} \ , \tag{43}$$

where $\overline{\mathcal{U}}$ is the normal component of $\vec{\mathcal{U}}$. Equation (**??**) then reads

$$(R^W)^{(k)} = (R^{(k)})^2 (\rho^{(k)} \overline{\mathcal{U}}^{(k)} / \rho^{inc} \overline{\mathcal{U}}^{inc}) \ . \tag{44}$$

Equation (**??**) gives the relation between the normal instantaneous energy flux R/T coefficient R^W and standard displacement coefficient R.

Taking the square roots of both sides of (**??**), we obtain the so-called *energy-normalized R/T coefficients* $\mathcal{R}^{(k)}$,

$$\mathcal{R}^{(k)} = [(R^W)^{(k)}]^{1/2} = R^{(k)} [\rho^{(k)} \overline{\mathcal{U}}^{(k)} / \rho^{inc} \overline{\mathcal{U}}^{inc}]^{1/2} \ . \tag{45}$$

Such energy-normalized R/T coefficients have been used broadly in the ray method in non-dissipative elastic media, see Červený (2001, Eq.(5.4.13)). The energy-normalized R/T coefficients have many advantages over the standard displacement coefficients. Inter alia, they are reciprocal, even though the standard displacement R/T coefficients are not reciprocal; see Chapman (1994), Thomson (1996), Červený (2001).

b) **Normal complex Poynting vector R/T coefficients**

Dividing (**??**) by $X_l^{inc} g_l^{inc*}$, the first term in (**??**), corresponding to the incident wave, equals one. The individual terms then have the meaning of the normal complex Poynting vector R/T coefficient R^F, given by the relation

$$(R^F)^{(k)} = |R^{(k)}|^2 (X_j^{(k)} g_j^{(k)*} / X_l^{inc} g_l^{inc*}) \ . \tag{46}$$

In this case, the conservation law (**??**), however, also contains the interaction terms. The interaction R/T coefficients are obtained from (**??**) or (**??**), by dividing them by $X_l^{inc} g_l^{inc*}$.

c) **Normal average energy flux R/T coefficients**

Dividing Eq. (**??**) by $\mathrm{Re}(X_l^{inc} g_l^{inc*})$, the first term in (**??**), corresponding to the incident wave, equals 1. The other terms then have the meaning of the normal average energy flux R/T coefficients R^S:

$$(R^S)^{(k)} = |R^{(k)}|^2 [\mathrm{Re}(X_j^{(k)} g_j^{(k)*}) / \mathrm{Re}(X_l^{inc} g_l^{inc*})] \ . \tag{47}$$

Conservation law (**??**) also contains interaction terms. The relevant interaction coefficients are obtained from (**??**), by dividing them by $\mathrm{Re}(X_l^{inc} g_l^{inc*})$. Note that all the R/T coefficients (including the interaction coefficients) are real-valued in this case. However, they do not equal the real parts of the analogous normal complex Poynting vector coefficients.

9 Isotropic dissipative media

The derived conservation laws and energy-based R/T coefficients can, of course, also be used in isotropic media. We only have to choose properly the eigenvectors $\vec{g}^{(1)}$, $\vec{g}^{(2)}$, $\vec{g}^{(3)}$, corresponding to S1, S2 and P waves. For dissipative isotropic media, $\vec{g}^{(i)}$ are complex-valued, and their determination is described in detail in Brokešová and Červený (1998). The expressions for X_i in isotropic dissipative media are obtained from the general relation (1), if we insert $c_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\delta_{ik}\delta_{jl} + \mu\delta_{il}\delta_{jk}$:

$$X_i = \lambda n_i(g_k p_k) + \mu p_i(g_k n_k) + \mu g_i(n_k p_k) \ . \tag{48}$$

The Lamé's elastic moduli λ and μ are pressumably complex-valued.

The general conservation laws (**??**), (**??**) and (**??**) can be used even if the propagation vector $\vec{P}^{inc}$ and the attenuation vector $\vec{A}^{inc}$ of the incident wave *are not coplanar* with the unit vector $\vec{n}$, normal to the interface. In this case, the system of six equations for the displacement R/T coefficients cannot be decomposed into two subsystems for P-SV and for SH waves. Also the number of interaction terms remains the same as in anisotropic media. $\overline{F}_1^{int}$ and $\overline{S}_1^{int}$ have six terms each, and $\overline{F}_2^{int}$ and $\overline{S}_2^{int}$ three terms each.

If $\vec{P}^{inc}$, $\vec{A}^{inc}$ and $\vec{n}$ *are coplanar*, the situation is considerably simplified. The system of six equations for the displacement R/T coefficients can be decomposed into two subsystems, for P-SV waves (four equations) and for SH waves (two equations).

Let us call the plane, defined by $\vec{P}^{inc}$, $\vec{A}^{inc}$ and $\vec{n}$, the plane of incidence. We choose the Cartesian coordinate system in such a way that x_2-axis is perpendicular to the plane of incidence and the x_3-axis coincides with $\vec{n}$. Then $n_1 = n_2 = 0$, $n_3 = 1$. We choose the eigenvector of the S2 wave, $\vec{g}^{(2)}$, perpendicular to the plane of incidence, so that $g_1^{(2)} = g_3^{(2)} = 0$, $g_2^{(2)} = 1$. We speak of SH waves in this case. The eigenvectors of the S1 waves (called SV waves) and P waves, $\vec{g}^{(1)}$ and $\vec{g}^{(3)}$, are complex-valued and are confined to the plane of incidence, so that $g_2^{(1)} = g_2^{(3)} = 0$. The slowness vector of all three waves (SV, SH, P) are also complex-valued and are confined to the plane of incidence, so that $p_2^{(1)} = p_2^{(2)} = p_2^{(3)} = 0$. For more details, see Ainslie and Burns (1995), Brokešová and Červený (1998), Krebes (1984).

In the following, we shall consider independently the two cases: a) the P-SV case, b) the SH case.

a) **P-SV case**

In this case, only P and SV waves exist. Using (**??**), we obtain the following relations for X_i:

$$X_1 \ = \ \mu(p_1 g_3 + p_3 g_1) \ ,$$

$$
\begin{aligned}
X_2 &= 0 \ , \\
X_3 &= \lambda(g_1p_1+g_3p_3)+2\mu p_3g_3 \ .
\end{aligned}
\tag{49}
$$

Conservation laws (**??**), (**??**) and (**??**) now simplify:

$$
\begin{aligned}
\overline{W}^{inc}+\overline{W}^{rP}+\overline{W}^{rSV} &= \overline{W}^{tP}+\overline{W}^{tSV} \ , \\
\overline{F}^{inc}+\overline{F}^{rP}+\overline{F}^{rSV}+\overline{F}_1^{int} &= \overline{F}^{tP}+\overline{F}^{tSV}+\overline{F}_2^{int} \ , \\
\overline{S}^{inc}+\overline{S}^{rP}+\overline{S}^{rSV}+\overline{S}_1^{int} &= \overline{S}^{tP}+\overline{S}^{tSV}+\overline{S}_2^{int} \ ,
\end{aligned}
\tag{50}
$$

The expressions for the individual terms in (**??**) are again given by (**??**), (**??**) and (**??**), taking into account (**??**). Similarly, interaction terms $\overline{F}_1^{int}$, $\overline{F}_2^{int}$, $\overline{S}_1^{int}$ and $\overline{S}_2^{int}$ are given by (**??**)–(**??**) and (**??**)–(**??**), using again (**??**). The number of interaction terms in (**??**) and (**??**) is, however, lower, as only P and SV waves are involved. The summation in expressions (**??**) and (**??**) for $\overline{F}_1^{int}$ and $\overline{S}_1^{int}$ runs only from 1 to 3, and for $\overline{F}_2^{int}$, $\overline{S}_2^{int}$ from 1 to 2.

b) **SH case**

In this case, only SH waves exist. Equation (**??**) yields

$$
X_1 = X_3 = 0 \ , \qquad X_2 = \mu p_3 \ . \tag{51}
$$

All the expressions are very simple in this case. Moreover, only one interaction term is obtained. For this reason, we shall give all expressions in full.

i) **Conservation law for the normal instantaneous energy flux**

Equations (**??**), (**??**) and (**??**) yield

$$
\overline{W}^{inc}+\overline{W}^{rSH}=\overline{W}^{tSH} \ , \tag{52}
$$

where

$$
\begin{aligned}
\overline{W}^{inc} &= \mu_1 p_3^{inc} \ , \\
\overline{W}^{rSH} &= (R^{SH})^2\mu_1 p_3^r \ , \\
\overline{W}^{tSH} &= (T^{SH})^2\mu_2 p_3^t \ .
\end{aligned}
\tag{53}
$$

ii) **Conservation law for the normal complex Poynting vector**

Equations (**??**), (**??**), (**??**) and (**??**) yield

$$
\overline{F}^{inc}+\overline{F}^{rSH}+\overline{F}_1^{int}=\overline{F}^{tSH} \ , \tag{54}
$$

where

$$
\begin{aligned}
\overline{F}^{inc} &= \mu_1 p_3^{inc} \ , \\
\overline{F}^{rSH} &= |R^{SH}|^2\mu_1 p_3^r \ , \\
\overline{F}^{tSH} &= |T^{SH}|^2\mu_2 p_3^t \ , \\
\overline{F}_1^{int} &= -2\mathrm{i}\mu_1 p_3^{inc}\mathrm{Im}(R^{SH}) \ .
\end{aligned}
\tag{55}
$$

iii) **Conservation law for the normal average energy flux**

Equations (**??**), (**??**), (**??**) and (**??**) yield

$$\overline{S}^{inc} + \overline{S}^{rSH} + \overline{S}_1^{int} = \overline{S}^{tSH} \ , \tag{56}$$

where

$$\begin{aligned}
\overline{S}^{inc} &= \mathrm{Re}(\mu_1 p_3^{inc}) \ , \\
\overline{S}^{rSH} &= |R^{SH}|^2 \mathrm{Re}(\mu_1 p_3^{r}) \ , \\
\overline{S}^{tSH} &= |T^{SH}|^2 \mathrm{Re}(\mu_2 p_3^{t}) \ , \\
\overline{S}_1^{int} &= 2\mathrm{Im}(\mu_1 p_3^{inc}) \mathrm{Im}(R^{SH}) \ .
\end{aligned} \tag{57}$$

In (**??**), (**??**) and (**??**), μ_1 is the complex-valued rigidity in the medium with the incident wave, μ_2 the complex-valued rigidity in the second medium, R^{SH} and T^{SH} denote standard SH displacement R/T coefficients, given by relations

$$R^{SH} = \frac{\mu_1 p_3^{inc} - \mu_2 p_3^t}{\mu_1 p_3^{inc} + \mu_2 p_3^t} \ , \qquad T^{SH} = \frac{2\mu_1 p_3^{inc}}{\mu_1 p_3^{inc} + \mu_2 p_3^t} \ . \tag{58}$$

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