Correlation functions of random media

Luděk Klimeš
Department of Geophysics, Charles University, Ke Karlovu 3, 121 16 Praha 2, Czech Republic, Fax: +420-2-2191 1214, E-mail: klimes@seis.karlov.mff.cuni.cz

Summary

In geophysics, the correlation functions of random media are of principal importance for understanding and inverting the properties of seismic waves propagating in geological structures. Unfortunately, the kinds of correlation functions inappropriate for the description of geological structures are often assumed and applied. The most frequently used types of correlation functions are thus summarized and reviewed in this paper, together with explanation of the physical meaning of their parameters.

A stationary random medium is assumed to be realized in terms of a white noise filtered by a spectral filter. The spectral filter is considered isotropic, in a simple general form enabling the random media used in geophysics to be specified. The medium correlation functions, corresponding to the individual special cases of the general random medium (Gaussian, exponential, von Kármán, self–affine, Kummer), are then derived and briefly discussed. The corresponding elliptically anisotropic correlation functions can simply be obtained by linear coordinate transforms.

Keywords

Random medium, covariance function, correlation function, scaling geology, fractal geology.

1 Introduction

In order to estimate the relation of the seismic model to the geological structure and, in consequence, to estimate the relation of the synthetic quantities, calculated in the seismic model, to reality, it is important to have an estimate of the medium covariance function. On large scales, the medium covariance function is of principal importance in refraction travel–time tomographic inversion (Tarantola 1987; Maurer, Holliger & Boerner 1998; Klimeš 2002a), especially when estimating the accuracy of the seismic model, its relation to the geological structure, or the covariance matrix describing the statistics of synthetic travel times. On smaller scales, correlation functions enable to estimate the scattering attenuation (e.g., Chernov 1960; Wu 1982) or to approximately upscale a rough distribution of material parameters and estimate the relation to a smoother distribution of effective material parameters (e.g., Shapiro & Kneib 1993; Shapiro, Schwarz & Gold 1996; Müller & Shapiro 2001).

In contrast to the importance of the correlation functions, the kinds of correlation functions inappropriate for the description of geological structures are applied in too many geophysical papers, whereas the kinds of correlation functions suitable for the description are applied rarely in the geophysical literature. For example, the physically reasonable values of the Hurst exponent N, describing the scaling properties of geological

In: Seismic Waves in Complex 3-D Structures, Report 11 (Department of Geophysics, Charles University, Prague, 2001), pp. 91–109; to appear in Pure and Applied Geophysics, **159** (2002)

structures, are $-\frac{1}{2} < N < 0$, whereas the two values most popular in the geophysical literature are $N = -\frac{3}{2}$ and $N = \frac{1}{2}$. The author thus has come to the conclusion that it is reasonable to briefly summarize the most frequently used types of correlation functions, together with explanation of the physical meaning of their parameters in geophysical applications.

We assume each material parameter of the geological structure to be composed of a heterogeneous mean value and a particular representation of the stationary (statistically homogeneous) random medium in this paper. The stationary random medium is assumed to be realized in terms of a white noise filtered by a spectral filter. In this way, the medium covariance function may be expressed in terms of the correlation function.

The aim of this paper is to summarize systematically and discuss briefly the types of random media used in geophysics. The generic spectral filter is considered isotropic, in a simple general form described by four constants, enabling the random media used in geophysics to be specified. The medium correlation functions, corresponding to the individual special cases of the general random medium, are then derived and briefly discussed. The corresponding elliptically anisotropic correlation functions can simply be obtained from their generic isotropic counterparts by linear coordinate transforms, see Section 3.2.

2 Correlation functions of material parameters

The material parameters of the *geological structure* are assumed to be realizations of random media with known statistical characteristics. This approach enables us to account satisfactorily for the differences between the geological structure and the *seismic model* of the structure, based on our quite incomplete information on the structure. The statistical characteristics represent our knowledge of the deterministic features of the structure and may change with more information on the structure. The randomness corresponds to the unknown details in the geological structure and may decrease with more information on the structure.

2.1 Material parameters

The geological structure is assumed to be expressed in terms of spatial distribution $u = u(\mathbf{x})$ of several *material parameters* u. For example, the material parameters may represent the density, P and S wave velocities or slownesses, or the elements of the stiffness matrix.

The *microscale heterogeneities* of these parameters (heterogeneities on scales shorter than the wavelength) may influence the propagation of seismic waves of a particular frequency in such a way that the propagation corresponds to some frequency–dependent *effective material parameters* $\overline{u}(\mathbf{x}, \omega)$ "locally homogeneous" on microscales. For the study of *macroscale heterogeneities* (e.g., seismic travel–time tomography) material parameters $u(\mathbf{x})$ may appropriately be replaced with the effective material parameters $\overline{u}(\mathbf{x}, \omega)$. The corresponding seismic model is then often called the *macromodel*, particularly in seismic prospecting. Hereinafter, "material parameter $u(\mathbf{x})$" may represent either any of the material parameters $u(\mathbf{x})$ or any of the effective material parameters $\overline{u}(\mathbf{x}, \omega)$ at a given frequency.

2.2 Mean value of the material parameter

Mean value

$$u_0(\mathbf{x}) = \langle u(\mathbf{x}) \rangle \tag{1}$$

of material parameter $u(\mathbf{x})$ describes the most probable value of the material parameter. The mean value $u_0(\mathbf{x})$ of the material parameter is a very smooth function in most cases.

The unknown difference

$$U(\mathbf{x}) = u(\mathbf{x}) - u_0(\mathbf{x}) \tag{2}$$

between the material parameter and its mean value $u_0(\mathbf{x})$ is considered to be a realization of the random quantity.

2.3 Generalized correlation function of the material parameter

The unknown difference between the material parameter and its mean value may be characterized in terms of the *medium covariance function*

$$C(\mathbf{x}_1, \mathbf{x}_2) = \langle U^*(\mathbf{x}_1)\, U(\mathbf{x}_2) \rangle \quad . \tag{3}$$

The asterisk * denoting complex conjugacy is included here for completeness only. We shall consider real–valued material parameters in this paper.

To characterize the properties of the medium covariance function better, we introduce the *generalized correlation function*

$$c^{\mathrm{G}}(\mathbf{y}, \mathbf{x}) = C(\mathbf{y} - \tfrac{1}{2}\mathbf{x}, \mathbf{y} + \tfrac{1}{2}\mathbf{x}) = \left\langle U^*(\mathbf{y} - \tfrac{1}{2}\mathbf{x})\, U(\mathbf{y} + \tfrac{1}{2}\mathbf{x}) \right\rangle \quad . \tag{4}$$

The first argument of $c^{\mathrm{G}}(\mathbf{y}, \mathbf{x})$ is the midpoint of the pair of points $\mathbf{x}_1 = \mathbf{y} - \frac{1}{2}\mathbf{x}$ and $\mathbf{x}_2 = \mathbf{y} + \frac{1}{2}\mathbf{x}$ in which the correlation of difference U is studied, whereas the second argument is the distance of the points. The generalized correlation function should vary very moderately with respect to the first argument, whereas it may vary very rapidly with respect to the second argument (distance of points), especially in the vicinity of $\mathbf{x} = \mathbf{0}$.

The Fourier transform of the generalized correlation function with respect to the second argument may be introduced, for instance, in the form

$$\widehat{c^{\mathrm{G}}}(\mathbf{y}, \mathbf{k}) = (2\pi)^{-d} \int_{-\infty}^{+\infty} \mathrm{d}\mathbf{x}\ \exp(-\mathrm{i}\mathbf{k}^{\mathrm{T}}\mathbf{x})\, c^{\mathrm{G}}(\mathbf{y}, \mathbf{x}) \quad , \tag{5}$$

where d is the Euclidean dimension of the space, e.g., $d = 3$ in 3-D. Here we have denoted

$$\int_{-\infty}^{+\infty} \mathrm{d}\mathbf{k} = \int_{-\infty}^{+\infty} \mathrm{d}k_1 \int_{-\infty}^{+\infty} \mathrm{d}k_2 \ldots \int_{-\infty}^{+\infty} \mathrm{d}k_d \quad . \tag{6}$$

The inverse Fourier transform of (5) is

$$c^{\mathrm{G}}(\mathbf{y}, \mathbf{x}) = \int_{-\infty}^{+\infty} \mathrm{d}\mathbf{k}\ \exp(\mathrm{i}\mathbf{k}^{\mathrm{T}}\mathbf{x})\, \widehat{c^{\mathrm{G}}}(\mathbf{y}, \mathbf{k}) \quad . \tag{7}$$

Relation to the Wigner distribution

The *Wigner distribution* of function $\varphi(\mathbf{y})$ is

$$\varphi^{\mathrm{W}}(\mathbf{y},\mathbf{k}) = (2\pi)^{-d} \int_{-\infty}^{+\infty} \mathrm{d}\mathbf{x}\ \exp(-\mathrm{i}\mathbf{k}^{\mathrm{T}}\mathbf{x})\ \varphi^{*}(\mathbf{y}-\tfrac{1}{2}\mathbf{x})\ \varphi(\mathbf{y}+\tfrac{1}{2}\mathbf{x}) \quad . \tag{8}$$

It may also be expressed in terms of the Fourier transform $\widehat{\varphi}(\mathbf{p})$ of $\varphi(\mathbf{y})$,

$$\varphi^{\mathrm{W}}(\mathbf{y},\mathbf{k}) = \int_{-\infty}^{+\infty} \mathrm{d}\mathbf{p}\ \exp(-\mathrm{i}\mathbf{p}^{\mathrm{T}}\mathbf{y})\ \widehat{\varphi}^{*}(-\mathbf{k}-\tfrac{1}{2}\mathbf{p})\ \widehat{\varphi}(-\mathbf{k}+\tfrac{1}{2}\mathbf{p}) \quad , \tag{9}$$

see, e.g., Bastiaans (1981) or Ryzhik, Papanicolaou & Keller (1996).

The Fourier transform (5) of the generalized correlation function is thus the mean value of the Wigner distribution of the unknown difference between the material parameter and its mean value,

$$\widehat{c^{\mathrm{G}}}(\mathbf{y},\mathbf{k}) = \left\langle U^{\mathrm{W}}(\mathbf{y},\mathbf{k}) \right\rangle \quad , \tag{10}$$

and the generalized correlation function is its inverse Fourier transform,

$$c^{\mathrm{G}}(\mathbf{y},\mathbf{x}) = \int_{-\infty}^{+\infty} \mathrm{d}\mathbf{k}\ \exp(\mathrm{i}\mathbf{k}^{\mathrm{T}}\mathbf{x})\ \left\langle U^{\mathrm{W}}(\mathbf{y},\mathbf{k}) \right\rangle \quad . \tag{11}$$

2.4 Correlation function of the stationary random medium

Random medium $u(\mathbf{x})$ is *stationary* if mean value $u_0(\mathbf{x})$ is constant and generalized correlation function $c^{\mathrm{G}}(\mathbf{y},\mathbf{x})$ is independent of the first argument, depending only on distance $\mathbf{x} = \mathbf{x}_1 - \mathbf{x}_2$,

$$c^{\mathrm{G}}(\mathbf{y},\mathbf{x}) = c(\mathbf{x}) \quad , \tag{12}$$

where $c(\mathbf{x})$ is the *correlation function* of the stationary random medium (Tarantola 1987). Note that we do not assume mean value $u_0(\mathbf{x})$ to be constant in this paper.

2.5 Synthetic realizations of a stationary random medium

(a) On a regular rectangular grid, we generate a pseudorandom realization $W(\mathbf{x})$ of the *white noise* of the desired statistical distribution of the functional values, with the *unit standard deviation*.

(b) Calculate the d-dimensional Fourier transform $\widehat{W}(\mathbf{k})$ of the white noise.

(c) Multiply the Fourier transform with the *spectral filter* $\widehat{F}(\mathbf{k})$,

$$\widehat{U}(\mathbf{k}) = \widehat{F}(\mathbf{k})\ \widehat{W}(\mathbf{k}) \quad . \tag{13}$$

(d) Calculate the d-dimensional inverse Fourier transform $U(\mathbf{x})$ of product $\widehat{U}(\mathbf{k})$. The multiplication factors of the forward and inverse Fourier transform may, of course, be different from equations (5) and (7).

(e) Add the appropriate mean value $u_0(\mathbf{x})$ to arrive at the desired pseudorandom realization,

$$u(\mathbf{x}) = u_0(\mathbf{x}) + U(\mathbf{x}) \quad . \tag{14}$$

The Fourier transform of the correlation function is then

$$\widehat{c}(\mathbf{k}) = (2\pi)^{-d}\ \widehat{F}^{*}(\mathbf{k})\ \widehat{F}(\mathbf{k}) \quad , \tag{15}$$

where $\widehat{F}^*(\mathbf{k})\widehat{F}(\mathbf{k})$ is the power spectrum of the filter. The corresponding correlation function is

$$\boxed{c(\mathbf{x}) = (2\pi)^{-d} \int\limits_{-\infty}^{+\infty} \mathrm{d}\mathbf{k} \ \exp(\mathrm{i}\mathbf{k}^{\mathrm{T}}\mathbf{x}) \ \widehat{F}^*(\mathbf{k}) \ \widehat{F}(\mathbf{k}) \quad . \qquad (16)}$$

3 Isotropic and elliptically anisotropic correlation functions

3.1 Isotropic correlation functions

The realizations $U(\mathbf{x})$ of a random medium with the *isotropic correlation function* may be obtained by multiplying the Fourier transform of the realizations of a white noise of unit standard deviation by *isotropic spectral filter*

$$\widehat{F}(\mathbf{k}) = \widehat{f}(k) \qquad (17)$$

with

$$k = (\mathbf{k}^{\mathrm{T}}\mathbf{k})^{\frac{1}{2}} \quad , \qquad (18)$$

and inversely Fourier transforming the products back into the space domain. Remember that $U(\mathbf{x})$ denotes just the random component taken with respect to the mean value, i.e. that $U(\mathbf{x})$ has zero mean value.

The power spectrum of the filter is then the function of k only,

$$P(k) = \widehat{f}^*(k)\widehat{f}(k) \quad . \qquad (19)$$

Assuming that a white noise has the unit standard deviation, the corresponding *isotropic correlation function* is then, see (16),

$$c(\mathbf{x}) = (2\pi)^{-d} \int\limits_{-\infty}^{+\infty} \mathrm{d}k_1 \cos(k_1 x_1) ... \int\limits_{-\infty}^{+\infty} \mathrm{d}k_d \cos(k_d x_d) \ P(k) \quad , \qquad (20)$$

where d is the Euclidean dimension of the space.

We may rotate, before integrating, the k_1 axis into the direction of vector $\mathbf{x}$ to arrive at

$$c(\mathbf{x}) = (2\pi)^{-d} \int\limits_{-\infty}^{+\infty} \mathrm{d}k_1 \cos(k_1 x) \int\limits_{-\infty}^{+\infty} \mathrm{d}k_2 ... \int\limits_{-\infty}^{+\infty} \mathrm{d}k_d \quad P(k) \quad , \qquad (21)$$

where

$$x = (\mathbf{x}^{\mathrm{T}}\mathbf{x})^{\frac{1}{2}} \quad . \qquad (22)$$

For $d > 1$, we introduce the distance r from the k_1 axis,

$$r = [(k_2)^2 + ... + (k_d)^2]^{\frac{1}{2}} \quad , \qquad (23)$$

and recall the equation

$$V_d(r) = \frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2}+1)} \quad r^d \qquad (24)$$

for the volume of the d–dimensional sphere of radius r. Differentiating the volume with respect to the radius, we obtain the surface of the sphere,

$$S_d(r) = \frac{d\ \pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2}+1)}\ r^{d-1} = \frac{2\ \pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}\ r^{d-1} \quad . \tag{25}$$

Integrating (21) over the surface of the $(d-1)$–dimensional sphere of radius r in subspace $k_1 = constant$, and taking into account that the integrands are constant along the surface, we arrive at

$$c(\mathbf{x}) = (2\pi)^{-d}\ \frac{4\ \pi^{\frac{d-1}{2}}}{\Gamma(\frac{d-1}{2})} \int_0^{+\infty} \mathrm{d}k_1 \cos(k_1 x) \int_0^{+\infty} \mathrm{d}r\ r^{d-2}\ P\Big(\sqrt{(r)^2+(k_1)^2}\Big) \quad . \tag{26}$$

We now switch from the integration over k_1 and r to the integration over k_1 and $k = \sqrt{(r)^2+(k_1)^2}$,

$$c(\mathbf{x}) = (2\pi)^{-d}\ \frac{4\ \pi^{\frac{d-1}{2}}}{\Gamma(\frac{d-1}{2})} \int_0^{+\infty} \mathrm{d}k\ k\ P(k) \int_0^{k} \mathrm{d}k_1 \cos(k_1 x) \left[k^2-(k_1)^2\right]^{\frac{d-3}{2}} \quad . \tag{27}$$

For $d > 1$, the integral with respect to k_1 may be calculated,

$$\int_0^{k} \mathrm{d}k_1 \cos(k_1 x) \left[k^2-(k_1)^2\right]^{\frac{d-3}{2}} = 2^{\frac{d-4}{2}}\ \pi^{\frac{1}{2}}\ \Gamma(\tfrac{d-1}{2})\ k^{\frac{d-2}{2}}\ x^{\frac{2-d}{2}}\ \mathrm{J}_{\frac{d-2}{2}}(kx) \quad , \tag{28}$$

where $\mathrm{J}_\nu(x)$ is the Bessel function and $\Gamma(x)$ is the Gamma function. Inserting (28) into (27), the correlation function may be expressed in terms of the Hankel transform of $k^{\frac{d-1}{2}} P(k) = k^{\frac{d-1}{2}} \widehat{f}^*(k)\widehat{f}(k)$,

$$\boxed{c(\mathbf{x}) = (2\pi)^{-\frac{d}{2}}\ x^{\frac{1-d}{2}} \int_0^{+\infty} \mathrm{d}k\ \sqrt{kx}\ \ \mathrm{J}_{\frac{d-2}{2}}(kx)\ k^{\frac{d-1}{2}}\ \widehat{f}^*(k)\widehat{f}(k) \quad . \qquad (29)}$$

Taking into account that

$$\mathrm{J}_{-\frac{1}{2}}(\xi) = \sqrt{\frac{2}{\pi}}\ \frac{\cos(\xi)}{\sqrt{\xi}} \quad , \qquad \mathrm{J}_{\frac{1}{2}}(\xi) = \sqrt{\frac{2}{\pi}}\ \frac{\sin(\xi)}{\sqrt{\xi}} \quad , \tag{30}$$

equation (29) may be used for $d \geq 1$, see (21) for $d = 1$. On the other hand, relations (30) enable to express Hankel transform (29) in the form of Fourier transform in 1-D,

$$c(\mathbf{x}) = \pi^{-1} \int_0^{+\infty} \mathrm{d}k \cos(kx)\ \widehat{f}^*(k)\widehat{f}(k) \quad , \tag{31}$$

and 3-D,

$$c(\mathbf{x}) = \frac{1}{2\pi^2\ x} \int_0^{+\infty} \mathrm{d}k \sin(kx)\ k\ \widehat{f}^*(k)\widehat{f}(k) \quad . \tag{32}$$

3.2 Elliptically anisotropic correlation functions

Equation (29) remains valid if we generalize norms (18) and (22) to

$$k = (\mathbf{k}^{\mathrm{T}}\mathbf{L}\mathbf{k})^{\frac{1}{2}} \quad , \qquad x = (\mathbf{x}^{\mathrm{T}}\mathbf{L}^{-1}\mathbf{x})^{\frac{1}{2}} \quad , \tag{33}$$

where the positive–definite symmetric scaling matrix $\mathbf{L}$ may account for the elliptical anisotropy of the correlation functions (Goff & Jordan 1988).

4 Examples of isotropic and elliptically anisotropic correlation functions

Most of the isotropic or elliptically anisotropic spectral filters (17) recently considered in the geophysical literature are special cases of filter

$$\boxed{\widehat{f}(k) = \kappa \; \left[a^{-2} + k^2\right]^{-\frac{d}{4}-\frac{N}{2}} \; \exp\left(-\frac{a_{\mathrm{G}}^2 k^2}{8}\right) \quad ,} \tag{34}$$

described by four constants κ, N, a_{G} and a. The individual special filters may be fitted by the relevant choice of *Hurst exponent* (Hurst parameter, Hurst number) N and two correlation lengths a_{G} (Gaussian correlation length, see Sections 4.1 and 4.4) and a (von Kármán correlation length, see Section 4.2).

Function (34) is the product of spectral filter

$$\widehat{f}_{\mathrm{S}_N}(k) = \kappa \; k^{-\frac{d}{2}-N} \quad , \tag{35}$$

corresponding to the self–affine random medium, with high–pass wavenumber filter

$$\widehat{f}_a(k) = \left[1 + (ak)^{-2}\right]^{-\frac{d}{4}-\frac{N}{2}} \tag{36}$$

and low–pass wavenumber filter

$$\widehat{f}_{a_{\mathrm{G}}}(k) = \exp\left(-\frac{a_{\mathrm{G}}^2 k^2}{8}\right) \quad , \tag{37}$$

see Figure 1.

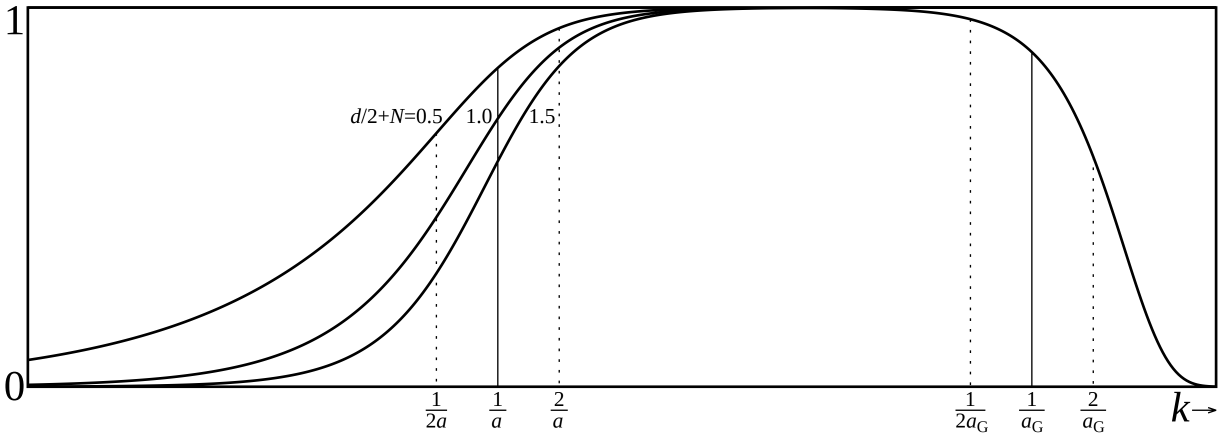

Figure 1. High–pass filter $\widehat{f}_a(k)$ controlled by correlation length a and low–pass Gaussian filter $\widehat{f}_{a_{\mathrm{G}}}(k)$ controlled by correlation length a_{G}. The horizontal wavenumber axis has a logarithmic scale.

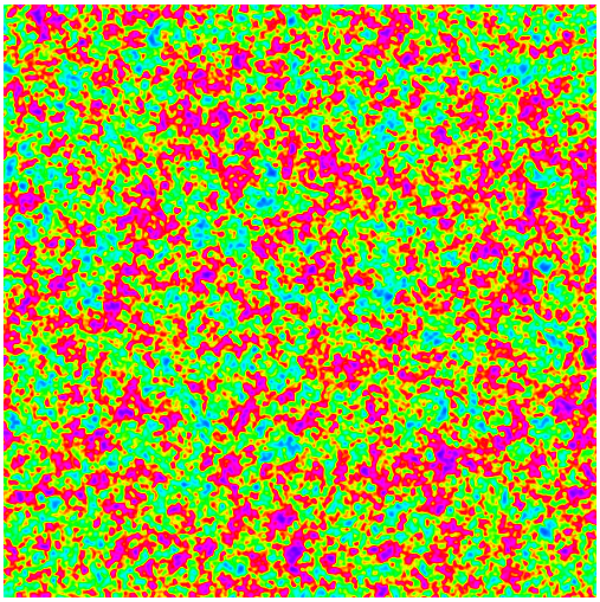

Figure 2. A 2-D representation of the random medium described by Hurst exponent $N = -0.2$ and correlation lengths $a_{\mathrm{G}} = 0.005\ H$ and $a = 0.020\ H$, where H is the dimension of the square sample. Yellow colour corresponds to the mean value. Compare with Figure 7 imagined without both small and large heterogeneities.

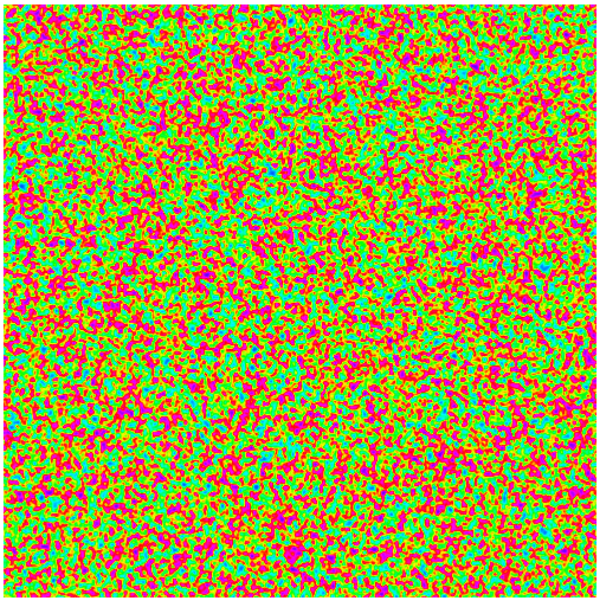

Figure 3. A 2-D representation of the Gaussian random medium ($N = -d/2 = -1$, $a = +\infty$) of correlation length $a_{\mathrm{G}} = 0.005\ H$, where H is the dimension of the square sample. The figure looks like a white noise if observed from a distance of 1.5 metre ($\approx 20H$) or more.

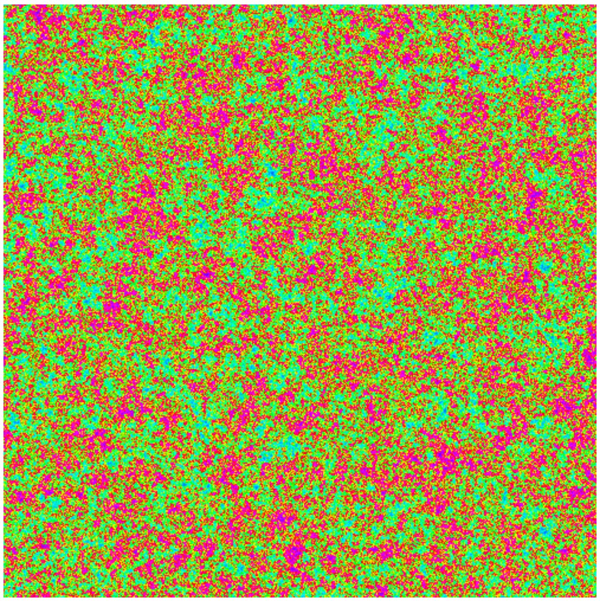

Figure 4. A 2-D representation of the von Kármán random medium ($a_{\mathrm{G}} = 0$) described by Hurst exponent $N = -0.2$ and correlation length $a = 0.020\ H$, where H is the dimension of the square sample. Compare with Figure 7 imagined without large heterogeneities.

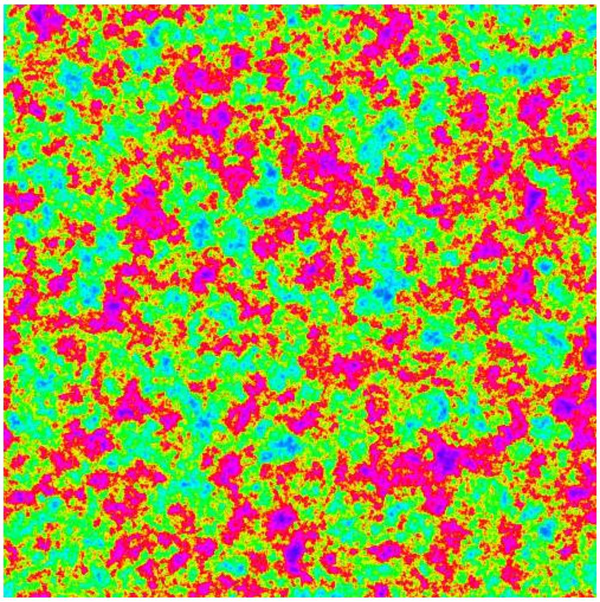

Figure 5. A 2-D representation of the exponential medium ($N = 1/2$, $a_{\mathrm{G}} = 0$) described by correlation length $a = 0.020\ H$, where H is the dimension of the square sample.

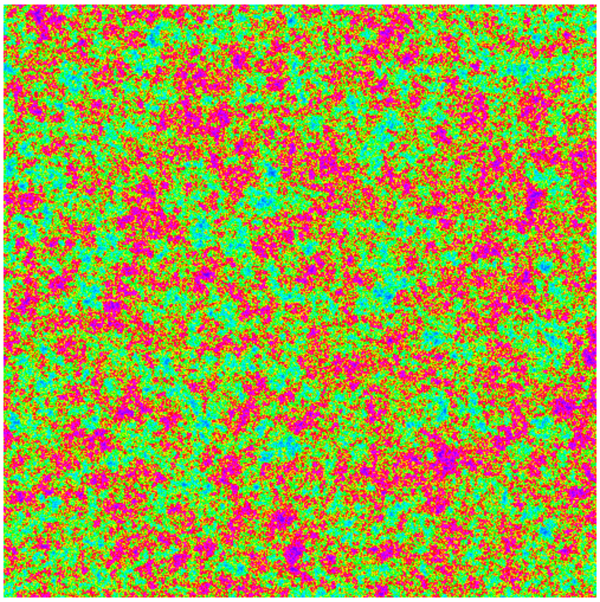

Figure 6. A 2-D representation of the von Kármán random medium ($a_{\mathrm{G}} = 0$) described by zero Hurst exponent $N = 0$ and correlation length $a = 0.020\ H$, where H is the dimension of the square sample.

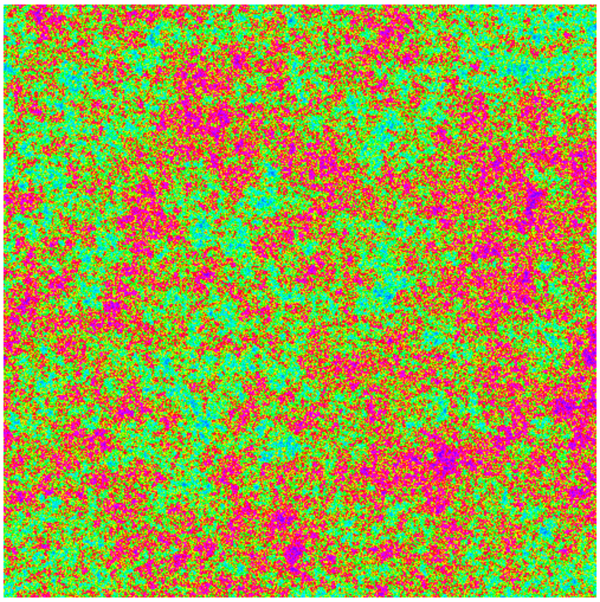

Figure 7. A 2-D representation of the self–affine random medium ($a_{\mathrm{G}} = 0$, $a = +\infty$) described by Hurst exponent $N = -0.2$.

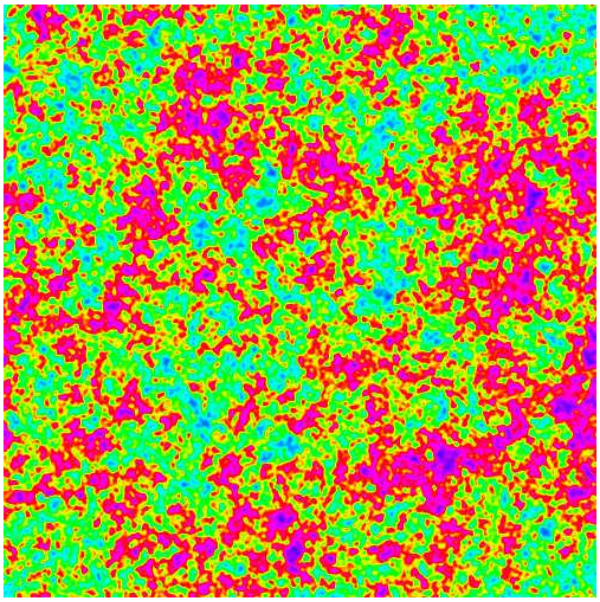

Figure 8. A 2-D representation of the Kummer random medium ($a = +\infty$) described by Hurst exponent $N = -0.2$ and correlation length $a_{\mathrm{G}} = 0.005\ H$, where H is the dimension of the square sample. Compare with Figure 7 imagined without small heterogeneities.

Note that, with regards to the discussion in Section 4.3, the self–affine random medium with Hurst exponents

$$-\tfrac{1}{2} < N < 0 \tag{38}$$

seems to be suitable for the description of the material properties of the geological structures. For the illustration of the random medium given by spectral filter (34) see Figure 2.

For example, one possibility of estimating the mean value $u_0(\mathbf{x})$ and parameters κ and N of the correlation function from travel times in the Western Bohemia region, under the assumption of a self–affine random medium, has been demonstrated by Klimeš (1996, 2002b). The results suggest that the value of Hurst exponent N, chosen in Figures 2, 4, 7 and 8, is acceptable, e.g., for the Western Bohemia region.

The application of the low–pass filter of correlation length a_G removes small–scale heterogeneities. The resulting random medium remains self–affine at distances considerably longer than correlation length a_G and describes the features of the geological structure seen by seismic waves of limited bandwidth rather than the entire geological structure.

The application of the high–pass filter of correlation length a suppresses large–scale heterogeneities. The resulting random medium remains self–affine at distances considerably shorter than correlation length a, and corresponds to the difference between the material parameter and its mean value rather than to the geological structure itself, provided that the mean value fits the heterogeneities considerably larger than correlation length a.

If both the low–pass and high–pass filters are applied, the random medium is self–affine at distances x between the correlation lengths,

$$a_\mathrm{G} \ll x \ll a \quad . \tag{39}$$

Correlation lengths a_G and a thus serve as the *inner and outer cutoff scales* of the self–affine random medium (Mandelbrot 1977).

4.1 Gaussian correlation function ($N = -\frac{d}{2}$)

Choosing Hurst exponent $N = -\frac{d}{2}$, we obtain a white noise filtered by low–pass filter (37). Spectral filter (34) then reads

$$\widehat{f}_\mathrm{G}(k) = \kappa \ \exp\left(-\frac{a_\mathrm{G}^2 k^2}{8}\right) \quad . \tag{40}$$

For $0 < d$, the Hankel transform of $k^{\frac{d-1}{2}} \exp\left(-\frac{a_\mathrm{G}^2 k^2}{4}\right)$ is

$$\int_0^{+\infty} \mathrm{d}k \ \sqrt{kx} \ \mathrm{J}_{\frac{d-2}{2}}(kx) \ k^{\frac{d-1}{2}} \exp\left(-\frac{a_\mathrm{G}^2 k^2}{4}\right) = 2^{\frac{d}{2}} \ a_\mathrm{G}^{-d} \ x^{\frac{d-1}{2}} \exp\left(-a_\mathrm{G}^{-2} x^2\right) \quad . \tag{41}$$

Inserting (40) and (41) into (29), we arrive at the Gaussian correlation function

$$c_\mathrm{G}(\mathbf{x}) = \kappa^2 \ \pi^{-\frac{d}{2}} \ a_\mathrm{G}^{-d} \ \exp\left(-a_\mathrm{G}^{-2} x^2\right) \quad , \tag{42}$$

or simply

$$c_\mathrm{G}(\mathbf{x}) = \sigma_\mathrm{G}^2 \ \exp\left(-a_\mathrm{G}^{-2} x^2\right) \tag{43}$$

if we put

$$\kappa = \sigma_{\mathrm{G}} \ \pi^{\frac{d}{4}} \ a_{\mathrm{G}}^{\frac{d}{2}} \quad . \tag{44}$$

The Gaussian correlation function describes, in principle, a low–pass filtered white noise, is too smooth to represent actual inhomogeneities in the earth (Wu & Aki 1985; Sato & Fehler 1998), and cannot explain observations of both seismic wave scattering and travel–time variations (Frankel & Clayton 1986). White noise is approached for $a_{\mathrm{G}} \rightarrow 0+$. The author thus does not see any relation of the Gaussian correlation function to the properties of geological structures.

For the illustration of the random medium with the Gaussian correlation function see Figure 3.

4.2 Von Kármán correlation functions ($a_{\mathrm{G}} = 0$)

For $a_{\mathrm{G}} = 0$, filter (34) simplifies to

$$\widehat{f}_{\mathrm{K}_N}(k) = \kappa \ \left[a^{-2} + k^2\right]^{-\frac{d}{4}-\frac{N}{2}} \quad . \tag{45}$$

For $0 \leq d$ and $-\frac{d+1}{2} < 2N$, the Hankel transform of $k^{\frac{d-1}{2}} \left[a^{-2} + k^2\right]^{-\frac{d}{2}-N}$ is

$$\int\limits_0^{+\infty} \mathrm{d}k \ \sqrt{kx} \ \mathrm{J}_{\frac{d-2}{2}}(kx) \ k^{\frac{d-1}{2}} \left[a^{-2} + k^2\right]^{-\frac{d}{2}-N}$$
$$= 2^{\frac{2-d}{2}-N} \ \left[\Gamma(\tfrac{d}{2} + N)\right]^{-1} \ a^N \ x^{\frac{d-1}{2}+N} \ \mathrm{K}_{-N}\left(\tfrac{x}{a}\right) \quad . \tag{46}$$

Inserting (45) and (46) into (29), and using relation

$$\mathrm{K}_{-N}(\xi) = \mathrm{K}_N(\xi) \tag{47}$$

for MacDonald functions K_N, we arrive at the von Kármán correlation function

$$c_{\mathrm{K}_N}(\mathbf{x}) = \kappa^2 \ 2^{1-d-N} \ \pi^{-\frac{d}{2}} \ \left[\Gamma(\tfrac{d}{2} + N)\right]^{-1} \ \left[ax\right]^N \ \mathrm{K}_N\left(a^{-1}x\right) \quad . \tag{48}$$

Choosing, for Hurst exponent $N \neq 0$,

$$\kappa^2 = \sigma_{\mathrm{K}}^2 \ 2^d \ \pi^{\frac{d}{2}} \ \left[\Gamma(|N|)\right]^{-1} \ \Gamma(\tfrac{d}{2} + N) \ a^{-2N} \quad , \tag{49}$$

we obtain the von Kármán correlation function in the form

$$c_{\mathrm{K}_N}(\mathbf{x}) = \sigma_{\mathrm{K}}^2 \ \frac{2}{\Gamma(|N|)} \ \left[\frac{x}{2a}\right]^N \ \mathrm{K}_N\left(\frac{x}{a}\right) \quad . \tag{50}$$

Right–hand side of equation (50) is σ_{K}^2 times the von Kármán function of argument $a^{-1}x$, see Frankel & Clayton (1986), Wu (1982, 1989) or Korn (1993). The von Kármán correlation functions are frequently used in the geophysical literature, but unfortunately, ignoring the negative Hurst exponents and often using inappropriate correlation lengths. For the illustration of the von Kármán random medium see Figure 4.

The asymptotic behaviour of the MacDonald functions K_N for very large arguments is

$$\mathrm{K}_N(\xi) \underset{\xi\rightarrow+\infty}{\approx} \sqrt{\frac{\pi}{2}} \ \frac{\exp(-\xi)}{\sqrt{\xi}} \tag{51}$$

and, consequently, the asymptotic behaviour of the von Kármán correlation functions (50) for very large arguments is

$$c_{\mathrm{K}_N}(\mathbf{x}) \underset{x\to+\infty}{\approx} \sigma_{\mathrm{K}}^2 \sqrt{\pi} \; [\Gamma(|N|)]^{-1} \left[\frac{x}{2a}\right]^{N-\frac{1}{2}} \exp\left(-\frac{x}{a}\right) \quad . \tag{52}$$

Note that relations (51) and (52) are exact for $N = \pm\frac{1}{2}$. The asymptotic behaviour of the MacDonald functions K_N for very small arguments is

$$\mathrm{K}_N(\xi) \underset{\xi\to 0+}{\approx} \tfrac{1}{2}\Gamma(|N|) \left[\frac{\xi}{2}\right]^{-|N|} \tag{53}$$

and, consequently, the asymptotic behaviour of the von Kármán correlation functions (50) for very small arguments is

$$\begin{aligned} N>0: &\quad c_{\mathrm{K}_N}(\mathbf{x}) \underset{x\to 0+}{\approx} \sigma_{\mathrm{K}}^2 \quad , \\ N<0: &\quad c_{\mathrm{K}_N}(\mathbf{x}) \underset{x\to 0+}{\approx} \sigma_{\mathrm{K}}^2 \left[\frac{x}{2a}\right]^{2N} \quad . \end{aligned} \tag{54}$$

The von Kármán medium is thus self–affine at distances considerably shorter than correlation length a.

In seismic practice, the von Kármán correlation functions with negative Hurst exponent $N<0$ may be appropriate for the description of the self–affine random medium in which the large heterogeneities (larger than the correlation length) are fitted well enough by mean value $u_0(\mathbf{x})$. In such a case, the von Kármán correlation functions describe small heterogeneities, unresolved by mean value $u_0(\mathbf{x})$, or by the seismic model provided that stands $u_0(\mathbf{x})$ for the seismic model.

4.2.1 Exponential correlation function ($N=\frac{1}{2}$, $a_{\mathrm{G}}=0$)

For the special case of Hurst exponent $N=\frac{1}{2}$, equation (45) takes the form

$$\widehat{f}_{\mathrm{E}}(k) = \widehat{f}_{\mathrm{K}_{\frac{1}{2}}}(k) = \kappa \left[a^{-2}+k^2\right]^{-\frac{d+1}{4}} \quad . \tag{55}$$

For $0<d$, the Hankel transform of $k^{\frac{d-1}{2}} \left[a^{-2}+k^2\right]^{-\frac{d+1}{2}}$ is

$$\int_0^{+\infty} \mathrm{d}k \, \sqrt{kx} \, \mathrm{J}_{\frac{d-2}{2}}(kx) \, k^{\frac{d-1}{2}} \left[a^{-2}+k^2\right]^{-\frac{d+1}{2}} = 2^{-\frac{d}{2}} \, \pi^{\frac{1}{2}} \left[\Gamma(\tfrac{d+1}{2})\right]^{-1} a \, x^{\frac{d-1}{2}} \exp(-a^{-1}x) \quad . \tag{56}$$

Inserting (55) and (56) into (29), we arrive at

$$c_{\mathrm{E}}(\mathbf{x}) = c_{\mathrm{K}_{\frac{1}{2}}}(\mathbf{x}) = \kappa^2 \, 2^{-d} \, \pi^{\frac{1-d}{2}} \left[\Gamma(\tfrac{d+1}{2})\right]^{-1} a \, \exp(-a^{-1}x) \quad . \tag{57}$$

Choosing

$$\kappa^2 = \sigma_{\mathrm{K}}^2 \, 2^d \, \pi^{\frac{d-1}{2}} \, \Gamma(\tfrac{d+1}{2}) \, a^{-1} \quad , \tag{58}$$

we obtain the exponential correlation function in the form

$$c_{\mathrm{E}}(\mathbf{x}) = c_{\mathrm{K}_{\frac{1}{2}}}(\mathbf{x}) = \sigma_{\mathrm{K}}^2 \ \exp(-a^{-1}x) \quad . \tag{59}$$

Equations (58) and (59) are special cases of (48) and (50) respectively, with

$$\mathrm{K}_{\frac{1}{2}}(\xi) = \sqrt{\tfrac{\pi}{2}}\ \xi^{-\frac{1}{2}} \ \exp(-\xi) \quad . \tag{60}$$

The exponential random medium is illustrated by Figure 5. It cannot explain observations of both seismic wave scattering and travel–time variations (Frankel & Clayton 1986), because both small–scale and large–scale heterogeneities are considerably suppressed. If the Hurst exponent is decreased from $N = \frac{1}{2}$ to $N = 0$ and the von Kármán correlation length a is increased towards infinity, the fit of the arrival–time and amplitude fluctuations at the NORSAR is getting better (Flatté & Wu 1988).

4.2.2 Zero von Kármán correlation function ($N = 0$, $a_{\mathrm{G}} = 0$)

The right–hand side of (49) goes to infinity for Hurst exponent $N \to 0$. This may be removed by rescaling (49) by factor $\frac{1}{2}\Gamma(|N|)$,

$$\kappa^2 = \sigma_{\mathrm{K}_0}^2 \ 2^{d-1} \ \pi^{\frac{d}{2}} \ \Gamma(\tfrac{d}{2}+N) \ a^{-2N} \quad . \tag{61}$$

The rescaled correlation function (50) then takes the form

$$c_{\mathrm{K}_N}(\mathbf{x}) = \sigma_{\mathrm{K}_0}^2 \ \left[\frac{x}{2a}\right]^N \ \mathrm{K}_N\left(\frac{x}{a}\right) \quad . \tag{62}$$

For the special case of zero Hurst exponent, $N = 0$,

$$c_{\mathrm{K}_0}(\mathbf{x}) = \sigma_{\mathrm{K}_0}^2 \ \mathrm{K}_0\left(a^{-1}x\right) \quad , \tag{63}$$

see Frankel & Clayton (1986). The random medium with the zero von Kármán correlation function is illustrated in Figure 6.

4.3 Self–affine random medium ($a_{\mathrm{G}} = 0$, $a = +\infty$)

The von Kármán medium is not self–affine over all scales, it is asymptotically self–affine at small distances $x \ll a$. The self–affine random medium may thus be obtained as the limiting case of the von Kármán medium for $a \to +\infty$,

$$\widehat{f}_{\mathrm{S}_N}(k) = \kappa \ k^{-\frac{d}{2}-N} \quad , \tag{64}$$

see (45). For $-\frac{d}{2} < 2N < 0$, the Hankel transform of $k^{\frac{d-1}{2}} k^{-d-2N}$ is

$$\int_0^{+\infty} \mathrm{d}k \ \sqrt{kx} \ \mathrm{J}_{\frac{d-2}{2}}(kx) \ k^{-\frac{d+1}{2}-2N} = 2^{-\frac{d}{2}-2N} \ \frac{\Gamma(|N|)}{\Gamma(\frac{d}{2}+N)} \ x^{\frac{d-1}{2}+2N} \quad . \tag{65}$$

Inserting (64) and (65) into (29), we arrive at the power–law correlation function

$$c_{\mathrm{S}_N}(\mathbf{x}) = \kappa^2 \ 2^{-d-2N} \ \pi^{-\frac{d}{2}} \ \frac{\Gamma(|N|)}{\Gamma(\frac{d}{2}+N)} \ x^{2N} \tag{66}$$

of the self–affine random medium. Choosing

$$\kappa^2 = \sigma_{\mathrm{S}}^2 \; 2^d \; \pi^{\frac{d}{2}} \; [\Gamma(|N|)]^{-1} \; \Gamma(\tfrac{d}{2}+N) \; \left[\tfrac{2}{L}\right]^{2N} \quad , \tag{67}$$

we obtain the power–law correlation function in the form of

$$c_{\mathrm{S}_N}(\mathbf{x}) = \sigma_{\mathrm{S}}^2 \; \left[\frac{x}{L}\right]^{2N} \quad . \tag{68}$$

Here L is some reference distance supplemented to preserve the correct physical units in the expressions. Multiplication factor $\left(\frac{2}{L}\right)^{2N}$ on the right–hand side of (67) substitutes here multiplication factor a^{-2N} which would approach zero on the right–hand side of (49) for $a \to +\infty$, see (54). Reference distance L is unnecessary, $L = 1$, if equations (33) are used to define x and k, because scaling matrix $\mathbf{L}$ can play the role of L^2.

The self–affine random medium seems to be the most appropriate description of the material properties of the geological structures, because the structures contain heterogeneities of all scales. Indeed it has been found that power spectra of a variety of geophysical variables show the power–law behaviour over considerably large ranges of wavenumbers (Pilkington & Todoeschuck 1990). Since correlation function (68) should decrease with distance x (Chernov 1960), the Hurst exponent should be negative,

$$N < 0 \quad , \tag{69}$$

in the self–affine random medium. Further limitations of the acceptable values of the Hurst exponent are imposed by the influence of heterogeneities of various scales on the properties of seismic waves.

In the inversion of travel–times, correlation functions are required for the calculation of the geometrical travel–time covariances. The line integrals of correlation function (68) across $\mathbf{x} = \mathbf{0}$, representing travel–time variances along infinitely thin rays, are finite for

$$-\tfrac{1}{2} < N \tag{70}$$

(Klimeš 1996, 2002b). Although "thick" rays, corresponding to finite frequencies, allow also of $-\frac{d}{2} < N \le -\frac{1}{2}$, the travel–time variances would increase considerably with frequency for these values of Hurst exponent N. The author does not consider such a strong velocity dispersion to be realistic.

On smaller scales, correlation functions enable to approximately upscale a rough distribution of material parameters and estimate the relation to a smoother distribution of effective material parameters. Müller & Shapiro (2001) derived the expressions for the mean travel–time delay and amplitude decay due to random perturbations of the squared slowness in the weak fluctuation regime. Assuming the self–affine random medium, the integrals in their expressions for the mean logarithmic acoustic wavefield are finite for $-\frac{1}{2} < N < \frac{1}{2}$ in 2-D and for $-\frac{1}{2} < N < 0$ in 3-D. Restriction (70) on the Hurst exponent is thus useful also on small scales.

Flatté & Wu (1988) demonstrated that decreasing the Hurst exponent from $N = \frac{1}{2}$ to $N = 0$ improves the fit of the arrival–time and amplitude fluctuations at the NORSAR. They did not continue with the negative Hurst exponents, but found the combination of two layers with the Hurst exponents of $N = -\frac{3}{2}$ and $N = \frac{1}{2}$ yielding better fit than $N = 0$. Note that their combination of the two correlation functions can reasonably be fit within the relevant inner and outer cutoff scales by a single correlation function with the negative Hurst exponent $-\frac{1}{2} < N < 0$. Let us also remark that the experimental values of the correlation function calculated from the velocity log in the granite at Fenton Hill at scales from 2.4 m to 24 m (Wu 1982) can be fit with the relative r.m.s. difference around 8% by the power–law correlation function with Hurst exponent $N = -0.3$.

Hurst exponent $N = -\frac{d}{2}$ yields a white noise. Negative Hurst exponents $-\frac{1}{2} < N < 0$ are suitable to characterize the material parameters of the geological structures. The limiting case of $N = 0$ is called a *flicker noise* (Schottky 1926). *Fractional Brown noises* obtained for positive Hurst exponent $0 < N < 1$ (Addison 1997) may be divided into *antipersistent fractional Brown noises* with $0 < N < \frac{1}{2}$ (Mandelbrot 1977), *Brown noise* with $N = \frac{1}{2}$ (e.g., Mandelbrot 1977; Turcotte 1989; Addison 1997), and *persistent fractional Brown noises* with $\frac{1}{2} < N < 1$ (Mandelbrot 1977), called also *black noises* (Addison 1997). In spite of that, in 2-D space, $d = 2$, Crossley & Jensen (1989) refer to the case of $N = -\frac{1}{2}$ as flicker noise and to the case of $N = 0$ as brown noise.

The self–affine random medium, with Hurst exponent N acceptable for the Western Bohemia region (Klimeš 2002b), is illustrated in Figure 7.

4.4 Kummer correlation functions ($a = +\infty$)

On the other hand, the effective material parameters may be appropriately described in terms of the self–affine random medium, filtered by the low–pass Gaussian smoothing function,

$$\widehat{f}_{\mathrm{F}N}(k) = \kappa\ k^{-\frac{d}{2}-N}\ \exp\left(-\frac{a_{\mathrm{G}}^2 k^2}{8}\right) \quad , \tag{71}$$

see (40) and (64). The Hankel transform of $k^{\frac{d-1}{2}} k^{-d-2N} \exp(-\frac{1}{4} a_{\mathrm{G}}^2 k^2)$ is

$$\int_0^{+\infty} \mathrm{d}k\ \sqrt{kx}\ \mathrm{J}_{\frac{d-2}{2}}(kx)\ k^{-\frac{d+1}{2}-2N} \exp\left(-\frac{a_{\mathrm{G}}^2 k^2}{4}\right) = \frac{\Gamma(|N|)}{\Gamma(\frac{d}{2})} \frac{a_{\mathrm{G}}^{2N}}{2^{\frac{d}{2}+2N}} x^{\frac{d-1}{2}}\ {}_1F_1(-N; \tfrac{d}{2}; -x^2 a_{\mathrm{G}}^{-2}) \, , \tag{72}$$

where ${}_1F_1(\alpha;\beta;\xi)$ is the *confluent hypergeometric function*, called also the *Kummer function*. Inserting (71) and (72) into (29), we arrive at correlation function

$$c_{\mathrm{F}N}(\mathbf{x}) = \kappa^2\ 2^{-d-2N}\ \pi^{-\frac{d}{2}}\ \frac{\Gamma(|N|)}{\Gamma(\frac{d}{2})}\ a_{\mathrm{G}}^{2N}\ {}_1F_1(-N; \tfrac{d}{2}; -x^2 a_{\mathrm{G}}^{-2}) \quad , \tag{73}$$

which we shall call the *Kummer correlation function*. Choosing

$$\kappa^2 = \sigma_{\mathrm{F}}^2\ 2^{d+2N}\ \pi^{\frac{d}{2}}\ [\Gamma(|N|)]^{-1}\ \Gamma(\tfrac{d}{2})\ a_{\mathrm{G}}^{-2N} \quad , \tag{74}$$

we obtain the Kummer correlation function in the form of

$$c_{\mathrm{F}N}(\mathbf{x}) = \sigma_{\mathrm{F}}^2\ {}_1F_1(-N; \tfrac{d}{2}; -x^2 a_{\mathrm{G}}^{-2}) \quad . \tag{75}$$

The Kummer medium is self–affine at distances considerably longer than correlation length a_{G}, with

$$\sigma_{\mathrm{F}}^2 = \frac{\Gamma(\frac{d}{2}+N)}{\Gamma(\frac{d}{2})} \left[\frac{a_{\mathrm{G}}}{L}\right]^{2N} \sigma_{\mathrm{S}}^2 \quad , \tag{76}$$

where σ_{S} corresponds to the self–affine random medium, see (67). On the other hand, the Kummer correlation function is smooth at short distances, with $c_{\mathrm{F}N}(\mathbf{0}) = \sigma_{\mathrm{F}}^2$.

The Kummer correlation functions may thus be very useful for the discrete numerical approximation of the self–affine correlation function (68), in order to avoid the aliasing due to the singularity related to infinitely small heterogeneities when using the fast Fourier transform.

For the illustration of the random medium with the Kummer correlation function see Figure 8.

4.5 Low–pass filtered self–affine random medium

Box–window low–pass filtered, self–affine random medium spectral filter (64) takes the form

$$\begin{aligned} k < K: \quad & \widehat{f}_{\mathrm{S}_N]}(k) = \kappa\ k^{-\frac{d}{2}-N} \quad , \\ k > K: \quad & \widehat{f}_{\mathrm{S}_N]}(k) = 0 \quad . \end{aligned} \tag{77}$$

Correlation function (29) then reads

$$c_{\mathrm{S}_N]}(\mathbf{x}) = \kappa^2\ (2\pi)^{-\frac{d}{2}}\ x^{\frac{1-d}{2}} \int\limits_0^K \mathrm{d}k\ \sqrt{kx}\ \ \mathrm{J}_{\frac{d-2}{2}}(kx)\ k^{\frac{-d-1}{2}-2N} \quad . \tag{78}$$

Changing the integration variable, this may be turned into

$$c_{\mathrm{S}_N]}(\mathbf{x}) = \kappa^2\ (2\pi)^{-\frac{d}{2}}\ x^{\frac{1-d}{2}}\ K^{\frac{-d+1}{2}-2N} \int\limits_0^1 \mathrm{d}l\ \sqrt{l\,Kx}\ \ \mathrm{J}_{\frac{d-2}{2}}(l\,Kx)\ l^{\frac{-d-1}{2}-2N} \quad . \tag{79}$$

The low–pass filtered self–affine random medium has been considered, e.g., by Flatté & Wu (1988).

4.5.1 Low–pass filtered self–affine medium: $N = -1$

For the special case of Hurst exponent $N = -1$,

$$\begin{aligned} k < K: \quad & \widehat{f}_{\mathrm{S}_{-1}]}(k) = \kappa\ k^{1-\frac{d}{2}} \quad , \\ k > K: \quad & \widehat{f}_{\mathrm{S}_{-1}]}(k) = 0 \quad , \end{aligned} \tag{80}$$

the Hankel transform of $l^{\frac{-d+2}{2}+\frac{1}{2}}$ in equation (79) may be calculated analytically to yield

$$c_{\mathrm{S}_{-1}]}(\mathbf{x}) = \kappa^2\ (2\pi)^{-\frac{d}{2}}\ x^{\frac{1-d}{2}}\ K^{\frac{-d+1}{2}+2} \left[\frac{2^{\frac{-d+2}{2}+1} (Kx)^{\frac{d-2}{2}-\frac{3}{2}}}{\Gamma\left(\frac{d-2}{2}\right)} - \frac{1}{\sqrt{Kx}}\ \mathrm{J}_{\frac{d}{2}}(Kx) \right] \ , \tag{81}$$

which finally reads

$$c_{\mathrm{S}_{-1}]}(\mathbf{x}) = \kappa^2\ (2\pi)^{-\frac{d}{2}} \left[\frac{2^{2-\frac{d}{2}}\ x^{-2}}{\Gamma\left(\frac{d-2}{2}\right)} - K^{2-\frac{d}{2}}\ x^{-\frac{d}{2}}\ J_{\frac{d}{2}}(Kx) \right] \quad . \tag{82}$$

In three–dimensional space, $d = 3$, this equation reads

$$c_{\mathrm{S}_{-1}]}(\mathbf{x}) = \kappa^2\ (2\pi)^{-\frac{3}{2}} \left[\frac{\sqrt{2}}{\sqrt{\pi}\ x^2} - \frac{\sqrt{2}}{\sqrt{\pi}}\ \frac{\cos(Kx)}{x^2} \right] = \frac{\kappa^2}{2\,\pi^2}\ \frac{1}{x^2}\ [1-\cos(Kx)] \quad . \tag{83}$$

For

$$\kappa = \sigma_{\mathrm{S}]} \; 2\pi \, K^{-1} \quad , \tag{84}$$

see (67), it takes the form

$$c_{\mathrm{S}_{-1}]}(\mathbf{x}) = \sigma_{\mathrm{S}]}^2 \; \frac{2\,[1-\cos(Kx)]}{(Kx)^2} \quad . \tag{85}$$

In two–dimensional space, $d = 2$, the special case discussed here limits to the low–pass filtered white noise of the next section.

4.5.2 Low–pass filtered white noise

For the special case of Hurst exponent $N = -\frac{d}{2}$, equation (77) reads

$$\begin{aligned} k < K: \quad & \widehat{f}_{\mathrm{W}]}(k) = \widehat{f}_{\mathrm{S}_{-\frac{d}{2}}]}(k) = \kappa \quad , \\ k > K: \quad & \widehat{f}_{\mathrm{W}]}(k) = \widehat{f}_{\mathrm{S}_{-\frac{d}{2}}]}(k) = 0 \quad , \end{aligned} \tag{86}$$

and the Hankel transform of $l^{\frac{d-2}{2}+\frac{1}{2}}$ in equation (79) may be calculated analytically to arrive at

$$c_{\mathrm{W}]}(\mathbf{x}) = \kappa^2 \; (2\pi)^{-\frac{d}{2}} \; x^{\frac{1-d}{2}} \; K^{\frac{d+1}{2}} \; \frac{1}{\sqrt{Kx}} \; \mathrm{J}_{\frac{d}{2}}(Kx) \quad , \tag{87}$$

which finally reads

$$c_{\mathrm{W}]}(\mathbf{x}) = \kappa^2 \; (2\pi)^{-\frac{d}{2}} \; \left(\frac{K}{x}\right)^{\frac{d}{2}} \; \mathrm{J}_{\frac{d}{2}}(Kx) \quad . \tag{88}$$

5 Conclusions

Several simple types of medium correlation functions, their mutual relation and physical meaning of their parameters has briefly been discussed. Among them, the self–affine random medium is the most appropriate for description of the material properties of geological structures, because the earth contains heterogeneities on all scales.

The Kummer correlation functions, with the Gaussian correlation length as small as the computing equipment allows, are recommended for numerical realizations of the self–affine random medium if we need to avoid the singularity related to infinitely small heterogeneities.

The von Kármán correlation functions with Hurst exponent $-\frac{1}{2} < N < 0$ may be useful for the description of small–scale random heterogeneities if the large–scale heterogeneities are known, i.e., are not random.

It is difficult to justify the application of the frequently used Gaussian correlation function and von Kármán correlation functions with a positive Hurst exponent (e.g., exponential correlation function).

Acknowledgements

I am indebted to Paul Spudich who provided me with his perfect code I used to calculate the representations of the random media presented here.

This research has been supported by the Grant Agency of the Czech Republic under Contracts 205/95/1465 and 205/01/0927, by the Grant Agency of the Charles University under Contract 237/2001/B-GEO/MFF, by the Ministry of Education of the Czech Republic within Research Project J13/98 113200004, and by the members of the consortium "Seismic Waves in Complex 3-D Structures" (see "http://seis.karlov.mff.cuni.cz /consort/main.htm").

References

Addison, P.S. (1997): Fractals and Chaos: An Illustrated Course. IOP Publishing, London.

Bastiaans, M.J. (1981): Signal description by means of a local frequency spectrum. In: Rhodes, W.T. et al. (eds.): Transformations in Optical Signal Processing. Proc. SPIE, vol. 373, Soc. Photo–Opt. Instrum. Eng., Bellingham, pp. 49–62.

Chernov, L.A. (1960): Wave Propagation in a Random Medium. McGraw-Hill, New York.

Crossley, D.J. & Jensen, O.G. (1989): Fractal velocity models in refraction seismology. Pure appl. Geophys., **131**, 61–76.

Flatté, S.M. & Wu, R-S. (1988): Small–scale structure the litosphere and astenosphere deduced from arrival time amplitude fluctuations. J. geophys. Res., **93B**, 6601–6614.

Frankel, A. & Clayton, R.W. (1986): Finite difference simulations of seismic scattering: Implications for the propagation of short–period seismic waves in the crust and models of crustal heterogeneity. J. geophys. Res., **91B**, 6465–6489.

Goff, J.A. & Jordan, T.H. (1988): Stochastic modeling of seafloor morphology: Inversion of sea beam data for second–order statistics. J. geophys. Res., **93B**, 13589–13608.

Klimeš, L. (1996): Correlation function of a self–affine random medium. In: Seismic Waves in Complex 3-D Structures, Report 4, pp. 25–38, Dep. Geophys. Charles Univ., Prague, online at "http://seis.karlov.mff.cuni.cz/consort/main.htm".

Klimeš, L. (2002a): Application of the medium covariance functions to travel–time tomography. Pure appl. Geophys., **159**, in press.

Klimeš, L. (2002b): Estimating the correlation function of a self–affine random medium. Pure appl. Geophys., **159**, in press.

Korn, M. (1993): Seismic waves in random media. J. appl. Geophys., **29**, 247–269.

Mandelbrot, B.B. (1977): The Fractal Geometry of Nature. W.H.Freeman and Co., New York.

Maurer, H, Holliger, K. & Boerner, D.E. (1998): Stochastic regularization: Smoothness or similarity? Geophys. Res. Lett., **25**, 2889–2892.

Müller, T.M. & Shapiro, S.A. (2001): Most probable seismic pulses in single realizations of two– and three–dimensional random media. Geophys. J. int., **144**, 83–95.

Pilkington, M. & Todoeschuck, J.P. (1990): Stochastic inversion for scaling geology. Geophys. J. int., **102**, 205–217.

Ryzhik, L., Papanicolaou, G. & Keller, J.B. (1996): Transport equations for elastic and other waves in random media. Wave Motion, **24**, 327–370.

Sato, H. & Fehler, M. (1998): Seismic Wave Propagation and Scattering in the Heterogeneous Earth. Springer-Verlag, New York.

Schottky, W. (1926): Small–shot effect and flicker effect. Phys. Rev., **28**, 74.

Shapiro, S.A. & Kneib, G. (1993): Seismic attenuation by scattering: theory and numerical results. Geophys. J. int., **114**, 373–191.

Shapiro, S.A., Schwarz, R. & Gold, N. (1996): The effect of random isotropic inhomogeneities on the phase velocity of seismic waves. Geophys. J. int., **127**, 783–794.

Tarantola, A. (1987): Inverse Problem Theory. Elsevier, Amsterdam.

Turcotte, D.L. (1989): Fractals in geology and geophysics. Pure appl. Geophys., **131**, 171–196.

Wu, R.S. (1982): Attenuation of short period seismic due to scattering. Geophys. Res. Lett., **9**, 9–12.

Wu, R.S. (1989): The perturbation method in elastic wave scattering. Pure appl. Geophys., **131**, 605–637.

Wu, R-S. & Aki, K. (1985): Elastic wave scattering by a random medium and the small–scale inhomogeneities in the litosphere. J. geophys. Res., **90B**, 10261–10273.