# Summation of paraxial Gaussian beams and of paraxial ray approximations in inhomogeneous anisotropic layered structures

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## Summary

Elementary high-frequency approximate solutions of the elastodynamic equation in anisotropic inhomogeneous media, connected with a ray $\Omega$ of a zero-order reflected/transmitted wave, are studied. These elementary solutions are called paraxial ray approximations and paraxial Gaussian beams, and are approximately valid not only along the ray $\Omega$, but also in its vicinity. The travel times of paraxial ray approximations are real-valued, but the travel times of paraxial Gaussian beams complex-valued outside the ray $\Omega$. For a fixed travel time, the phase fronts of all elementary solutions are mutually tangent at the relevant point of the ray $\Omega$. With any ray $\Omega$, a three-parameteric system of paraxial ray approximations and a six-parameteric system of paraxial Gaussian beams is connected.

More general solutions of the elastodynamic equation can be obtained by summation of elementary solutions connected with rays passing in the vicinity of the receiver, multiplied by some weighting functions. A general superposition integral is derived which can be used both for the summation of paraxial ray approximations and for summation of paraxial Gaussian beams. The superposition integral is over ray parameters of the near-by rays. It removes certain singularities of standard ray method (caustics, etc.). The integral may be used even for isotropic media, pressure waves in fluids, radio-waves, etc. Individual quantities in the superposition integral can be computed by dynamic ray tracing in wavefront orthonormal coordinates, or by dynamic ray tracing in Cartesian coordinates. The wave under consideration may be generated by an initial surface with the variable initial travel time along it, by a point source with an arbitrary radiation function, and so forth.

The receiver $R$ may be situated arbitrarily in the model, including structural interfaces and the Earth's surface. The superposition is performed over points $R_\gamma$, situated on near-by rays specified by ray parameters $\gamma_1, \gamma_2$. Points $R_\gamma$ on near-by rays may be

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chosen in different ways. It is customary (but not necessary) to introduce a target surface (possibly curved), passing through the receiver. The points $R_\gamma$ then represent the end-points or intersections of near-by rays with the target surface. The superposition integrals, however, work even if receivers are not situated on the target surface, but are close to it. For suitably chosen parameters of paraxial ray approximations, the superposition integral yields the Maslov-Chapman integral.

For wave fields generated by a point source with a suitably chosen radiation function, the superposition integrals yield the superposition Green functions, i.e. the paraxial ray approximation Green functions or paraxial Gaussian beam Green function. As a special case, the Maslov-Chapman Green function is obtained.

**Keywords**

Anisotropic inhomogeneous layered structures. Paraxial ray approximation. Paraxial Gaussian beams. Summation of paraxial ray approximations. Summation of paraxial Gaussian beams. Superposition Green functions. Maslov-Chapman method.

# 1 Introduction

We shall construct approximate high-frequency solutions of the elastodynamic equations in anisotropic inhomogeneous media, valid not only along rays, but also in the paraxial vicinity of these rays. We shall call these elementary solutions the paraxial ray approximations and paraxial Gaussian beams. We shall also use these elementary solutions, connected with the individual rays, as corner stones in the superposition integrals, representing more general solutions of the elastodynamic equation. The summation of paraxial ray approximations and the summation of paraxial Gaussian beams remove certain singularities of the standard ray method.

Consider an arbitrary reflected, transmitted and converted wave, propagating in a 3-D, laterally varying, anisotropic layered structure. We select arbitrarily one of its rays and denote it by $\Omega$. Using the standard, zero-order, ray computations, we can find travel times and vectorial ray amplitudes of the elementary wave under consideration along $\Omega$. Consequently, we can also construct the ray-theory displacement vector of this wave along $\Omega$. For high frequencies, this displacement vector satisfies approximately the elastodynamic equation.

We now wish to find the approximate elementary solution of the elastodynamic equation, corresponding to the wave under consideration, valid not only along ray $\Omega$, but also in its vicinity. We shall call such a solution *the paraxial ray approximation, connected with ray* $\Omega$.

The basic procedure in the computation of the paraxial ray approximation connected with ray $\Omega$ is the dynamic ray tracing along $\Omega$. By dynamic ray tracing, we determine important $2\times2$ matrices $\mathbf{Q}, \mathbf{P}$ and $\mathbf{M} = \mathbf{P}\mathbf{Q}^{-1}$. The $2\times2$ matrix $\mathbf{Q}$ is known as the matrix of geometrical spreading, as the amplitudes are inversely proportional to $[\det\mathbf{Q}]^{1/2}$. The $2 \times 2$ matrix $\mathbf{M}$ represents the matrix of second derivatives of the travel time field with respect to coordinates tangent to the wavefront. Matrix $\mathbf{M}$ can be used to determine the "quadratic expansion" of the travel time field of the elementary wave under consideration in the vicinity of $\Omega$. We also speak of paraxial vicinity of ray $\Omega$, and of paraxial travel times. A similar (but linear) paraxial expansion can be also found for the paraxial slowness vector. In many applications it would also be very useful to know the ray amplitudes in the paraxial vicinity of $\Omega$. The situation is, however, more complex in this case. The amplitudes along the central ray are inversely proportional to $[\det \mathbf{Q}]^{1/2}$, so that it would be necessary to determine the expansion of quantity $[\det \mathbf{Q}]^{-1/2}$. This is, however, impossible in the framework of standard dynamic ray tracing. Dynamic ray tracing along $\Omega$ yields only $\mathbf{Q}$ along $\Omega$, but not in the vicinity of $\Omega$. There are two possibilities of determining the distribution of $\mathbf{Q}$ in the vicinity of $\Omega$: a) Computation of new rays in the vicinity of $\Omega$. We can then compute $\mathbf{Q}'s$ by the dynamic ray tracing along the new rays, and perform interpolation between near-by rays. See Eisner and Pšenčík (1996). b) Application of "higher-order paraxial methods", outlined in Klimeš (1997), Červený (2000), Section 4.7.6. Such higher-order paraxial methods, however, have not yet been numerically investigated.

In layered media, there would be an additional difficulty connected with the amplitude variations, caused by the variations of the R/T coefficients from one ray to another. As a consequence of all these difficulties, paraxial amplitudes are usually considered constant in the paraxial vicinity of $\Omega$, and equal to the amplitudes on the central ray. We shall call the final expressions for the paraxial displacement vector, constructed in this way, *paraxial ray approximations*, and discuss them in Section 2.

The equations for the time-harmonic paraxial displacement vector can also be used to find some more general solutions of the elastodynamic equation, closely concentrated around central ray $\Omega$ for high frequencies $\omega$. Such solutions can be constructed in various ways. Here we shall briefly discuss one form of these solutions, known as *paraxial Gaussian beams*, or simply *Gaussian beams*. See Section 3.

Both the paraxial ray approximations and the paraxial Gaussian beams play an important role in various applications. The most important application consists in the construction of more general solutions of the elastodynamic equation by their summation. The integral superposition of paraxial ray approximations or of paraxial Gaussian beams yields only an approximate high-frequency solution of the elastodynamic equation, but removes certain singularities of the standard ray method (caustic region, critical region, etc.). See Section 4 for a detailed explanation of superposition integrals, Sections 5 and 6 for their discussion, Section 7 for the summation of paraxial ray approximations, Section 8 for the alternative versions of the superposition integrals and Section 9 for the superposition integrals in 2-D models. Note that the summation of paraxial ray approximations (Section 7) yields results close or equal to the Maslov-Chapman theory.

In the whole paper, we shall consider an inhomogeneous anisotropic non-dissipative layered structure, and a ray $\Omega$ of an arbitrary high-frequency elastic body wave propagating in the structure. We shall now introduce several important terms and symbols related to rays in smooth inhomogeneous anisotropic media that we shall need in the following. For more details, see Červený (2000), Sections 3.6 and 4.14. The basic role in the anisotropic ray method is played by a $3 \times 3$ Christoffel matrix $\Gamma_{ik}(x_i, p_i^{(x)}) = a_{ijkl} p_j^{(x)} p_l^{(x)}$, where $a_{ijkl} = c_{ijkl}/\rho$ are density normalized elastic parameters, and $p_i^{(x)}$ are Cartesian components of the slowness vector, perpendicular to the wavefront. We denote the three eigenvalues and relevant eigenvectors of the Christoffel matrix by $G_m(x_i, p_i^{(x)})$ and $\vec{g}^{(m)}(x_i, p_i^{(x)})$, $m = 1, 2, 3$. We consider only a non-degenerate case $G_1(x_i, p_i^{(x)}) \neq G_2(x_i, p_i^{(x)}) \neq G_3(x_i, p_i^{(x)})$. Three elementary waves may propagate in a smooth inhomogeneous anisotropic medium, corresponding to $m = 1, 2$ and 3. We speak of qS1, qS2 and qP waves. The eikonal equation of any of these three waves is given by the relation

$$G_m(x_i, p_i^{(x)}) = 1 \; . \tag{1.1}$$

We can also introduce a Hamiltonian $\mathcal{H}(x_i, p_i^{(x)})$ by the relation $\mathcal{H}(x_i, p_i^{(x)}) = \frac{1}{2}(G_m(x_i, p_i^{(x)}) - 1)$. Then the eikonal equation (1.1) takes the classical form:

$$\mathcal{H}(x_i, p_i^{(x)}) = 0 \; . \tag{1.2}$$

As the eikonal equations are valid for any $m$ ($m = 1, 2, 3$), we shall omit the index $m$ in the symbols for $G_m$ and $\vec{g}^{(m)}$.

From the eikonal equation, we can easily derive the ray tracing system:

$$\mathrm{d}x_i/\mathrm{d}T = \partial\mathcal{H}/\partial p_i^{(x)} = \mathcal{U}_i^{(x)} \; , \qquad \mathrm{d}p_i^{(x)}/\mathrm{d}T = -\partial\mathcal{H}/\partial x_i = \eta_i^{(x)} \; . \tag{1.3}$$

Here $\mathcal{U}_i^{(x)}$ and $\eta_i^{(x)}$ are given by relations

$$\mathcal{U}_i^{(x)} = a_{ijkl} p_l^{(x)} g_j^{(x)} g_k^{(x)} \; , \qquad \eta_i^{(x)} = -\tfrac{1}{2} \frac{\partial a_{jkln}}{\partial x_i} p_k^{(x)} p_n^{(x)} g_j^{(x)} g_l^{(x)} \; . \tag{1.4}$$

The Cartesian components $g_k^{(x)}$ of eigenvectors $\vec{g}$ corresponding to the relevant eigenvalue $G$ can be obtained as solutions of equations

$$(\Gamma_{jk} - G\delta_{jk}) g_k^{(x)} = 0 \; , \qquad g_j^{(x)} g_j^{(x)} = 1 \; . \tag{1.5}$$

$\mathcal{U}_i^{(x)}$ represent the Cartesian components of the group velocity vector $\vec{\mathcal{U}}$, tangent to the ray. The four vectors $\vec{\mathcal{U}}$, $\vec{\eta}$, $\vec{p}$ and $\vec{g}$ will be used broadly in the whole paper. All these four vectors are used in ray tracing, so that we can consider them to be known at any point of ray $\Omega$. Also, we can determine the group velocity $\mathcal{U}$ and phase velocity $\mathcal{C}$ at any point of $\Omega$; $\mathcal{U} = |\vec{\mathcal{U}}|$, $\mathcal{C} = (p_k^{(x)} p_k^{(x)})^{-1/2}$. $\mathcal{C}$ and $\mathcal{U}$ depend both on position and on the direction of the slowness vector $\vec{p}$.

At any point of ray $\Omega$, we introduce the wavefront orthonormal coordinate system $y_1, y_2, y_3$, with its origin at that point. We denote its unit basis vectors by $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$. Here $\vec{e}_3$ is perpendicular to the wavefront, $\vec{e}_3 = \mathcal{C}\vec{p}$. The basis vectors $\vec{e}_1$, $\vec{e}_2$ are mutually perpendicular and tangent to the wavefront. Along $\Omega$, $\vec{e}_1$ and $\vec{e}_2$ satisfy the ordinary differential equations

$$\mathrm{d}\vec{e}_I/\mathrm{d}T = -(\vec{p}\cdot\vec{p})(\vec{e}_I \cdot \mathrm{d}\vec{p}/\mathrm{d}T)\vec{p} \; , \tag{1.6}$$

where $T$ is a monotonic variable along $\Omega$, representing the travel time. The $3 \times 3$ transformation matrix from the wavefront orthonormal coordinates $y_i$ to the Cartesian coordinates $x_k$ is denoted by $\hat{\mathbf{H}}$. The elements of this important transformation matrix are given by relations

$$H_{ik} = \partial x_i/\partial y_k = \partial y_k/\partial x_i = e_{ki} \; , \tag{1.7}$$

where $e_{ki}$ is the $i$-th Cartesian component of the unit vector $\vec{e}_k$. As soon as the $3 \times 3$ matrix $\hat{\mathbf{H}}$ is known, it is simple to calculate the wavefront orthonormal components of vectors $\vec{\mathcal{U}}$, $\vec{\eta}$, $\vec{p}$, etc., from their Cartesian components:

$$\mathcal{U}_i^{(y)} = H_{ni}\mathcal{U}_n^{(x)} \ , \quad \eta_i^{(y)} = H_{ni}\mathcal{U}_n^{(x)} \ , \quad p_i^{(y)} = H_{ni}p_n^{(x)} \ . \tag{1.8}$$

Note that $p_I^{(y)} = 0$, $p_3^{(y)} = \mathcal{C}^{-1}$, $\mathcal{U}_I^{(y)} \neq 0$, $\mathcal{U}_3^{(y)} = \mathcal{C}$ along $\Omega$ ($I = 1, 2$). In isotropic media, $\mathcal{U}_I^{(y)} = 0$, but this is not the case of anisotropic media.

Let us now consider an orthonomic system of rays, parameterized by two ray parameters $\gamma_1, \gamma_2$. We introduce the $2 \times 2$ matrices $\mathbf{Q}$ and $\mathbf{P}$, with elements

$$Q_{IJ} = (\partial y_I / \partial \gamma_J)_{T=const.} \ , \quad P_{IJ} = (\partial p_I^{(y)} / \partial \gamma_J)_{T=const.} \ . \tag{1.9}$$

These matrices can be determined along the ray $\Omega$ by dynamic ray tracing, if they are known at one point on $\Omega$. Both matrices $\mathbf{Q}$ and $\mathbf{P}$ are very useful in the paraxial ray methods. Particularly, they are simply related to the $2 \times 2$ matrix $\mathbf{M}$ of second derivatives of the travel time field with respect to $y_1$ and $y_2$ (for $y_3 = 0$):

$$\mathbf{M} = \mathbf{P}\mathbf{Q}^{-1} \ . \tag{1.10}$$

Solving the dynamic ray tracing system twice, we can determine the $4 \times 4$ *ray propagator matrix* (in wavefront orthonormal coordinates) $\mathbf{\Pi}(R, S)$, at any point $R$ of the ray $\Omega$. We shall use the notation

$$\mathbf{\Pi}(R, S) = \begin{pmatrix} \mathbf{Q}_1(R, S) & \mathbf{Q}_2(R, S) \\ \mathbf{P}_1(R, S) & \mathbf{P}_2(R, S) \end{pmatrix} . \tag{1.11}$$

The initial conditions at $S$ are given by simple relation $\mathbf{\Pi}(S, S) = \mathbf{I}$, where $\mathbf{I}$ is the $4 \times 4$ identity matrix.

The $4 \times 4$ ray propagator matrix $\mathbf{\Pi}(R, S)$ has found many important applications in paraxial ray methods. For example, the solution of the dynamic ray tracing system $\mathbf{Q}(R)$, $\mathbf{P}(R)$ at a point $R$ on $\Omega$ can be expressed in terms of any initial conditions $\mathbf{Q}(S)$, $\mathbf{P}(S)$ at point $S$ on $\Omega$ as follows:

$$\begin{pmatrix} \mathbf{Q}(R) \\ \mathbf{P}(R) \end{pmatrix} = \mathbf{\Pi}(R, S) \begin{pmatrix} \mathbf{Q}(S) \\ \mathbf{P}(S) \end{pmatrix} . \tag{1.12}$$

For more details on dynamic ray tracing in wavefront orthonormal coordinates and on the ray propagator matrix, see Červený (1999, 2000).

A brief note to the mathematical notations used in the paper. For matrices, we have used both the component and matrix notation. The capital letter indices take the values of 1 and 2, lower-case indices 1, 2 and 3. The Einstein summation convention is used. Alternatively, the matrices are denoted by boldface letters. To distinguish between $2 \times 2$ and $3 \times 3$ matrices, we use the circumflex $\hat{}$ above the letter for $3 \times 3$ matrices. If the same symbol $\mathbf{M}$ is used for both $2 \times 2$ and $3 \times 3$ matrices, matrix $\mathbf{M}$ denotes the upper left-hand minor of $\hat{\mathbf{M}}$. Also the $3 \times 1$ and $3 \times 2$ matrices are supplemented by a circumflex above the letter. In the $4 \times 4$ matrices, the circumflex is not used, as this notation cannot cause any misunderstanding.

# 2 Paraxial ray approximation for the displacement vector

In this section, we shall construct an elementary high-frequency solution of the elastodynamic equation connected with the ray $\Omega$, situated in an inhomogeneous anisotropic medium. We shall call it the *paraxial ray approximation* and denote the relevant displacement vector by $\vec{u}^{par}$. Of course, the elementary solutions are only approximate.

Let us consider a point $R$, situated on the ray $\Omega$, and a point $R^{'}$, situated in the paraxial vicinity of $R$ (possibly outside ray $\Omega$). Then the elementary solution $\vec{u}^{par}(R^{'})$ is given by the relation:

$$\vec{u}^{par}(R^{'}) = \vec{U}(R)\exp[-\mathrm{i}\omega(t - T(R^{'},R))] \ . \tag{2.1}$$

The amplitude vector $\vec{U}(R)$ can be separated into two factors. The first factor does not depend on the paraxial ray field, but only on quantities connected with the ray $\Omega$:

$$U_i^{(x)}(R) = U_i^{\Omega}(R)[\det \mathbf{Q}(S)/\det \mathbf{Q}(R)]^{1/2} \ . \tag{2.2}$$

We shall call the vector $\vec{U}^{\Omega}(R)$, with Cartesian components $U_i^{\Omega}(R)$, the *spreading-free amplitude vector*. It is given by the expression:

$$U_i^{\Omega}(R) = [\rho(S)\mathcal{C}(S)/\rho(R)\mathcal{C}(R)]^{1/2}\mathcal{R}^C g_i(R)g_k(S)U_k^{\Omega}(S) \ . \tag{2.3}$$

Here $\rho$ is the density, $\mathcal{C}$ the phase velocity, $\mathcal{R}^C$ the product of normalized R/T coefficients along the ray $\Omega$ and $\vec{g}$ the polarization vector. The spreading-free amplitude $U_i^{\Omega}(R)$ is the same for any paraxial ray approximation (2.1) connected with the ray $\Omega$, and is not influenced by the paraxial ray field. Contrary to it, the second factor in (2.2) is influenced by the paraxial ray field. If the point $R$ is situated on a structural interface or on the Earth's surface, $U_i^{\Omega}(R)$ should be multiplied by the relevant conversion coefficient. $U_i^{\Omega}(R)$ does not include the phase shift due to caustics along $\Omega$ from $S$ to $R$; the phase shift is formally included in the factor $[\det \mathbf{Q}(S)/\det \mathbf{Q}(R)]^{1/2}$. For more details, see Červený (2000), Section 5.4.

Let us now discuss the travel-time function $T(R', R)$. It represents the paraxial travel time at $R^{'}$, corresponding to the selected elementary wave, expressed in terms of the travel time at $R$. Formally, the simplest expression for $T(R, R^{'})$ is obtained in the wavefront orthonormal coordinates $y_I$:

$$T(R^{'},R) = T(R) + \tfrac{1}{2}\mathbf{y}^T(R^{'})\mathbf{M}(R)\mathbf{y}(R^{'}) \ , \quad \mathbf{y} = (y_1, y_2)^T \ . \tag{2.4}$$

Unfortunately, this expression can be used only if point $R^{'}$ is situated in the plane tangent to the wavefront at $R$ on $\Omega$ (plane $y_3 = 0$). A considerably more flexible relation for $T(R^{'}, R)$ is obtained in general Cartesian coordinates $x_i$. Denote the Cartesian coordinates of points $R$ and $R^{'}$ by $x_i(R)$, and $x_i(R^{'})$, and introduce $x_i(R^{'}, R) = x_i(R^{'}) - x_i(R)$. Then the quadratic expansion in terms of $x_i(R^{'}, R)$ is as follows:

$$T(R^{'},R) = T(R) + x_i(R^{'},R)p_i^{(x)}(R) + \tfrac{1}{2}x_i(R^{'},R)x_j(R^{'},R)M_{ij}^{(x)}(R) \ . \tag{2.5}$$

The elements of the $3\times 3$ matrix $\hat{\mathbf{M}}^{(x)}(R)$ represent the second partial derivatives of the travel-time field with respect to coordinates $x_i, x_j$. The $3 \times 3$ matrix $\hat{\mathbf{M}}^{(x)}(R)$ can be expressed in terms of the $2 \times 2$ matrix $\mathbf{M}(R) = \mathbf{P}(R)\mathbf{Q}^{-1}(R)$ as follows:

$$\hat{\mathbf{M}}^{(x)}(R) = \hat{\hat{\mathbf{H}}}(R) \left( \begin{array}{ccc} \multicolumn{2}{c}{\mathbf{M}(R)} & \begin{array}{c} (\mathcal{C}^{-1}\eta_1^{(y)})_R \\ (\mathcal{C}^{-1}\eta_2^{(y)})_R \end{array} \\ (\mathcal{C}^{-1}\eta_1^{(y)})_R & (\mathcal{C}^{-1}\eta_2^{(y)})_R & (\mathcal{C}^{-2}\eta_i^{(y)}\mathcal{U}_i^{(y)})_R \end{array} \right) \hat{\hat{\mathbf{H}}}^T(R) \ . \tag{2.6}$$

Here $\mathcal{U}_i^{(y)}$ and $\eta_i^{(y)}$ are given by (1.8), with (1.4) and (1.7). The $3 \times 3$ matrix $\hat{\hat{\mathbf{H}}}(R)$ has the same meaning as $\hat{\mathbf{H}}(R)$, only the elements $H_{nI} = e_{In}$ are replaced by $e_{In}^A = e_{In} - \mathcal{C}^{-1}\mathcal{U}_I^{(y)}e_{3n}$:

$$\hat{\hat{\mathbf{H}}} = \left( \begin{array}{ccc} e_{11} - \mathcal{C}^{-1}\mathcal{U}_1^{(y)}e_{31} & e_{21} - \mathcal{C}^{-1}\mathcal{U}_2^{(y)}e_{31} & e_{31} \\ e_{12} - \mathcal{C}^{-1}\mathcal{U}_1^{(y)}e_{32} & e_{22} - \mathcal{C}^{-1}\mathcal{U}_2^{(y)}e_{32} & e_{32} \\ e_{13} - \mathcal{C}^{-1}\mathcal{U}_1^{(y)}e_{33} & e_{23} - \mathcal{C}^{-1}\mathcal{U}_2^{(y)}e_{33} & e_{33} \end{array} \right) \ . \tag{2.7}$$

Note that $e_{3n} = \mathcal{C}p_n^{(x)}$, so that $e_{In}^A = e_{In} - \mathcal{U}_I^{(y)}p_n^{(x)}$. For derivation and physical explanation of physical meaning of vectors $\vec{e}_I^{\,A}$ $(I = 1, 2)$ see Červený (2000), Section 4.14.8.

As we can see from (2.2), (2.5) and (2.6), the only two quantities in (2.1) which depend on the paraxial ray field are $[\det \mathbf{Q}(S)/\det \mathbf{Q}(R)]^{1/2}$ and $\mathbf{M}(R)$. It can be shown that both these two quantities can be expressed in terms of $\mathbf{M}(S)$. Using (1.11) and (1.12), we obtain:

$$\begin{aligned} \det \mathbf{Q}(R)/\det \mathbf{Q}(S) &= \det[\mathbf{Q}_1(R,S) + \mathbf{Q}_2(R,S)\mathbf{M}(S)] \ , \qquad (2.8) \\ \mathbf{M}(R) &= [\mathbf{P}_1(R,S) + \mathbf{P}_2(R,S)\mathbf{M}(S)][\mathbf{Q}_1(R,S) + \mathbf{Q}_2(R,S)\mathbf{M}(S)]^{-1} \ . \end{aligned}$$

Here $\mathbf{Q}_1(R,S)$, $\mathbf{Q}_2(R,S)$, $\mathbf{P}_1(R,S)$ and $\mathbf{P}_2(R,S)$ are $2 \times 2$ minors of the (pressumably known) $4 \times 4$ ray propagator matrix $\mathbf{\Pi}(R,S)$, see (1.11).

Note that the $2 \times 2$ matrices $\mathbf{M}(R)$ and $\mathbf{M}(S)$ are related to the $2 \times 2$ matrices of wavefront curvatures at $R$ and $S$, $\mathbf{M}(R) = \mathcal{C}^{-1}(R)\mathbf{K}(R)$, $\mathbf{M}(S) = \mathcal{C}^{-1}(S)\mathbf{K}(S)$.

Equations of this section can be directly used even for isotropic media; we only insert $\mathcal{C} = V$ and $\mathcal{U}_I^{(y)} = 0$. Then $\hat{\hat{\mathbf{H}}} = \hat{\mathbf{H}}$, see (2.7), and $\eta_i^{(y)}\mathcal{U}_i^{(y)} = \mathcal{C}\eta_3^{(y)}$ in (2.6). The quantities $\eta_i^{(x)}$ are known from standard ray tracing, $\eta_i^{(x)} = -V^{-1}\partial V/\partial x_i$, and are transformed to $\eta_i^{(y)}$ using the relation $\eta_i^{(y)} = H_{ni}\eta_n^{(x)}$. The wavefront orthonormal coordinates also reduce to local ray-centered coordinates, with the basis vectors $\vec{e}_1$ and $\vec{e}_2$ perpendicular to the ray. The expression (2.3) for the spreading free amplitude also requires some slight modification, see Červený (2000).

Thus, to construct any paraxial ray approximation (2.1) connected with the known central ray $\Omega$, we have to specify $\mathbf{M}(S)$. Point $S$ on $\Omega$ may be chosen arbitrarily. Matrix $\mathbf{M}(S)$ must be symmetrical. A consequence of the symplectic properties of propagator matrix $\mathbf{\Pi}(R,S)$ is that $\mathbf{M}(R)$ is symmetrical along the whole central ray $\Omega$. Consequently, $\mathbf{M}(R)$ has two real-valued eigenvalues, $M_1(R)$ and $M_2(R)$. We can relate eigenvalues $M_1(S)$ and $M_2(S)$ to the principal curvatures of the wavefront at $S$, $K_1(S)$ and $K_2(S)$, $K_I(S) = \mathcal{C}(S)M_I(S)$.

As we can see, any value of $\mathbf{M}(S)$ specifies one paraxial ray approximation (2.1). Thus, the complete system of paraxial ray approximations (2.1), connected with one central ray $\Omega$, is three-parameteric. The relevant three parameters are $M_{11}(S)$, $M_{22}(S)$, and $M_{12}(S) = M_{21}(S)$.

The curvature of the wavefront of the paraxial ray approximation (2.1), connected with central ray $\Omega$, equals $\mathbf{K}(R) = \mathcal{C}(R)\mathbf{M}(R)$ at point $R$. Consequently, we have a three-parameteric system of paraxial wavefronts connected with the selected central ray $\Omega$ at point $R$. All these paraxial wavefronts are mutually tangent at $R$ on $\Omega$, as they have a common slowness vector at $R$. Assume that central ray $\Omega$ belongs to an orthonomic system of rays, corresponding to some elementary wave. We can then also construct the actual wavefront, corresponding to the elementary wave under consideration at $R$ on $\Omega$. Denote its curvature at $R$ on $\Omega$ by $\mathbf{K}^a(R)$. Then $\mathbf{K}^a(R)$ is one curvature of the three-parameteric system of paraxial curvatures $\mathbf{K}(R)$ and all other paraxial wavefronts are tangent to it at $R$ on $\Omega$. These paraxial wavefronts, tangent to the actual wavefront, are often called Snell's wavefronts. But there is no uniqueness in this terminology.

# 3 Paraxial Gaussian beams

We again consider central ray $\Omega$ situated in an anisotropic elastic laterally varying layered structure, and real-valued travel time $T(R)$ along central ray $\Omega$. The paraxial high-frequency solution (2.1) of the elastodynamic equation can be generalized by allowing complex-valued solutions $\mathbf{Q}$ and $\mathbf{P}$ of the dynamic ray tracing system. Consequently, also matrix $\mathbf{M} = \mathbf{P}\mathbf{Q}^{-1}$ and quantity $\det \mathbf{Q}$ are complex-valued. We shall use the following notation:

$$\mathbf{M} = \mathrm{Re}(\mathbf{M}) + \mathrm{i}\ \mathrm{Im}(\mathbf{M})\ . \tag{3.1}$$

Assuming that $\mathrm{Im}(\mathbf{M})$ is positive definite, the paraxial high-frequency solution of the elastodynamic equation, alternative to (2.1), is closely concentrated about the central ray and *represents a beam.* In wavefront orthonormal coordinates, the relevant displacement vector is given by the relation, see (2.1) and (2.4),

$$\begin{aligned} \vec{u}^{\,\mathrm{beam}}(R^{'}) &= \vec{U}(R)\exp[-\mathrm{i}\omega(t - T(R) - \tfrac{1}{2}\mathbf{y}^T(R^{'})\mathbf{M}(R)\mathbf{y}(R^{'})] \\ &= \vec{U}(R)\exp[-\mathrm{i}\omega(t - T(R) - \tfrac{1}{2}\mathbf{y}^T(R^{'})(\mathrm{Re}\mathbf{M}(R))\mathbf{y}(R^{'})] \\ &\times\ \exp[-\tfrac{1}{2}\omega\mathbf{y}^T(R^{'})(\mathrm{Im}\mathbf{M}(R))\mathbf{y}(R^{'})]\ . \end{aligned} \tag{3.2}$$

Here $\vec{U}(R)$ is expressed in terms of $U_i^\Omega(R)$ using (2.2) and (2.3). The spreading-free amplitude $U_i^\Omega(R)$ remains exactly the same as in (2.3), the matrices $\mathbf{M}(R)$, $\mathbf{Q}(R)$ and $\mathbf{Q}(S)$ in (2.2) and (3.2) are, however, complex-valued. In Cartesian coordinates, we use (2.5) with (2.6) and (3.1).

Similarly as in the case of paraxial displacement vector (2.1), it is sufficient to know $\mathbf{M}(S)$ to determine $(\det \mathbf{Q}(R)/\det \mathbf{Q}(S))^{-1/2}$ and $\mathbf{M}(R)$ along the whole ray $\Omega$ in (3.2). Relations (2.8) remain valid even for complex-valued $\mathbf{Q}$ and $\mathbf{M}$. Complex-valued matrices $\mathbf{M}(S)$ and $\mathbf{Q}(S)$, however, must satisfy three conditions: a) $\mathbf{Q}(S)$ is regular, i.e. $\det(\mathbf{Q}(S)) \neq 0$ and $\det(\mathbf{M}(S)) \neq \infty$, b) $\mathbf{M}(S)$ is symmetrical, c) $\mathrm{Im}(\mathbf{M}(S))$ is positive definite. It can be proved that the following three important properties are then satisfied along the whole ray $\Omega$, at any $R$: i) $\mathbf{Q}(R)$ is regular everywhere, i.e. $\det \mathbf{Q}(R) \neq 0$, ii) $\mathbf{M}(R)$ is symmetrical, iii) $\mathrm{Im}(\mathbf{M}(R))$ is positive definite. The proof of these conclusions for isotropic media can be found in Červený and Pšenčík (1983b). For anisotropic media, the proof is analogous. The argument of square root $[\det \mathbf{Q}(R)/\det \mathbf{Q}(S)]^{-1/2} = [\det(\mathbf{Q}_1(R,S) + \mathbf{Q}_2(R,S)\mathbf{M}(S)]^{-1/2}$ in (2.2) is determined in the following way: a) It equals zero at $S$, b) It varies continuously along ray $\Omega$. The solutions (3.2) of the elastodynamic equation satisfying the above three conditions are known as *paraxial Gaussian beams*, or simply *Gaussian beams*.

Thus, once $\mathbf{M}(S)$ satisfies the Gaussian beam conditions at any point $S$ of ray $\Omega$, (3.2) represents the Gaussian beam along the whole ray $\Omega$. *Once a Gaussian beam - always a Gaussian beam.* The Gaussian beam is regular everywhere, even at caustic points. As the amplitudes are inversely proportional to $[\det \mathbf{Q}(R)]^{1/2} \neq 0$, they remain finite along the whole ray $\Omega$, including the caustic points. This is the important difference between the paraxial ray approximation (2.1) and the paraxial Gaussian beam (3.2).

What is the meaning of matrices $\mathrm{Re}(\mathbf{M}(R))$ and $\mathrm{Im}(\mathbf{M}(R))$? Matrix $\mathrm{Re}(\mathbf{M}(R))$ describes the *geometrical properties of the phase front* of the Gaussian beam at $R$. As $\mathrm{Re}(\mathbf{M}(R))$ is always symmetrical, its eigenvalues $M_1^R(R)$, $M_2^R(R)$ are always real. Quantities $K_1(R) = \mathcal{C}(R)M_1^R(R)$ and $K_2(R) = \mathcal{C}(R)M_2^R(R)$ represent the principal curvatures of the phase front of the Gaussian beam.

Matrix $\mathrm{Im}(\mathbf{M}(R))$ controls the *amplitude profile of the Gaussian beam* in the plane tangent to the wavefront at $R$. As matrix $\mathrm{Im}(\mathbf{M}(R))$ is positive definite and symmetrical, it has two real-valued positive eigenvalues $M_1^I(R)$, $M_2^I(R)$, and the amplitude profile is Gaussian in the paraxial vicinity of ray $\Omega$. This is the reason why the solutions (3.2), with $\mathrm{Im}(\mathbf{M}(R)) \neq \mathbf{0}$, are called *paraxial Gaussian beams.* Alternatively, they are also known as *solutions closely concentrated in the vicinity of ray* $\Omega$. With the increasing square of distance from ray $\Omega$, the amplitudes of the beam decrease exponentially. The exponential decrease is frequency-dependent, it is steeper for higher frequencies. Quadratic curve $\frac{1}{2}\omega\mathbf{y}^T(\mathrm{Im}(\mathbf{M}(R))\mathbf{y} = 1$ in the plane tangent to the wavefront at $R$ represents a *spot ellipse* for frequency $\omega$. Along the spot ellipse, the amplitudes of the Gaussian beam equal $e^{-1}\vec{U}(R)$.

Thus, we can construct a six-parameteric system of paraxial Gaussian beams (3.2) connected with any one central ray $\Omega$. Parameters $\mathrm{Re}(M_{11}(S))$, $\mathrm{Re}(M_{22}(S))$ and $\mathrm{Re}(M_{12}(S))$ control the shape of the phase-front of the Gaussian beam at the initial point $S$. Parameters $\mathrm{Im}(M_{11}(S))$, $\mathrm{Im}(M_{22}(S))$ and $\mathrm{Im}(M_{12}(S))$ control the width of the Gaussian beam at the initial point $S$. The real-valued travel time $T(R)$, the real-valued slowness vector $\vec{p}(R)$, and spreading-free amplitudes $U_i^\Omega(R)$ are the same in the whole system of Gaussian beams, and also the same as in the three-parameteric system of paraxial approximations (2.1). Outside the central ray $\Omega$, the travel time is complex-valued.

To guarantee the fulfilment of Gaussian beam conditions a)–c) at $S$, $M_1^I(S)$ and $M_2^I(S)$ must not vanish and must be finite. On the contrary, $M_1^R(S)$ and $M_2^R(S)$ must be finite, but both of them may vanish (plane phase-front at $S$, with a Gaussian windowing of amplitudes). For the limiting case of infinitely broad Gaussian beams ($M_1^I(S) \to 0$ and $M_2^I(S) \to 0$), the solution (3.2) does not represent the Gaussian beam, but standard paraxial ray approximation (2.1). In 2-D problems, also *ribbon Gaussian beams* are often used, for which $M_1^I(S)$ is nonvanishing, but $M_2^I(S)$ vanishes.

Matrix $\mathrm{Im}\mathbf{M}$ varies along the central ray. We can determine these variations from the second equation of (2.8). The paraxial Gaussian beams may be narrow in some parts of ray $\Omega$, but very broad in some other region. In the latter region, where the width of Gaussian beams is large, the validity conditions for the paraxial ray method are violated, and the accuracy of the paraxial expressions derived here is low.

As the travel time is complex-valued outside the central ray of the beam, the corresponding rays in the vicinity of the central ray can be interpreted as complex rays. In this way, the Gaussian beams may be understood as *bundles of complex rays* (Keller and Streifer, 1971; Deschamps, 1971; Felsen and Marcuvitz, 1973; Felsen, 1976). Both the positions of the points along the ray and the ray-centered components of the slowness vector are complex-valued for complex rays. The interpretation of Gaussian beams as bundles of complex rays is closely connected with the idea of displacing a source into a complex coordinate space (Felsen, 1976, 1984; Wu, 1985; Norris, 1986).

Several alternative approaches can be used to derive the equations for high-frequency elastodynamic paraxial Gaussian beams. For a detailed derivation of 3-D paraxial Gaussian beams from the elastodynamic equation in a smooth isotropic elastic medium without interfaces see Červený and Pšenčík (1983b). The elastodynamic HF Gaussian beams are derived there as asymptotic HF one-way solutions of the elastodynamic equation, concentrated close to the rays of P and S waves. In this case, the elastodynamic equation is reduced to a parabolic equation which further leads to a matrix Riccati equation for complex-valued matrix $\mathbf{M}$, and to the transport equation for the amplitude factor along the central ray. Finally, the matrix Riccati equation for $\mathbf{M}$ yields the dynamic ray tracing system for $\mathbf{Q}$ and $\mathbf{P}$.

For more details on paraxial Gaussian beams in isotropic media see Babich (1968), Kirpichnikova (1971), Babich and Buldyrev (1972), Babich and Kirpichnikova (1974), Babich and Popov (1981), Popov (1982), Červený, Popov and Pšenčík (1982), Červený and Pšenčík (1983a,b, 1984). For anisotropic media, see Hanyga (1986). By a simple extension of the above approach, we obtain *Hermite-Gaussian beams* which have a more complex amplitude profile in the plane perpendicular to the central ray, see Siegman (1973), Klimeš (1983). They represent "higher modes" of Gaussian beams.

Another extension of the paraxial Gaussian beams yields *paraxial Gaussian wave packets*, moving along rays. See Arnaud (1971a,b), Babich and Ulin (1981a,b), Ralston (1983), Katchalov (1984), Klimeš (1984b), Norris, White and Schrieffer (1987), Klimeš (1989a). For anisotropic media, see Norris (1987). The most natural way to derive and study the Gaussian wave packets is to use the space-time ray method.

Let us add one important note regarding Gaussian beams. Whereas rays are only *mathematical trajectories*, Gaussian beams may represent *physical objects*. They may be generated by physical sources and investigated experimentally. Particularly important in practical applications are very narrow beams. The disadvantage of the elastodynamic Gaussian beams discussed here is their large width in certain regions. There is no physical mechanism in our treatment which would keep the Gaussian beam narrow along the whole ray. Such mechanisms, however, exist, but are mostly non-linear. The most important non-linear mechanism is based on the dependence of the propagation velocity on amplitudes, assuming the amplitudes are large. The velocities of propagation decrease with increasing amplitudes. As the highest amplitudes are concentrated close to the center of the beam, a small low-velocity channel is formed along the central ray keeping the beam narrow even along long rays (waveguide effects).

# 4 Summation methods

The spherical wave in a homogeneous medium can be expressed as the superposition of plane waves using the Weyl integral, and as the superposition of cylindrical waves using the Sommerfeld integral. Both these superposition integrals are exact. These integrals play an important role in 1-D isotropic media, where they can be used to compute numerically and/or asymptotically the wave fields generated by a point source and the relevant Green functions. Using the elastodynamic representation theorems, the Green functions may further be applied to compute the wave field in 1-D isotropic models even in more general situations (volume sources, etc.).

In 2-D and 3-D, laterally varying, layered, isotropic or anisotropic structures, such exact integral expressions for the wave field generated by a point source are not generally available. It is, however, possible to construct useful expressions for the wave field by integral superposition of asymptotic ray-based solutions. These superposition integrals are not exact, but they often represent a uniform asymptotic solution of the problem under study, valid even in certain singular regions of the ray method.

The superposition integrals may be expressed either in the time domain or in the frequency domain. We shall present two general forms of the frequency-domain superposition integrals, which can be used in many applications. For time-domain versions of these integrals see Spencer, Chapman and Cragh (1997), Chapman (2000), Červený (2000), Section 6.2.6. The first of them is based on the summation of paraxial ray approximations. The time-domain versions of the resulting superposition integrals are then close or equal to the Maslov-Chapman integrals. In 1-D models, they yield the WKBJ integrals. The second form is based on the summation of paraxial Gaussian beams.

It should be emphasized that the individual contributions in the superposition integrals, corresponding to paraxial ray approximations or to paraxial Gaussian beams, represent approximate solutions of the elastodynamic equation. Consequently, the superposition integral also represents an approximate high-frequency solution of the elastodynamic equation.

Let us consider an elementary wave propagating in a laterally varying, layered, isotropic or anisotropic, 3-D structure, and the relevant orthonomic system of rays $\Omega(\gamma_1, \gamma_2)$, parameterized by two ray parameters $\gamma_1, \gamma_2$. On each ray, we specify one initial point $S_\gamma$, at which some initial conditions are specified. We assume that dynamic ray tracing has been performed along ray $\Omega(\gamma_1, \gamma_2)$ from initial point $S_\gamma$, and that the $4 \times 4$ ray propagator matrix $\mathbf{\Pi}(R_\gamma, S_\gamma)$ is known along $\Omega(\gamma_1, \gamma_2)$ at any point $R_\gamma$ situated on $\Omega(\gamma_1, \gamma_2)$. The initial points $S_\gamma$ of rays $\Omega(\gamma_1, \gamma_2)$ are distributed along a smooth initial surface $\Sigma^0$, along a smooth initial line $C^0$, or coincide at a common point $S^0$ (central ray field with a point source at $S^0$). We also assume that the $2 \times 2$ matrices $\mathbf{Q}^a(S_\gamma)$, $\mathbf{P}^a(S_\gamma)$ and $\mathbf{M}^a(S_\gamma) = \mathbf{P}^a(S_\gamma)\mathbf{Q}^{a-1}(S_\gamma)$, corresponding to the actual ray field $\Omega(\gamma_1, \gamma_2)$, are known at $S_\gamma$. To emphasize the fact that these matrices correspond to the actual ray field, we use "$a$" in the superscript.

With each ray $\Omega(\gamma_1, \gamma_2)$, we can connect a three-parameteric system of paraxial ray approximations, specified by a $2 \times 2$ real-valued symmetric matrix $\mathbf{M}(S_\gamma)$, and/or a six-parameteric system of paraxial Gaussian beams, specified by a $2 \times 2$ complex-valued symmetric matrix $\mathbf{M}(S_\gamma)$ with a positive definite imaginary part, see Sections 2 and

3. The $2 \times 2$ matrices $\mathbf{Q}(S_\gamma)$, $\mathbf{P}(S_\gamma)$ and $\mathbf{M}(S_\gamma)$ correspond to any of these paraxial solutions. Let us emphasize that $\mathbf{Q}^a(S_\gamma)$, $\mathbf{P}^a(S_\gamma)$ and $\mathbf{M}^a(S_\gamma)$ correspond to the actual ray field, and are fixed for the elementary wave under consideration. Matrices $\mathbf{Q}(S_\gamma)$, $\mathbf{P}(S_\gamma)$ and $\mathbf{M}(S_\gamma)$, however, should be specified in some other way. Their selection corresponds to the selection of the paraxial ray approximation or paraxial Gaussian beams we wish to use in the superposition integrals. In all superposition integrals we shall present, it is sufficient to specify $\mathbf{M}(S_\gamma)$, not $\mathbf{Q}(S_\gamma)$ and $\mathbf{P}(S_\gamma)$. We shall show, however, that $\mathbf{M}(S_\gamma)$ must be specified in such way that $\mathbf{M}(S_\gamma) \neq \mathbf{M}^a(S_\gamma)$.

## 4.1 Superposition integrals

We wish to determine the wave field of the elementary wave $\vec{u}(R, \omega)$ at a fixed receiver point $R$. We do not have to know the ray which passes through $R$. The wavefield at $R$ is calculated by a weighted superposition of paraxial ray approximations or paraxial Gaussian beams connected with rays $\Omega(\gamma_1, \gamma_2)$ passing in the vicinity of $R$. Consequently, two-point ray tracing is not required in the whole procedure.

In the frequency domain, the superposition integral for the Cartesian component $u_i^{(x)}(R, \omega)$ of the displacement vector $\vec{u}(R, \omega)$ reads:

$$u_i^{(x)}(R, \omega) = \iint_{\mathcal{D}} \Psi(\gamma_1, \gamma_2) U_i^{(x)}(R_\gamma) \exp[\mathrm{i}\omega T(R, R_\gamma)] \mathrm{d}\gamma_1 \mathrm{d}\gamma_2 \; . \tag{4.1}$$

Factor $\exp[-\mathrm{i}\omega t]$ is omitted in (4.1). The integral is over the rays specified by ray parameters $\gamma_1, \gamma_2$; $\mathcal{D}$ denotes the region of ray parameters under consideration. We will not be too concerned with the size of the region $\mathcal{D}$; it may be different in different problems. However, we have to remember that the boundaries of region $\mathcal{D}$ may generate spurious arrivals in the computation of synthetic seismograms; see Thomson and Chapman (1986) for a detailed treatment. Function $\Psi(\gamma_1, \gamma_2)$ is the weighting function and its various forms will be given later. Point $R_\gamma$ is situated on the same ray $\Omega(\gamma_1, \gamma_2)$ as $S_\gamma$. It is convenient to choose $R_\gamma$ on $\Omega(\gamma_1, \gamma_2)$ as close to the fixed point $R$ as possible. The function $U_i^{(x)}(R_\gamma) \exp[\mathrm{i}\omega T(R, R_\gamma)]$ in (4.1) represents the paraxial ray approximation (or the paraxial Gaussian beam) connected with ray $\Omega(\gamma_1, \gamma_2)$. To calculate $U_i^{(x)}(R_\gamma)$ and $T(R, R_\gamma)$, we can use (2.2)–(2.7). The travel-time function $T(R, R_\gamma)$ represents the travel time at $R$, calculated by the paraxial ray methods from the travel time $T(R_\gamma)$ at $R_\gamma$, situated on a near-by ray $\Omega(\gamma_1, \gamma_2)$.

Let us emphasize again that the ray passing through $R$ does not have to be known. Actually, such a ray need not exist, for example if $R$ is situated in the shadow region of the elementary wave under consideration. On the contrary, several such rays may exist in the case of multipathing. In both these cases, superposition integral (4.1) is well defined.

In the actual computations, $T(R, R_\gamma)$ may be represented in various coordinate systems. As an example, see (2.5) with (2.6) and (2.7) in Cartesian coordinates. It is customary, but not necessary to introduce a "target" surface $\Sigma^R$ passing through $R$, and consider points $R_\gamma$ situated on this surface $\Sigma^R$. For more details refer to Section 4.3.

Integral (4.1) expresses the wave field at point $R$ by a weighted superposition of the paraxial ray approximations or paraxial Gaussian beams, connected with the neighbouring rays, passing close to $R$. Due to this, we can expect the integrals to smooth the singular behavior in caustic region, critical regions, etc. Moreover, we can expect the summation of synthetic seismograms not to be as sensitive to the minor details in the approximation of the structural model as the standard ray synthetic seismograms. This is, in fact, the reason why the superposition integrals are used.

The superposition integral (4.1) may be also expressed in a slightly different, alternative form. We use the standard zero-order ray-theory complex-valued vectorial amplitude $U_i^{ray}(R_\gamma)$, and take into account that

$$U_i^{(x)}(R_\gamma) = \Psi_0(\gamma_1, \gamma_2) U_i^{ray}(R_\gamma) \ , \tag{4.2}$$

where

$$\Psi_0(\gamma_1, \gamma_2) = \left[\frac{\det \mathbf{Q}^a(R_\gamma)}{\det \mathbf{Q}^a(S_\gamma)}\right]^{1/2} \left[\frac{\det \mathbf{Q}(S_\gamma)}{\det \mathbf{Q}(R_\gamma)}\right]^{1/2} \ . \tag{4.3}$$

The arguments of both square roots in (4.3) are determined in a standard way. The superposition integral (4.1) then reads

$$u_i^{(x)}(R, \omega) = \iint_{\mathcal{D}} \Phi(\gamma_1, \gamma_2) U_i^{ray}(R_\gamma) \exp[\mathrm{i}\omega T(R, R_\gamma)] \mathrm{d}\gamma_1, \mathrm{d}\gamma_2 \ . \tag{4.4}$$

The relation between the weighting functions $\Psi(\gamma_1, \gamma_2)$ in (4.1) and $\Phi(\gamma_1, \gamma_2)$ in (4.4) is as follows:

$$\Phi(\gamma_1, \gamma_2) = \Psi(\gamma_1, \gamma_2) \Psi_0(\gamma_1, \gamma_2) \ . \tag{4.5}$$

In the following, we shall mostly use the superposition integral (4.4) and determine directly the weighting function $\Phi(\gamma_1, \gamma_2)$. Strictly speaking, the superposition integral (4.4) does not represent the expansion into paraxial ray approximations or into paraxial Gaussian beams, as the ray-theory amplitude factor $U_i^{ray}(R_\gamma)$ is different from $U_i^{(x)}(R_\gamma)$. The differences in amplitudes are, however, only formal, see (4.2), and are compensated by the weighting functions. The most important exponential factor $\exp[\mathrm{i}\omega T(R, R_\gamma)]$ in (4.4) is exactly the same as in (4.1). For this reason, we shall continue to speak on the summation of paraxial ray approximations and on the summation of paraxial Gaussian beams, even if we consider the superposition integral (4.4).

## 4.2 Determination of the weighting function $\Phi(\gamma_1, \gamma_2)$

We shall now present a very simple derivation of the weighting function $\Phi(\gamma_1, \gamma_2)$, based on the asymptotic high-frequency treatment of (4.4). In regular regions of the ray field, the asymptotic high-frequency treatment of superposition integral (4.4) should yield the same result as the standard ray method. Consequently, $\Phi(\gamma_1, \gamma_2)$ can be determined by matching both these solutions.

If we wish to determine weighting function $\Phi(\gamma_1, \gamma_2)$, we can assume point $R$ to be situated in a region covered regularly by rays $\Omega(\gamma_1, \gamma_2)$. One of these rays passes through point $R$. We denote this ray by $\Omega_0$, the relevant ray parameters by $\gamma_{10}, \gamma_{20}$, and the initial point of $\Omega_0$ on $\Sigma^0$ by $S$.

In the determination of the weighting function $\Phi(\gamma_1, \gamma_2)$, we shall use the wavefront orthonormal coordinates $y_i$, with $T(R, R_\gamma)$ given by (2.4). The reason is that the solution is formally very simple and straightforward in this case. We do not loose any generality by this approach. Appropriate paraxial modifications can be made later in the resulting superposition integral. Particularly, we can express $T(R,R_\gamma)$ in Cartesian coordinates and use (2.5), with (2.6) and (2.7) in these integrals.

Before we evaluate superposition integral (4.4) for high frequencies $\omega$, we shall transform it to a more useful form, suitable for its asymptotic evaluation. As the travel time $T(R_\gamma)$ in the expression for $T(R,R_\gamma)$ given by (2.4) is different for different rays $\Omega(\gamma_1,\gamma_2)$, we shall express it in terms of $T(R)$. Point $R$ is pressumably situated in the regular ray region, so that we can use

$$T(R) = T(R_\gamma) + \tfrac{1}{2}\mathbf{y}^T(R)\mathbf{M}^a(R_\gamma)\mathbf{y}(R) \ ,$$

where $\mathbf{M}^a(R_\gamma)$ corresponds to the actual ray field. Inserting this into expression (2.4) for $T(R,R_\gamma)$, $T(R,R_\gamma) = T(R_\gamma) + \frac{1}{2}\mathbf{y}^T(R)\mathbf{M}(R_\gamma)\mathbf{y}^R(R)$, yields

$$T(R,R_\gamma) = T(R) + \tfrac{1}{2}\mathbf{y}^T(R)[\mathbf{M}(R_\gamma) - \mathbf{M}^a(R_\gamma)]\mathbf{y}(R) \ .$$

The superposition integral (4.4) then reads

$$u_i^{(x)}(R,\omega) = \exp[\mathrm{i}\omega T(R)] \iint_{\mathcal{D}} \Phi(\gamma_1,\gamma_2) U_i^{ray}(R_\gamma) \exp[\tfrac{1}{2}\mathrm{i}\omega \mathbf{y}^T(R)\boldsymbol{\mathcal{M}}(R_\gamma)\mathbf{y}(R)]\mathrm{d}\gamma_1\mathrm{d}\gamma_2 \ , \tag{4.6}$$

where the $2\times 2$ matrix $\boldsymbol{\mathcal{M}}$ is given by the relation

$$\boldsymbol{\mathcal{M}}(R_\gamma) = \mathbf{M}(R_\gamma) - \mathbf{M}^a(R_\gamma) \ . \tag{4.7}$$

This integral, valid in regular ray regions, is very general. It may be used to calculate weighting function $\Phi(\gamma_1,\gamma_2)$ both for isotropic and anisotropic media, for the summation of paraxial ray approximations (real-valued $\mathbf{Q},\mathbf{P},\mathbf{M}$) and for the summation of paraxial Gaussian beams (complex-valued $\mathbf{Q},\mathbf{P},\mathbf{M}$).

For high-frequencies $\omega$, the main contribution of the superposition integral (4.6) comes from the vicinity of ray $\Omega_0(\gamma_{10},\gamma_{20})$, passing through the receiver point $R$. Consequently, we can put approximately $U_i^{ray}(R_\gamma) \doteq U_i^{ray}(R)$, and $\Phi(\gamma_1,\gamma_2) \doteq \Phi(\gamma_{10},\gamma_{20})$. As a target surface $\Sigma^R$, we shall use the plane $y_1,y_2$, tangent to the wavefront at $R$, with the origin at $R$. In this case, we can use the following relation for $T(R,R_\gamma)$:

$$T(R,R_\gamma) = T(R) + \tfrac{1}{2}\mathbf{y}^T(R_\gamma)\boldsymbol{\mathcal{M}}(R)\mathbf{y}(R_\gamma) \ .$$

The superposition integral (4.4) can be then approximately expressed in the following way:

$$\begin{aligned} u_i^{(x)}(R,\omega) &\doteq \exp[\mathrm{i}\omega T(R)]\ \Phi(\gamma_{10},\gamma_{20})\ U_i^{ray}(R) \\ &\times \iint_{\mathcal{D}} \exp[\tfrac{1}{2}\mathrm{i}\omega \mathbf{y}^T(R_\gamma)\boldsymbol{\mathcal{M}}(R)\mathbf{y}(R_\gamma)]\mathrm{d}\gamma_1\mathrm{d}\gamma_2 \ . \end{aligned} \tag{4.8}$$

To compute (4.8), we shall exploit the known integral, see Appendix A:

$$\iint_{-\infty}^{\infty} \exp[\tfrac{1}{2}\mathrm{i}\omega \mathbf{y}^T\mathbf{W}\mathbf{y}]\mathrm{d}y_1\mathrm{d}y_2 = (2\pi/\omega)[-\det\mathbf{W}]^{-1/2} \ . \tag{4.9}$$

Here $y_1,y_2$ are Cartesian coordinates, $\mathbf{y} = (y_1,y_2)^T$ and $\mathbf{W}$ is a constant $2\times 2$ matrix with $\det\mathbf{W} \neq 0$. Matrix $\mathbf{W}$ may be also complex-valued, with a positive-definite imaginary part. The argument of $[-\det \mathbf{W}]^{-1/2}$ in (4.9) is given by the following relations, see (A.22):

$$\begin{aligned} &\mathrm{Re}[-\det \mathbf{W}]^{1/2} > 0 && \text{for } \mathrm{Im}\mathbf{W} \neq \mathbf{0} \ , \\ &[-\det \mathbf{W}]^{1/2} = |\det \mathbf{W}|^{1/2} \exp[-\mathrm{i}\tfrac{\pi}{4}\mathrm{Sgn}\mathbf{W}] && \text{for } \mathrm{Im}\mathbf{W} = \mathbf{0} \ . \end{aligned} \tag{4.10}$$

As usual, Sgn$\mathbf{W}$ denotes the signature of the real-valued matrix $\mathbf{W}$; it equals the number of its positive eigenvalues minus the number of its negative eigenvalues. Thus, it equals 2, 0 or -2.

Several comments to (4.9). For real-valued $\mathbf{W}$, the result (4.9) with (4.10) is well known from the method of stationary phase, see Bleistein (1984). For complex-valued $\mathbf{W}$, the computation of (4.9) is based on a simultaneous diagonalization of $2\times 2$ matrices Re$\mathbf{W}$ and Im$\mathbf{W}$, with Im$\mathbf{W}$ positive definite, and on a consequent transformation of the double integral (4.9) into a product of two relevant single integrals. Such a diagonalization is possible if Im$\mathbf{W}$ is positive definite, but this is just the case of paraxial Gaussian beams. For more details on the simultaneous diagonalization of $\mathbf{W}$ and on the computation of (4.9) see Appendix A.

Transforming the integration variables $\gamma_1, \gamma_2$ in (4.8) into integration variables $y_1, y_2$, and using (4.9), we obtain

$$u_i^{(x)}(R,\omega) \doteq (2\pi/\omega)\Phi(\gamma_{10},\gamma_{20})[-\det \boldsymbol{\mathcal{M}}(R)]^{-1/2}|\det \mathbf{Q}^a(R)|^{-1} U_i^{ray}(R) \exp[\mathrm{i}\omega T(R)] \ . \tag{4.11}$$

Matching (4.11) with the standard zero-order ray-theory solution $u_i^{(x)}(R,\omega) = U_i^{ray}(R)\exp[\mathrm{i}\omega T(R)]$ yields the following expression for the weighting function $\Phi(\gamma_{10},\gamma_{20})$:

$$\Phi(\gamma_{10},\gamma_{20}) = (\omega/2\pi)[-\det \boldsymbol{\mathcal{M}}(R)]^{1/2}|\det \mathbf{Q}^a(R)| \ . \tag{4.12}$$

As point $R$ may be situated on an arbitrary ray $\Omega(\gamma_1,\gamma_2)$, we can use the same expression for point $R_\gamma$, situated on the ray $\Omega(\gamma_1,\gamma_2)$,

$$\Phi(\gamma_1,\gamma_2) = (\omega/2\pi)[-\det \boldsymbol{\mathcal{M}}(R_\gamma)]^{1/2}|\det \mathbf{Q}^a(R_\gamma)| \ . \tag{4.13}$$

This is the final expression for $\Phi(\gamma_1,\gamma_2)$. The argument of $[-\det \boldsymbol{\mathcal{M}}(R_\gamma)]^{1/2}$ is given by (4.10). Inserting (4.13) with (2.4) or (2.5) into (4.4) yields the final form of the superposition integral. See Sections 5 and 6 for its more detailed discussion.

The integral (4.9) cannot be applied to the computation of the superposition integral (4.4) if $\det \boldsymbol{\mathcal{M}}(R) = 0$. In other words, the results are not valid for receiver points $R$, satisfying the relation

$$\det[\mathbf{M}(R) - \mathbf{M}^a(R)] = 0 \ , \tag{4.14}$$

see (4.7). Receiver points $R$, at which (4.14) is satisfied, are usually called the *pseudocaustic points*. The pseudocaustic points play an important role only in the superposition of paraxial ray approximations.

## 4.3 Travel-time function $T(R, R_\gamma)$

The most flexible form of the superposition integral (4.4) is obtained if the travel-time function $T(R,R_\gamma)$ is expressed in Cartesian coordinates:

$$T(R,R_\gamma) = T(R_\gamma) + x_i(R,R_\gamma)p_i^{(x)}(R_\gamma) + \tfrac{1}{2}x_i(R,R_\gamma)x_j(R,R_\gamma)M_{ij}^{(x)}(R_\gamma) \ , \tag{4.15}$$

see (2.5). The $3 \times 3$ matrix $\hat{\mathbf{M}}^{(x)}(R_\gamma)$ is expressed in terms of the $2 \times 2$ matrix $\mathbf{M}(R_\gamma)$ by (2.6) and (2.7). The $2 \times 2$ matrix $\mathbf{M}(R_\gamma)$ is free and may be chosen in various ways. See Section 4.4.

In the framework of the paraxial ray methods, points $R_\gamma$ may be chosen arbitrarily on rays $\Omega(\gamma_1, \gamma_2)$, but close to $R$. The only requirement is that the distance $|\vec{x}(R) - \vec{x}(R_\gamma)|$ is small and that the terms higher than quadratic may be neglected.

Commonly, some target surface $\Sigma^R$ is introduced, along which the endpoints $R_\gamma$ of rays $\Omega(\gamma_1, \gamma_2)$ and the receiver $R$ are situated. However, the superposition integral (4.4), with (4.13) and (4.15), can be used even in considerably more general configurations. For example, we can consider a target surface $\Sigma^R$, along which only the endpoints $R_\gamma$ of rays $\Omega(\gamma_1, \gamma_2)$ are situated, but not the receiver $R$. Then the superposition integral (4.4) remains valid even for receivers situated outside the target surface $\Sigma^R$ (but close to it). Actually, the target surface need not be constructed at all, if the quantities required in the superposition integral are stored at sufficiently dense systems of discrete points along rays $\Omega(\gamma_1, \gamma_2)$. Then the superposition integral (4.4) may be applied to a receiver situated arbitrarily in the model. The points $R_\gamma$ may be suitably selected among discrete points on rays $\Omega(\gamma_1, \gamma_2)$ passing close to receiver $R$. At each ray $\Omega(\gamma_1, \gamma_2)$, we choose one point $R_\gamma$. For example, we can take the discrete point $R_\gamma$ for which $|\vec{x}(R) - \vec{x}(R_\gamma)|$ is the smallest.

## 4.4 Specification of matrix M

Superposition integral (4.4) is influenced by the choice of the $2 \times 2$ matrix $\mathbf{M}$. As explained above, $\mathbf{M}$ may be specified at $S_\gamma$, at $R_\gamma$, or at any other point on ray $\Omega(\gamma_1, \gamma_2)$. Using the ray propagator matrix, $\mathbf{M}$ can be determined at any point of ray $\Omega(\gamma_1, \gamma_2)$, as soon as it is known at any other point of the ray. It is most common to specify it at $R_\gamma$ or $S_\gamma$. If we specify it at $R_\gamma$, it is possible to use the superposition integral (4.4) with (4.15) and (4.13) directly. If it is specified at $S_\gamma$, we have to supplement the above equations with the continuation relations (2.8).

We shall first present two simple examples of summation integrals with $\mathbf{M}$ specified at $S_\gamma$. For the expansion of the wave field at a smooth initial surface $\Sigma^0$ into locally plane waves, we use $\mathbf{M}(S_\gamma) = \mathbf{0}$. If we wish to expand the same wave field into locally plane waves with a Gaussian amplitude profiles, we use $\mathrm{Re}\mathbf{M}(S_\gamma) = \mathbf{0}$ and $\mathrm{Im}\mathbf{M}(S_\gamma)$ positive definite. The same choices can be used in the expansion of the wave field generated by a point source into locally plane waves, or into locally plane waves with Gaussian amplitude profiles.

It is however, very common to choose $\mathbf{M}$ at points $R_\gamma$. We can again consider locally plane waves at $R_\gamma$, with $\mathbf{M}(R_\gamma) = \mathbf{0}$, or locally plane waves with a Gaussian amplitude windowing, using $\mathrm{Re}\mathbf{M}(R_\gamma) = \mathbf{0}$, $\mathrm{Im}\mathbf{M}(R_\gamma) \neq \mathbf{0}$. In this case, $\mathrm{Im}\mathbf{M}(R_\gamma)$ must be positive definite.

Let us now consider a smoothly curved target surface $\Sigma^R$, and assume that all termination points $R_\gamma$ of rays $\Omega(\gamma_1, \gamma_2)$ are situated on this surface. Surface $\Sigma^R$ may represent a formal surface in a smooth medium, a structural interface inside the model, a free surface of the model, etc. Note that the receiver point $R$ need not be situated on the target surface $\Sigma^R$, but must be close to it.

Before we describe a suitable choice of $\mathrm{Re}\mathbf{M}(R_\gamma)$ at the target surface $\Sigma^R$, we shall present one approximate equation for $T(R, R_\gamma)$, which will be useful in the following. The travel time expansion (4.15) may be used for local Cartesian coordinates $z_i$ introduced as follows: The origin of the local Cartesian coordinate system $z_i$ is situated at $R_\gamma$ on $\Sigma^R$, with the axes $z_1, z_2$ situated in a plane tangent to the target surface $\Sigma^R$ at $R_\gamma$. The position of receiver point $R$ on the target surface $\Sigma^R$ is then specified by $z_i(R)$, with $z_3(R)$ pressumably small. The quadratic expansion of the travel-time function $T(R, R_\gamma)$ in terms of $z_I, z_J$ is for small $z_3(R)$ approximately given by the relation, see (4.15),

$$\begin{aligned} T(R, R_\gamma) &= T(R_\gamma) + \mathbf{z}^T(R)\mathbf{p}^{(z)}(R_\gamma) + \tfrac{1}{2}\mathbf{z}^T(R)\mathbf{F}(R_\gamma)\mathbf{z}(R) \\ &= T(R_\gamma) + \hat{\mathbf{z}}^T(R)\hat{\mathbf{p}}^{(z)}(R_\gamma) + \tfrac{1}{2}\mathbf{z}^T(R)\bar{\mathbf{F}}(R_\gamma)\mathbf{z}(R) \ . \end{aligned} \qquad (4.16)$$

Here the $2 \times 2$ matrix $\mathbf{F}(R_\gamma)$ is given by the relation, which immediately follows from (4.15) for $z_3 = -\frac{1}{2}z_I z_J D_{IJ}(R_\gamma)$,

$$\mathbf{F}(R_\gamma) = (\mathbf{G} - \mathbf{A}^{an})\mathbf{M}(\mathbf{G} - \mathbf{A}^{an})^T + \mathbf{E} - p_3^{(z)}\mathbf{D} \ . \qquad (4.17)$$

See also Červený (2000), Eq. (4.14.54). All quantities in (4.17) are taken at $R_\gamma$. The $2 \times 2$ matrices $\mathbf{G}(R_\gamma)$, $\mathbf{A}^{an}(R_\gamma)$, $\mathbf{E}(R_\gamma)$ and $\mathbf{D}(R_\gamma)$ have the following meaning:

The $3 \times 3$ matrix $\hat{\mathbf{G}}$ is a transformation matrix from wavefront orthonormal coordinates $y_l$ to the local Cartesian coordinates $z_k$, $G_{kl} = \partial z_k/\partial y_l = \partial y_l/\partial z_k$.

The $2 \times 2$ "anisotropy" matrix $\mathbf{A}^{an}$ is given by the relation $A^{an}_{IJ} = p_I^{(z)}\mathcal{U}_n^{(y)}$, where $p_I^{(z)}$ are $z_I$-components of the slowness vector $\vec{p}$.

The $2 \times 2$ "inhomogeneity" matrix $\mathbf{E}$ is given by the relation

$$E_{IJ} = \mathcal{C}^{-1}[G_{I3}G_{JM}\eta_M^{(y)} + G_{IK}G_{J3}\eta_K^{(y)} + G_{I3}G_{J3}(\eta_3^{(y)} - \mathcal{C}^{-1}\eta_N^{(y)}\mathcal{U}_N^{(y)})] \ . \qquad (4.18)$$

Here $\eta_i^{(y)} = H_{ni}\eta_n^{(x)}$, $\mathcal{U}_i^{(y)} = H_{ni}\mathcal{U}_n^{(x)}$, and $\mathcal{U}_i^{(x)}$ and $\eta_i^{(x)}$ are known from ray tracing. See also (1.4) and (1.8). .

An alternative expression for the inhomogeneity matrix $\mathbf{E}$, expressed in Cartesian coordinates, is as follows:

$$E_{IJ} = Z_{kI}Z_{jJ}[p_k^{(x)}\eta_j^{(x)} + p_j^{(x)}\eta_k^{(x)} - p_j^{(x)}p_k^{(x)}\eta_i^{(x)}\mathcal{U}_i^{(x)}] \ . \qquad (4.19)$$

Here $Z_{kl}$ are components of the $3 \times 3$ transformation matrix $\hat{\mathbf{Z}}$ from local Cartesian coordinates $z_i$ to the general Cartesian coordinates $x_i$, $Z_{kl} = \partial x_k/\partial z_l = \partial z_l/\partial x_k$.

Finally, $\mathbf{D}$ is the $2 \times 2$ curvature matrix of the target surface $\Sigma^R$.

Similarly, $\bar{\mathbf{F}}(R_\gamma)$ is given by (4.17), with the last term excluded,

$$\bar{\mathbf{F}}(R_\gamma) = (\mathbf{G} - \mathbf{A}^{an})\mathbf{M}(\mathbf{G} - \mathbf{A}^{an})^T + \mathbf{E} \ . \qquad (4.20)$$

The relation between the two equations (4.17) and (4.20) is obvious, if we take into account that $z_3 = -\frac{1}{2}z_I z_J D_{IJ}(R_\gamma)$.

Returning back from local Cartesian coordinates $z_i$ to the general Cartesian coordinates $x_i$ in the second equation of (4.16), we again obtain a simple expression (4.15) for $T(R, R_\gamma)$, where $M_{ij}^{(x)}(R_\gamma)$ is given by the relation:

$$M_{ij}^{(x)}(R_\gamma) = Z_{iI}(R_\gamma)Z_{jJ}(R_\gamma)\bar{F}_{IJ}(R_\gamma) \ . \qquad (4.21)$$

Here $Z_{kl}$ are again the components of the $3 \times 3$ transformation matrix $\hat{\mathbf{Z}}$.

Note that the matrix $\bar{\mathbf{F}}(R_\gamma)$ in (4.21) is free and may be chosen arbitrarily, instead of $\mathbf{M}(R_\gamma)$. The only difference is that $\mathbf{M}(R_\gamma)$ is related to the plane tangent to the wavefront at $R_\gamma$, but $\bar{\mathbf{F}}(R_\gamma)$ is related to the plane tangent to the target surface $\Sigma^R$. The transformation matrix $\hat{\mathbf{Z}}$ merely specifies the frame in which $\bar{\mathbf{F}}(R_\gamma)$ is expressed.

Using (4.17) or (4.20), we can also express $\boldsymbol{\mathcal{M}}(R_\gamma)$ in a useful form, see (4.7):

$$\boldsymbol{\mathcal{M}} = (\mathbf{G}-\mathbf{A}^{an})^{-1}(\mathbf{F}-\mathbf{F}^{a})(\mathbf{G}-\mathbf{A}^{an})^{-1T} = (\mathbf{G}-\mathbf{A}^{an})^{-1}(\bar{\mathbf{F}}-\bar{\mathbf{F}}^{a})(\mathbf{G}-\mathbf{A}^{an})^{-1T} \ . \quad (4.22)$$

The superscript "$a$" again specifies the actual ray field. Note also that $\det(\mathbf{G}-\mathbf{A}^{an}) = (\mathcal{U}/\mathcal{C})\cos\delta$, where $\delta$ is the angle between the ray and the normal to $\Sigma^R$ at $R_\gamma$.

Now we shall discuss in a greater detail a choice of $\mathrm{Re}\mathbf{M}(R_\gamma)$, which removes approximately the quadratic terms from the expansion (4.16) of $\mathrm{Re}T(R,R_\gamma)$, if the receiver $R$ is situated on the target surface $\Sigma^R$. We consider (4.16) and choose $\mathrm{Re}\mathbf{M}(R_\gamma)$ in such a way that $\mathrm{Re}\bar{\mathbf{F}}(R_\gamma) = \mathbf{0}$. We take into account (4.20), and obtain

$$\mathrm{Re}\mathbf{M}(R_\gamma) = -(\mathbf{G}-\mathbf{A}^{an})^{-1}\mathbf{E}(\mathbf{G}-\mathbf{A}^{an})^{-1T} \ . \quad (4.23)$$

All matrices in (4.23) are again taken at $R_\gamma$, and have the same meaning as in (4.17)–(4.20).

An important note: If the specification (4.23) is used, the quadratic terms from the expansion (4.16) of $\mathrm{Re}T(R,R_\gamma)$ is fully removed only if the receiver $R$ is also situated on the target surface $\Sigma^R$. For receivers $R$ situated outside $\Sigma^R$ (but close to it), the specification is again useful, as it supresses considerably $\mathrm{Re}T(R,R_\gamma)$. It, however, does not remove $\mathrm{Re}T(R,R_\gamma)$ completely.

The choice (4.23) also removes approximately the quadratic terms from the expansion of $\mathrm{Re}T(R,R_\gamma)$ in Cartesian coordinates (4.15), so that (4.15) reads

$$\begin{aligned} T(R,R_\gamma) \ &\doteq \ T(R_\gamma) + x_i(R,R_\gamma)p_i^{(x)}(R_\gamma) \\ &+ \ \tfrac{1}{2}\mathrm{i}x_i(R,R_\gamma)x_j(R,R_\gamma)\mathrm{Im}M_{ij}^{(x)}(R_\gamma) \ . \end{aligned} \quad (4.24)$$

Thus, if the summation of paraxial ray approximation is considered, (4.24) yields a very simple relation

$$T(R,R_\gamma) = T(R_\gamma) + x_i(R,R_\gamma)p_i^{(x)}(R_\gamma) \ . \quad (4.25)$$

This is a well-known ansatz expression, used broadly in the Maslov-Chapman method, and corresponding to transform methods. In the summation of paraxial Gaussian beams, $\mathrm{Im}\hat{\mathbf{M}}^{(x)}(R_\gamma) \neq \mathbf{0}$ in (4.24). We can choose a positive-definite $2 \times 2$ matrix $\mathrm{Im}\mathbf{M}(R_\gamma)$ arbitrarily, and use (2.6) to determine $\mathrm{Im}\hat{\mathbf{M}}^{(x)}(R_\gamma)$. Matrix $\mathrm{Im}\mathbf{M}(R_\gamma)$ controls the width of Gaussian beams under consideration. For infinitely broad paraxial Gaussian beams, $\mathrm{Im}\mathbf{M}(R_\gamma) = \mathbf{0}$, and the superposition of paraxial Gaussian beams becomes the superposition of paraxial ray approximations.

# 5 Superposition integrals: wavefront orthonormal coordinates

In this section, we shall present the final form of the superposition integral, assuming that the dynamic ray tracing along rays is performed in wavefront orthonormal coordinates $y_i$ (or in ray-centered coordinates $q_i$ in isotropic media). The results of dynamic ray tracing are the $2 \times 2$ matrices $\mathbf{Q}^a$, $\mathbf{P}^a$ and $\mathbf{M}^a = \mathbf{P}^a \mathbf{Q}^{a-1}$. The superposition integral is as follows, see (4.4), (4.10), (4.13) and (4.15):

$$u_i^{(x)}(R,\omega) = \frac{\omega}{2\pi} \iint_{\mathcal{D}} U_i^{ray}(R_\gamma)[-\det \boldsymbol{\mathcal{M}}(R_\gamma)]^{1/2} |\det \mathbf{Q}^a(R_\gamma)| \exp[\mathrm{i}\omega T(R,R_\gamma)] \mathrm{d}\gamma_1 \mathrm{d}\gamma_2 \ . \tag{5.1}$$

Here

$$\boldsymbol{\mathcal{M}}(R_\gamma) = \mathbf{M}(R_\gamma) - \mathbf{M}^a(R_\gamma) \ , \tag{5.2}$$

and the argument of $[-\det \boldsymbol{\mathcal{M}}(R_\gamma)]^{1/2}$ is given by relations, see (4.10):

$$\begin{array}{ll} \mathrm{Re}[-\det \boldsymbol{\mathcal{M}}]^{1/2} > 0 & \text{for } \mathrm{Im}\boldsymbol{\mathcal{M}} \neq \mathbf{0} \ , \\ [-\det \boldsymbol{\mathcal{M}}]^{1/2} = |\det \boldsymbol{\mathcal{M}}|^{1/2} \exp[-\mathrm{i}\frac{\pi}{4}\mathrm{Sgn}\boldsymbol{\mathcal{M}}] & \text{for } \mathrm{Im}\boldsymbol{\mathcal{M}} = \mathbf{0} \ . \end{array} \tag{5.3}$$

The travel-time function $T(R,R_\gamma)$, expressed in Cartesian coordinates, is as follows:

$$T(R,R_\gamma) = T(R_\gamma) + x_i(R,R_\gamma) p_i^{(x)}(R_\gamma) + \tfrac{1}{2} x_i(R,R_\gamma) x_j(R,R_\gamma) M_{ij}^{(x)}(R_\gamma) \ . \tag{5.4}$$

The $3 \times 3$ matrix $M_{ij}^{(x)}(R_\gamma)$ is given by relations,

$$\hat{\mathbf{M}}^{(x)}(R_\gamma) = \hat{\hat{\mathbf{H}}}(R_\gamma) \begin{pmatrix} \multicolumn{2}{c}{\mathbf{M}(R_\gamma)} & \begin{matrix} (\mathcal{C}^{-1}\eta_1^{(y)})_{R_\gamma} \\ (\mathcal{C}^{-1}\eta_2^{(y)})_{R_\gamma} \end{matrix} \\ (\mathcal{C}^{-1}\eta_1^{(y)})_{R_\gamma} & (\mathcal{C}^{-1}\eta_2^{(y)})_{R_\gamma} & (\mathcal{C}^{-2}\eta_i^{(y)}\mathcal{U}_i^{(y)})_{R_\gamma} \end{pmatrix} \hat{\hat{\mathbf{H}}}^T(R_\gamma) \ , \tag{5.5}$$

$$\hat{\hat{\mathbf{H}}} = \begin{pmatrix} (e_{11} - \mathcal{C}^{-1}\mathcal{U}_1^{(y)} e_{31})_{R_\gamma} & (e_{21} - \mathcal{C}^{-1}\mathcal{U}_2^{(y)} e_{31})_{R_\gamma} & e_{31}(R_\gamma) \\ (e_{12} - \mathcal{C}^{-1}\mathcal{U}_1^{(y)} e_{32})_{R_\gamma} & (e_{22} - \mathcal{C}^{-1}\mathcal{U}_2^{(y)} e_{32})_{R_\gamma} & e_{32}(R_\gamma) \\ (e_{13} - \mathcal{C}^{-1}\mathcal{U}_1^{(y)} e_{33})_{R_\gamma} & (e_{23} - \mathcal{C}^{-1}\mathcal{U}_2^{(y)} e_{33})_{R_\gamma} & e_{33}(R_\gamma) \end{pmatrix} \ . \tag{5.6}$$

see (2.6) and (2.7). The wavefront orthonormal components of $\vec{\mathcal{U}}$ and $\vec{\eta}$, $\mathcal{U}_i^{(y)}$ and $\eta_i^{(y)}$, are given by (1.8) with (1.4), and $e_{ij}$ by (1.7). If a smoothly curved target surface $\Sigma^R$ is considered, we can use a simple relation (4.21) for $\hat{\mathbf{M}}^{(x)}(R_\gamma)$, instead of (5.5) and (5.6).

The superposition integral (5.1), with (5.2)–(5.6), is valid for any inhomogeneous anisotropic elastic layered structure, which may be bounded or unbounded. It may be, however, applied even to isotropic elastic media and to fluid media. For isotropic media, we perform dynamic ray tracing in ray-centered coordinates, and put $\mathcal{U}_I^{(y)} = 0$ in (5.5) and (5.6). For pressure waves in fluid models, we only replace vectorial components $u_i^{(x)}(R,\omega)$ and $U_i^{ray}(R_\gamma)$ by scalars $p(R,\omega)$ and $P^{ray}(R_\gamma)$.

Actually, the superposition integral (5.1) with (5.2)–(5.6) can be applied to any other type of wavefield, for which the ray-theory amplitudes $U_i^{ray}(R_\gamma)$ can be computed, and the dynamic ray tracing can be performed along rays to compute $\mathbf{Q}^a(R_\gamma)$ and $\mathbf{M}^a(R_\gamma)$. For example, this applies to radio waves.

The wave field under consideration may correspond to any multiply-reflected zero-order ray-theory body wave. It may be also converted at structural interfaces. The wave may be generated at a smoothly curved initial surface, at a smooth initial line, or by a point source.

The superposition integral (5.1) may be applied to receiver $R$ situated arbitrarily in the model, including structural interfaces or Earth's surface. The expression for the ray amplitude $U_i^{ray}(R_\gamma)$ must contain a proper conversion coefficient in this case. The receiver $R$ may be also situated in regions of a complicated ray field, e.g. in regions where the ray field has several branches, separated by caustic surfaces, etc. All rays used in the summation, however, must correspond to the selected elementary wave, specified by the same ray code.

The points $R_\gamma$ may be situated arbitrarily on rays $\Omega(\gamma_1, \gamma_2)$, but in the paraxial (quadratic) vicinity of $R$. For example, assume that the receiver $R$ is situated in a smooth medium, covered continuously by rays $\Omega(\gamma_1, \gamma_2)$, and that rays $\Omega(\gamma_1, \gamma_2)$ are discretized. Then point $R_\gamma$ on ray $\Omega(\gamma_1, \gamma_2)$ may be chosen as that of the discrete points of $\Omega(\gamma_1, \gamma_2)$ which is closest to the receiver $R$.

It is also very common to consider a target surface $\Sigma^R$, passing through the receiver $R$ (or close to it). The points $R_\gamma$ are then the end-points of rays $\Omega(\gamma_1, \gamma_2)$ on $\Sigma^R$, or the intersection points of rays $\Omega(\gamma_1, \gamma_2)$ with $\Sigma^R$. If the receiver $R$ is situated at a structural interface (or close to it), it is common to consider the structural interface as a target surface. The same applies to the Earth's surface.

We shall now briefly discuss the choice of $\mathbf{M}(R_\gamma)$. There is some degree of freedom in the choice of $\mathbf{M}(R_\gamma)$, which may, to some extent, be chosen arbitrarily. Moreover, $\mathbf{M}$ can be specified at different points of rays $\Omega(\gamma_1, \gamma_2)$, as the dynamic ray tracing may be used to recalculate $\mathbf{M}(R_\gamma)$ along ray $\Omega(\gamma_1, \gamma_2)$. The most straightforward is to choose matrix $\mathbf{M}$ directly at points $R_\gamma$. Then the superposition integral (5.1) can be used as it stands; no recalculation of $\mathbf{M}$ along $\Omega(\gamma_1, \gamma_2)$ is needed.

We shall not discuss here the problem of the choice of $\mathbf{M}(R_\gamma)$ in details, we shall only briefly mention two important options. For more details, see Červený (1985b), Klimeš (1989b).

The most important would be the *optimum choice* of $\mathbf{M}(R_\gamma)$, which would minimize the error of computations. Such choice has been a subject of investigations for a long time, see Klimeš (1989b). Another important possibility applies to the summation along a target surface $\Sigma^R$. We can choose $\mathrm{Re}\mathbf{M}(R_\gamma)$ in such a way that it suppresses the quadratic terms from the expansion of $\mathrm{Re}(T(R, R_\gamma))$. Matrix $\mathrm{Re}\mathbf{M}(R_\gamma)$ is then given by (4.23). This choice suppresses quadratic oscillatory functions from the exponential function in the superposition integral, and yields more stable results. Further, if $\mathrm{Im}\mathbf{M}(R_\gamma) = \mathbf{0}$ is chosen along $\Sigma^R$, the terms in the exponential function are only linear, similarly as in transform methods. Such a choice of $\mathbf{M}(R_\gamma)$ then yields the Maslov-Chapman integrals, see Section 7.

For completeness, we shall present here expressions for the ray amplitudes $U_i^{ray}(R_\gamma)$ in two most important situations in seismological applications, corresponding to the elementary wave generated at an initial surface $\Sigma^0$ and to the elementary wave generated by a point source.

For an **initial surface** $\Sigma^0$, the ray amplitude is given by the relation

$$U_i^{ray}(R_\gamma) = \Big[\frac{\rho(S_\gamma)\mathcal{C}(S_\gamma)|\det \mathbf{Q}^a(S_\gamma)|}{\rho(R_\gamma)\mathcal{C}(R_\gamma)|\det \mathbf{Q}^a(R_\gamma)|}\Big]^{1/2}\mathcal{R}^C g_i(R_\gamma)g_k(S_\gamma)U_k^\Omega(S_\gamma)\exp[\mathrm{i}T^c(R_\gamma,S_\gamma)] \ . \tag{5.7}$$

Here $T^c(R_\gamma,S_\gamma)$ is the phase shift due to caustics along ray $\Omega$ between $S_\gamma$ and $R_\gamma$, $g_k(S_\gamma)$ and $g_i(R_\gamma)$ are Cartesian components of the eigenvectors $\vec{g}$ at $S_\gamma$ and $R_\gamma$. The basis unit vectors $\vec{e}_1(S_\gamma)$ and $\vec{e}_2(S_\gamma)$ at $S_\gamma$ may be chosen arbitrarily. Of course, they must be perpendicular to $\vec{e}_3(S_\gamma)=\mathcal{C}(S_\gamma)\vec{p}(S_\gamma)$, and also mutually perpendicular. Along ray $\Omega$, $\vec{e}_1$ and $\vec{e}_2$ are calculated using (1.6). All other symbols have the same meaning as in (2.3). The initial values of $\mathbf{Q}^a$ and $\mathbf{P}^a$ at $S_\gamma$ for the dynamic ray tracing are given by relations:

$$\mathbf{Q}^a(S_\gamma) = (\mathbf{G}-\mathbf{A}^{an})^T \ , \quad \mathbf{P}^a(S_\gamma) = (\mathbf{G}-\mathbf{A}^{an})^{-1}(\mathbf{T}^0+p_3^{(z)}\mathbf{D}-\mathbf{E}) \ . \tag{5.8}$$

The $2\times 2$ matrices $\mathbf{G},\mathbf{A}^{an},\mathbf{E}$ and $\mathbf{D}$ in (5.8) correspond to the initial point $S_\gamma$ on the initial surface $\Sigma^0$, and have the same meaning as in (4.17). The $2\times 2$ matrix $\mathbf{T}^0$ represents the second derivatives of the initial travel time along $\Sigma^0$, with respect to local Cartesian coordinates $z_I, z_J$. The local Cartesian coordinates $z_i$ at $S_\gamma$ on $\Sigma^0$ are introduced in a standard way, with the $z_3$-axis oriented along the normal to the initial surface $\Sigma^0$ at $S_\gamma$.

For a **point source at** $S_\gamma$, the ray amplitude is given by the relation:

$$U_i^{ray}(R_\gamma) = \Big[\frac{\rho(S_\gamma)\mathcal{C}(S_\gamma)}{\rho(R_\gamma)\mathcal{C}(R_\gamma)}\Big]^{1/2}\frac{\mathcal{G}(S_\gamma;\gamma_1,\gamma_2)}{\mathcal{L}(R_\gamma,S_\gamma)}\mathcal{R}^C g_i(R_\gamma)\exp[\mathrm{i}T^c(R_\gamma,S_\gamma)] \ . \tag{5.9}$$

Here $T^c(R_\gamma,S_\gamma)$ is again the phase shift due to caustics situated along ray $\Omega$ between $S_\gamma$ and $R_\gamma$, and $\mathcal{G}(S_\gamma;\gamma_1,\gamma_2)$ is the radiation function of the point source. The normalized geometrical spreading $\mathcal{L}(R_\gamma,S_\gamma)$ is given by the relation

$$\mathcal{L}(R_\gamma,S_\gamma) = |\det \mathbf{Q}_2(R_\gamma,S_\gamma)|^{1/2} = |\det \mathbf{Q}^a(R_\gamma)/\det \mathbf{P}^a(S_\gamma)|^{1/2} \ . \tag{5.10}$$

$\mathbf{Q}_2(R_\gamma,S_\gamma)$ is a $2\times 2$ submatrix of the ray propagator matrix $\mathbf{\Pi}(R_\gamma,S_\gamma)$, see (1.11). It should be noted that $\mathcal{L}(R_\gamma,S_\gamma)$ is reciprocal. For more details, see Červený (2000), Section 5.4.

The radiation function $\mathcal{G}(S_\gamma;\gamma_1\gamma_2)$ in (5.9) depends on the type of the point source (moment-tensor point source, single-force point source, and so forth). We shall present here only the most important radiation function, corresponding to a unit single force, oriented along the $x_n$-axis:

$$\mathcal{G}(S_\gamma;\gamma_1,\gamma_2) = \frac{g_n(S_\gamma)}{4\pi\rho(S_\gamma)\mathcal{C}(S_\gamma)}\exp[\tfrac{1}{2}\mathrm{i}\pi\sigma_0(S_\gamma)] \ . \tag{5.11}$$

Here $g_n(S_\gamma)$ is the $n$-th Cartesian component of the eigenvector $\vec{g}(S_\gamma)$, and $\frac{1}{2}\pi\sigma_0 = \frac{1}{2}\pi(1-\frac{1}{2}\mathrm{Sgn}\mathbf{D}^S(S_\gamma))$ is the point-source phase shift. $\mathbf{D}^S(S_\gamma)$ is the curvature matrix of the slowness surface at the source, in the appropriate direction. $\mathrm{Sgn}\mathbf{D}^S(S_\gamma)$ denotes the signature of the $2\times 2$ matrix $\mathbf{D}^S(S_\gamma)$. If the slowness surface is concave at $S_\gamma$ in the appropriate direction, $\sigma_0=0$. Thus, $\sigma_0=0$ in isotropic media.

Using (5.11) and (5.9), we obtain the amplitude of the ray-theory elastodynamic Green function $G_{in}(R_\gamma,S_\gamma,\omega)$:

$$U_i^{ray}(R_\gamma) = \frac{g_n(S_\gamma)g_i(R_\gamma)\exp[\mathrm{i}T^G(R_\gamma,S_\gamma)]}{[\rho(S_\gamma)\rho(R_\gamma)\mathcal{C}(S_\gamma)\mathcal{C}(R_\gamma)]^{1/2}\mathcal{L}(R_\gamma,S_\gamma)}\mathcal{R}^C \ . \tag{5.12}$$

$T^G(R_\gamma, S_\gamma) = T^c(R_\gamma S_\gamma) + \frac{1}{2}\pi\sigma_0(S_\gamma)$ represents the complete phase shift due to caustics, which includes also the point-source phase shift.

The superposition integral (5.1), with $U_i^{ray}(R_\gamma)$ given by (5.12), gives the *superposition Green function.* For $\mathrm{Im}\mathbf{M}(R_\gamma) = \mathbf{0}$, we can call it the *paraxial ray approximation Green function*, and for $\mathrm{Im}\mathbf{M}(R_\gamma)$ positive definite the *paraxial Gaussian beam Green function.*

The initial conditions for the dynamic ray tracing in wavefront orthonormal coordinates at a point source $S_\gamma$ are as follows:

$$\mathbf{Q}^a(S_\gamma) = \mathbf{0}\ , \quad \mathbf{P}^a(S_\gamma) = \frac{1}{\mathcal{C}(i_0, \phi_0)} \begin{pmatrix} 1 & 0 \\ 0 & \sin i_0 \end{pmatrix} . \tag{5.13}$$

In (5.13), $\gamma_1$ and $\gamma_2$ represent the take-off angles $i_0$ and $\phi_0$, with $\vec{p}^{\,0}(S_\gamma) \equiv \mathcal{C}^{-1}[\sin i_0 \cos\phi_0, \sin i_0 \sin\phi_0,\ \cos i_0]$, $\vec{e}_1(S_\gamma) \equiv [\cos i_0\ \cos\phi_0, \cos i_0\ \sin\phi_0, -\sin i_0]$ and $\vec{e}_2(S_\gamma) \equiv [-\sin\phi_0, \cos\phi_0, 0]$.

It is very interesting to see that the superposition integral (5.1) does not require the computation of the whole ray propagator matrices along rays $\Omega(\gamma_1, \gamma_2)$ from $S_\gamma$ to $R_\gamma$, if $\mathbf{M}(R_\gamma)$ is specified at $R_\gamma$. In this case, it is sufficient to perform the dynamic ray tracing only once, for initial conditions $\mathbf{Q}^a(S_\gamma)$ and $\mathbf{P}^a(S_\gamma)$ at $S_\gamma$. For the specification of $\mathbf{Q}^a(S_\gamma)$ and $\mathbf{P}^a(S_\gamma)$ at a curved initial surface with an arbitrary distribution of the initial travel time, see (5.8), and for analogous specification at a point source see (5.13). The computation of the whole ray propagator matrix $\mathbf{\Pi}(R_\gamma, S_\gamma)$ is needed only if we wish to specify $\mathbf{M}$ at $S_\gamma$, see (2.8), or at any other point between $S_\gamma$ and $R_\gamma$.

The integrand of the superposition integral (5.1) is finite everywhere, including the caustic point at $R_\gamma$. If $R_\gamma$ is a caustic point, $|\det \mathbf{Q}^a(R_\gamma)| = 0$ and $U_i^{ray}(R_\gamma)$ is infinite. The expression $U_i^{ray}(R_\gamma)[-\det \boldsymbol{\mathcal{M}}(R_\gamma)]^{1/2}|\det \mathbf{Q}^a(R_\gamma)|$, however, remains finite for arbitrary $\mathbf{M}(R_\gamma)$.

A simple alternative version of the superposition integral (5.1) is as follows:

$$u_i^{(x)}(R, \omega) = \frac{\omega}{2\pi} \iint_{\mathcal{D}} U_i^{ray}(R_\gamma)[-\det \boldsymbol{\mathcal{N}}(R_\gamma)]^{1/2} \exp[\mathrm{i}\omega T(R, R_\gamma)] \mathrm{d}\gamma_1 \mathrm{d}\gamma_2\ , \tag{5.14}$$

where the $2 \times 2$ matrix $\boldsymbol{\mathcal{N}}(R_\gamma)$ is given by the relation:

$$\boldsymbol{\mathcal{N}}(R_\gamma) = \mathbf{Q}^{aT}(\mathbf{M} - \mathbf{M}^a)\mathbf{Q}^a = -\mathbf{Q}^{aT}\mathbf{P}^a + \mathbf{Q}^{aT}\mathbf{M}\mathbf{Q}^a\ . \tag{5.15}$$

All quantities are taken at $R_\gamma$. The argument of $[-\det \boldsymbol{\mathcal{N}}(R_\gamma)]^{1/2}$ is again given by (5.3), and the travel-time function $T(R, R_\gamma)$ by (5.4), with (5.5) and (5.6).

Several comments to the summation of paraxial Gaussian beams and to the choice of the $2 \times 2$ symmetric matrix $\mathrm{Im}\mathbf{M}$: The superposition integral (5.1) represents the summation of paraxial Gaussian beams if $\mathrm{Im}\mathbf{M}$ is positive definite. Matrix $\mathrm{Im}\mathbf{M}$ controls the amplitude profile of paraxial Gaussian beams used in the expansion (their width, etc.). For $\mathrm{Im}\mathbf{M} = \mathbf{0}$ (i.e., for infinitely broad Gaussian beams), (5.1) represents the summation of paraxial ray approximations, see Section 7. Even small values of $\mathrm{Im}\mathbf{M}$, however, increase the stability of computations, suppress pseudocaustic points and spurious arrivals. The main disadvantage of the Gaussian beam summation solutions is that they depend on the free parameters (i.e. on the widths of the Gaussian beams) in certain singular regions. With sufficiently broad Gaussian beams (small

$\mathrm{Im}\mathbf{M}(R_\gamma)$), they yield the wave field in caustic and critical regions correctly, but not for edge-diffractions. The computation of edge diffractions requires the use of very narrow Gaussian beams close to the edge. Moreover, spurious saddle points can arise in strongly laterally inhomogeneous media and degrade the beam solutions, see White, Norris, Bayliss and Burridge (1987). The optimum choice of $\mathrm{Im}\mathbf{M}$ which would minimize the error of the computation is not known. The problem of the choice of $\mathrm{Im}\mathbf{M}$ in the superposition integrals has been broadly discussed in seismological literature, but still requires further research. See, e.g., Klimeš (1989b).

The expansion of a high-frequency wave field into paraxial Gaussian beams in isotropic media was first proposed by Babich and Pankratova (1973). See also Katchalov and Popov (1981), Popov (1982), Červený, Popov and Pšenčík (1982), Červený and Pšenčík (1983a,b,1984), Červený (1983,1985a,b), Katchalov, Popov and Pšenčík (1983), Červený and Klimeš (1984), Konopásková and Červený (1984), Madariaga (1984), Nowack and Aki (1984), Klimeš (1984a,b,1989b), Müller (1984), Fertig and Pšenčík (1985), Madariaga and Papadimitriou (1985), Norris (1986), White, Norris, Bayliss and Burridge (1987), Katchalov and Popov (1988), Weber (1988), Wang and Waltham (1995). A review with extensive literature up to 1985 can be found in Červený (1985b). The application of paraxial Gaussian wave packets instead of paraxial Gaussian beams in the summation procedure requires additional integration in the superposition integrals, see Ralston (1983), Klimeš (1984b,1989a).

Alternatively to the summation of Gaussian beams, it is also possible to use the so-called *coherent-state transform*, see Klauder (1987), Foster and Huang (1991). We then speak of the *coherent-state method*, and of the *coherent-state approximation*. For a detailed explanation, many references and applications, see Thomson (in press).

# 6 Superposition integrals: Cartesian coordinates

In this section, we shall assume that the dynamic ray tracing was performed in Cartesian coordinates. By dynamic ray tracing in Cartesian coordinates, we determine the $3\times 2$ matrices $\hat{\mathbf{Q}}^{(x)a}$ and $\hat{\mathbf{P}}^{(x)a}$, with components:

$$Q^{(x)a}_{iJ} = (\partial x_i/\partial\gamma_J)_T \ , \qquad P^{(x)a}_{iJ} = (\partial p^{(x)}_i/\partial\gamma_J)_T \ . \tag{6.1}$$

The subscript $T$ means that the derivatives are taken along the wavefront. Thus, the $3\times 2$ matrices

$$\hat{\mathbf{Q}}^{(x)a} = \begin{pmatrix} (Q^{(x)a}_{11})_T & (Q^{(x)a}_{12})_T \\ (Q^{(x)a}_{21})_T & (Q^{(x)a}_{22})_T \\ (Q^{(x)a}_{31})_T & (Q^{(x)a}_{32})_T \end{pmatrix} \ , \quad \hat{\mathbf{P}}^{(x)a} = \begin{pmatrix} (P^{(x)a}_{11})_T & (P^{(x)a}_{12})_T \\ (P^{(x)a}_{21})_T & (P^{(x)a}_{22})_T \\ (P^{(x)a}_{31})_T & (P^{(x)a}_{32})_T \end{pmatrix} \ , \tag{6.2}$$

are assumed to be known. We wish to modify the superposition integral in such a way that the $2\times 2$ matrices $\mathbf{Q}^a$, $\mathbf{P}^a$ are replaced by the $3\times 2$ matrices $\hat{\mathbf{Q}}^{(x)a}$, $\hat{\mathbf{P}}^{(x)a}$. Thus, we need to know relations between $\mathbf{Q}^a$, $\mathbf{P}^a$ and $\hat{\mathbf{Q}}^{(x)a}$, $\hat{\mathbf{P}}^{(x)a}$. Certain useful relations between $\mathbf{Q}^a$, $\mathbf{P}^a$ and $\hat{\mathbf{Q}}^{(x)a}$, $\hat{\mathbf{P}}^{(x)a}$ are given in Červený (2000), Sections 4.2.2 and 4.2.3.

In the following, we shall need only two simple relations:

$$\mathbf{Q}^a = \hat{\mathbf{H}}^T\hat{\mathbf{Q}}^{(x)a} \ , \quad \mathbf{Q}^{aT}\mathbf{P}^a = \hat{\mathbf{Q}}^{(x)aT}\hat{\mathbf{P}}^{(x)a} \ . \tag{6.3}$$

Here $\hat{\mathbf{H}}$ is a $3 \times 2$ transformation matrix from $y_I$ to $x_i$:

$$\hat{\mathbf{H}} = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \\ H_{31} & H_{32} \end{pmatrix} = \begin{pmatrix} e_{11} & e_{21} \\ e_{12} & e_{22} \\ e_{13} & e_{23} \end{pmatrix} . \tag{6.4}$$

$e_{1j}$ and $e_{2j}$ are Cartesian components, of the basis vectors $\vec{e}_1$, $\vec{e}_2$, see (1.7). Here, however, we consider $\hat{\mathbf{H}}$ to be a $3 \times 2$, not $3 \times 3$ matrix.

Thus, we can again use the superposition integral (5.14), where the expression (5.15) for the $2 \times 2$ matrix $\boldsymbol{\mathcal{N}}(R_\gamma)$ is modified as follows:

$$\boldsymbol{\mathcal{N}}(R_\gamma) = -\mathbf{Q}^{(x)aT}\hat{\mathbf{P}}^{(x)a} + \hat{\mathbf{Q}}^{(x)aT}\hat{\mathbf{H}}\mathbf{M}\hat{\mathbf{H}}^T\hat{\mathbf{Q}}^{(x)a} . \tag{6.5}$$

For the travel-time function $T(R, R_\gamma)$, we can use again (5.4), with (5.5) and (5.6). In these equations, $\mathcal{U}_i^{(y)}$ and $\eta_i^{(y)}$ can be expressed in terms of $\mathcal{U}_i^{(x)}$ and $\eta_i^{(x)}$, using (1.8) and (1.4). Also, $\mathcal{U}_i^{(y)}\eta_i^{(y)} = \mathcal{U}_i^{(x)}\eta_i^{(x)}$. The expression for $T(R, R_\gamma)$ can be expressed in many alternative forms (perhaps more concise), but we shall not discuss them here. We only present one simple and useful relation for $e_{Ik}^A$:

$$e_{Ik}^A = e_{Ik} - \mathcal{C}^{-1}\mathcal{U}_I^{(y)}e_{3k} = e_{Ik} - p_k^{(x)}\mathcal{U}_I^{(y)} = e_{In}(\delta_{nk} - \mathcal{U}_n^{(x)}p_k^{(x)}) . \tag{6.6}$$

This relation simplifies expression (5.6).

The expression for $T(R, R_\gamma)$ simplifies considerably if a smooth target surface $\Sigma^R$ is considered. Then we can use (5.4) with (4.21), and consider $\bar{F}_{IJ}(R_\gamma)$ as free parameters in the expansion. For more details, see Eqs. (7.9)–(7.13) in Section 7.

What remains is to specify also the geometrical spreading factors in the expression for ray amplitude $U_i^{ray}(R_\gamma)$, in terms of $\hat{\mathbf{Q}}^{(x)a}$ and $\hat{\mathbf{P}}^{(x)a}$. Such expressions are, however, well-known in seismological literature, see Chapman (2000), for example.

A comment to the choice of unit vectors $\vec{e}_1(R_\gamma)$, $\vec{e}_2(R_\gamma)$, $\vec{e}_3(R_\gamma)$. As we know, $\vec{e}_3(R_\gamma)$, can be expressed in terms of the slowness vector $\vec{p}(R_\gamma)$ as follows: $e_{3k}(R_\gamma) = \mathcal{C}(R_\gamma)p_k^{(x)}(R_\gamma)$. The two remaining unit vectors, $\vec{e}_1(R_\gamma)$ and $\vec{e}_2(R_\gamma)$, may be chosen arbitrarily at $R_\gamma$. The only requirement is that they are perpendicular to $\vec{e}_3(R_\gamma)$, and that they are also mutually perpendicular. In other words, $\vec{e}_1(R_\gamma)$ and $\vec{e}_2(R_\gamma)$ are tangent to the wavefront at $R_\gamma$.

Thus, the dynamic ray tracing in Cartesian coordinates and in wavefront orthonormal coordinates can be used quite alternatively in the superposition integrals. This applies both to the superposition of paraxial ray approximations ($\mathrm{Im}\mathbf{M}(R_\gamma) = \mathbf{0}$) and superposition of paraxial Gaussian beams ($\mathrm{Im}\mathbf{M}(R_\gamma)$ positive definite).

# 7 Maslov-Chapman integrals

In this section, we shall discuss the superposition of paraxial ray approximations ($\mathrm{Im}\mathbf{M}(R_\gamma) = \mathbf{0}$). Only at the end of the section, we extend the results also for complex-valued $\mathbf{M}(R_\gamma)$, i.e. for the superposition of Gaussian beams. For $\mathrm{Im}\mathbf{M}(R_\gamma) = \mathbf{0}$, the superposition integral (5.14) yields

$$u_i^{(x)}(R, \omega) = \frac{\omega}{2\pi}\iint_{\mathcal{D}} U_i^{ray}(R_\gamma)|\det\boldsymbol{\mathcal{N}}(R_\gamma)|^{1/2}\exp[\mathrm{i}\omega T(R, R_\gamma) - \mathrm{i}\tfrac{\pi}{4}\mathrm{Sgn}\boldsymbol{\mathcal{N}}(R_\gamma)]\mathrm{d}\gamma_1\mathrm{d}\gamma_2 , \tag{7.1}$$

where

$$\boldsymbol{\mathcal{N}}(R_\gamma) = -\mathbf{Q}^{aT}\mathbf{P}^a + \mathbf{Q}^{aT}\mathbf{M}\mathbf{Q}^a \ . \tag{7.2}$$

$T(R, R_\gamma)$ is given by (5.4) with (5.5) and (5.6), with $\mathrm{Im}\mathbf{M} = \mathbf{0}$ in (5.5).

The superposition integral (7.1) is still very general. We shall discuss now only a special case of (7.1), in which the points $R_\gamma$ are distributed along a target surface $\Sigma^R$ passing through $R$, and in which $\mathbf{M}(R_\gamma)$ is specified by (4.23) at $R_\gamma$. Then the second derivatives of the travel-time field $T(R, R_\gamma)$ along the target surface $\Sigma^R$ vanish, and $T(R, R_\gamma)$ simplifies considerably. Actually, it is given by the linear expansion (4.25), similarly as in the transform methods. The superposition integral (7.1) then reads:

$$\begin{aligned} u_i^{(x)}(R,\omega) &= \frac{\omega}{2\pi}\iint_{\mathcal{D}} U_i^{ray}(R_\gamma)|\det\boldsymbol{\mathcal{N}}(R_\gamma)|^{1/2} \\ &\times \ \exp[\mathrm{i}\omega(T(R_\gamma) + x_i(R,R_\gamma)p_i^{(x)}(R_\gamma)) - \mathrm{i}\tfrac{\pi}{4}\mathrm{Sgn}\boldsymbol{\mathcal{N}}(R_\gamma)]\mathrm{d}\gamma_1\mathrm{d}\gamma_2 \ , \end{aligned} \tag{7.3}$$

where

$$\boldsymbol{\mathcal{N}}(R_\gamma) = -[\mathbf{Q}^{aT}\mathbf{P}^a + \mathbf{Q}^{aT}(\mathbf{G}-\mathbf{A}^{an})^{-1}\mathbf{E}(\mathbf{G}-\mathbf{A}^{an})^{-1T}\mathbf{Q}^a] \ . \tag{7.4}$$

This is the final form of the superposition integral, in which the weighting function $\boldsymbol{\mathcal{N}}(R_\gamma)$ is expressed in terms of $2\times 2$ matrices $\mathbf{Q}^a$, $\mathbf{P}^a$, obtained by dynamic ray tracing in wavefront orthonormal coordinates. The $2\times 2$ matrices $\mathbf{G}$, $\mathbf{A}^{an}$ and $\mathbf{E}$ have the same meaning as in Section 4.4.

Now we wish to express $\boldsymbol{\mathcal{N}}(R_\gamma)$ in terms of $3\times 2$ matrices $\hat{\mathbf{Q}}^{(x)a}$ and $\hat{\mathbf{P}}^{(x)a}$, see (6.2). These $3\times 2$ matrices can be directly obtained by dynamic ray tracing in Cartesian coordinates.

We shall again consider a target surface $\Sigma^R$ inclined with respect to the wavefront at $R_\gamma$, which may be curved. We introduce $(Q_{iJ}^{(x)a})_{\Sigma^R}$ and $(P_{iJ}^{(x)a})_{\Sigma^R}$, representing the derivatives with respect to $\gamma_J$, taken along the target surface $\Sigma^R$. Their derivation is simple:

$$\begin{aligned} (Q_{iJ}^{(x)a})_{\Sigma^R} &= \left(\frac{\partial x_i}{\partial\gamma_J}\right)_{\Sigma^R} = \left(\frac{\partial x_i}{\partial\gamma_J}\right)_T + \left(\frac{\partial x_i}{\partial T}\right)_{ray}\left(\frac{\partial T}{\partial\gamma_J}\right)_{\Sigma^R} = Q_{iJ}^{(x)a} + \mathcal{U}_i^{(x)}\left(\frac{\partial T}{\partial\gamma_J}\right)_{\Sigma^R} \ , \\ (P_{iJ}^{(x)a})_{\Sigma^R} &= \left(\frac{\partial p_i^{(x)}}{\partial\gamma_J}\right)_{\Sigma^R} = \left(\frac{\partial p_i^{(x)}}{\partial\gamma_J}\right)_T + \left(\frac{\partial p_i^{(x)}}{\partial T}\right)_{ray}\left(\frac{\partial T}{\partial\gamma_J}\right)_{\Sigma^R} = P_{iJ}^{(x)a} + \eta_i^{(x)}\left(\frac{\partial T}{\partial\gamma_J}\right)_{\Sigma^R} \ . \end{aligned}$$

We now denote the normal unit vector to $\Sigma^R$ at $R_\gamma$ by $\vec{n}$. Multiplying the first equation by $n_i$, and taking into account that $n_i(\partial x_i/\partial\gamma_J)_{\Sigma^R} = 0$, we obtain

$$\left(\frac{\partial T}{\partial\gamma_J}\right)_{\Sigma^R} = -\frac{n_i}{n_j\mathcal{U}_j^{(x)}}\ Q_{iJ}^{(x)a} \ .$$

This yields the final expressions for $(Q_{iJ}^{(x)a})_{\Sigma^R}$ and $(P_{iJ}^{(x)a})_{\Sigma^R}$:

$$(Q_{iJ}^{(x)a})_{\Sigma^R} = \left(\delta_{ik} - \frac{\mathcal{U}_i^{(x)}n_k}{\mathcal{U}_j^{(x)}n_j}\right)Q_{kJ}^{(x)a} \ , \quad (P_{iJ}^{(x)a})_{\Sigma^R} = P_{iJ}^{(x)a} - \frac{\eta_i^{(x)}n_k}{\mathcal{U}_j^{(x)}n_j}Q_{kJ}^{(x)a} \ . \tag{7.5}$$

We remind the reader that $Q_{kJ}^{(x)a}$ and $P_{kJ}^{(x)a}$ in (7.5) are computed by dynamic ray tracing in Cartesian coordinates.

We now wish to express $\boldsymbol{\mathcal{N}}(R_\gamma)$ in terms of $(\hat{\mathbf{Q}}^{(x)a})_{\Sigma^R}$ and $(\hat{\mathbf{P}}^{(x)a})_{\Sigma^R}$, given by (7.5). As a starting point, we can use either (6.5) or (7.4). We shall use (7.4). Similarly as in Section 4.4, we introduce the local Cartesian coordinate system $z_i$ connected with the target surface $\Sigma^R$ in such a way that the $z_3$-axis corresponds to the normal $\vec{n}$ to $\Sigma^R$ at $R_\gamma$.

First, we shall use the relation,

$$\mathbf{Q}^{aT}\mathbf{P}^a = (\hat{\mathbf{Q}}^{(x)a})^T_{\Sigma^R}(\hat{\mathbf{P}}^{(x)a})_{\Sigma^R} - (\hat{\mathbf{Q}}^{(x)a})^T_{\Sigma^R}\hat{\mathbf{U}}(\hat{\mathbf{Q}}^{(x)a})_{\Sigma^R} \; . \tag{7.6}$$

This equation can be obtained, after some manipulation, from (7.5) and (6.3). The $3 \times 3$ matrix $\hat{\mathbf{U}}$ is given by relation $\hat{\mathbf{U}} = \hat{\mathbf{Z}}\mathbf{E}\hat{\mathbf{Z}}^T$, where $\mathbf{E}$ is the inhomogeneity matrix given by (4.18) and $\hat{\mathbf{Z}}$ is a $3 \times 2$ transformation matrix from local Cartesian coordinates $z_J$ to general Cartesian coordinates $x_i$, $Z_{iJ} = \partial x_i/\partial z_J$. We further use the relation

$$(\mathbf{G} - \mathbf{A}^{an})^{-1T}\mathbf{Q}^a = \hat{\mathbf{Z}}^T(\hat{\mathbf{Q}}^{(x)a})_{\Sigma^R} \; , \tag{7.7}$$

which can be proved by direct inspection. Inserting (7.6) and (7.7) into (7.4) yields the final very simple result:

$$\boldsymbol{\mathcal{N}}(R_\gamma) = -(\hat{\mathbf{Q}}^{(x)a})^T_{\Sigma^R}(\hat{\mathbf{P}}^{(x)a})_{\Sigma^R} \; . \tag{7.8}$$

Here $(\hat{\mathbf{Q}}^{(x)a})_{\Sigma^R}$ and $(\hat{\mathbf{P}}^{(x)a})_{\Sigma^R}$ are $3 \times 2$ matrices with elements $(Q^{(x)a}_{iJ})_{\Sigma^R}$ and $(P^{(x)a}_{iJ})_{\Sigma^R}$, given by (7.5). The expression (7.8) corresponds fully to that given by Chapman (2000) for the weighting function in the Maslov integral in anisotropic media.

The Maslov method was introduced to seismology and applied to the computation of synthetic seismograms (Maslov seismograms) by Chapman. The Maslov integrals are also known as *Maslov-Chapman integrals*, to acknowledge the basic contribution of C.H. Chapman to the application of the method to seismic wave propagation. The relevant expressions for the Maslov-Chapman integrals have mostly been derived in a more sophisticated way than outlined here, using pseudo-differential and Fourier integral operators in a mixed $x_i, p_i$-phase space. The Maslov-Chapman technique yields satisfactory results even in certain singular regions of the ray method (Airy caustics, critical regions, etc.).

For a more detailed derivation and discussion of the Maslov-Chapman integrals and for many applications in seismology, see Chapman and Drummond (1982), Klimeš (1984b), Ziolkowski and Deschamps (1984), Chapman (1985,2000), Thomson and Chapman (1985), Tromp and Dahlen (1993), Guest and Kendall (1993), Kendall and Thomson (1993), Brown (1994), Huang and West (1997), Huang, West and Kendall (1998), Huang, Kendall, Thomson and West (1998). Klimeš (1984b) was the first who explained the Maslov-Chapman integrals as a limiting case of the Gaussian beam summation integrals, for Im$\mathbf{M} \rightarrow \mathbf{0}$.

A note on the problem of pseudocaustic points. Integral (7.1) fails at pseudocaustic points, at which $\det[\mathbf{M}(R_\gamma) - \mathbf{M}^a(R_\gamma)] = 0$, see (4.14). We remind the reader that the standard ray-theory expressions fail at caustic points, where $\det \mathbf{Q}^a(R_\gamma) = 0$. If the caustic and pseudocaustic points are well separated, it is possible to use the superposition integrals at caustic points (and close to them), and the standard ray-theory expressions at pseudocaustic points (and close to them). It is possible to obtain uniformly valid solutions by blending together the two asymptotic solutions with weighting functions, (also called neutralizers), see Maslov (1965), Chapman and Drummond

(1982). Here we shall only consider the situation appropriate to our case. We introduce two smooth weighting functions $w_1(\gamma_1,\gamma_2)$ and $w_2(\gamma_1,\gamma_2)$ on target surface $\Sigma^R$, corresponding to the end-points $R_\gamma$ of the rays $\Omega(\gamma_1,\gamma_2)$ on $\Sigma^R$. We choose $w_1(\gamma_1,\gamma_2)$ and $w_2(\gamma_1,\gamma_2)$ so that $w_1(\gamma_1,\gamma_2)+w_2(\gamma_1,\gamma_2)=1$ everywhere on $\Sigma^R$. We then assign the weighting functions to the two asymptotic solutions, and take them to be unity where the relevant asymptotic solution is valid, to be zero where it is invalid, and to vary smoothly from 0 to 1 inbetween. This method has yielded very accurate results in many important applications. Nevertheless, the pseudocaustic points may appear very close to the caustic points in certain situations, and the phase partitioning becomes difficult. For a detailed discussion refer to Kendall and Thomson (1993), where also other methods of treating the pseudocaustic problem are proposed.

At the end of this section, we shall present a more general form of the Maslov-Chapman integral (7.3), using the same notation as in (7.3). This more general form can be used even for the summation of paraxial Gaussian beams, and for $\mathrm{Re}\mathbf{M}(R_\gamma)$ specified in a different way then by (4.23).

We assume that all points $R_\gamma$ are distributed along a smooth target surface $\Sigma^R$, but allow $\mathbf{M}(R_\gamma)$ to be complex-valued, with a positive-definite $\mathrm{Im}\mathbf{M}(R_\gamma)$. Using the same procedure as in this section, we obtain the *extended Maslov-Chapman integral*:

$$u_i^{(x)}(R,\omega)=\frac{\omega}{2\pi}\iint_{\mathcal{D}}U_i^{ray}(R_\gamma)[-\det\boldsymbol{\mathcal{N}}(R_\gamma)]^{1/2}\exp[\mathrm{i}\omega T(R,R_\gamma)]\mathrm{d}\gamma_1\mathrm{d}\gamma_2\ , \tag{7.9}$$

where

$$T(R,R_\gamma)=T(R_\gamma)+\hat{\mathbf{x}}^T(R,R_\gamma)\mathbf{p}^{(x)}(R_\gamma)+\tfrac{1}{2}\hat{\mathbf{x}}^T(R,R_\gamma)\hat{\mathbf{Z}}(R_\gamma)\bar{\mathbf{F}}(R_\gamma)\hat{\mathbf{Z}}^T(R_\gamma)\hat{\mathbf{x}}(R,R_\gamma)\ , \tag{7.10}$$

see (4.21), and

$$\boldsymbol{\mathcal{N}}(R_\gamma)=-(\hat{\mathbf{Q}}^{(x)a})^T_{\Sigma^R}(\hat{\mathbf{P}}^{(x)a})_{\Sigma^R}+(\hat{\mathbf{Q}}^{(x)a})^T_{\Sigma^R}\hat{\mathbf{Z}}\bar{\mathbf{F}}\hat{\mathbf{Z}}^T(\hat{\mathbf{Q}}^{(x)a})_{\Sigma^R}\ . \tag{7.11}$$

The argument of $[-\det\boldsymbol{\mathcal{N}}(R_\gamma)]^{1/2}$ is determined using (5.3), $(\hat{\mathbf{Q}}^{(x)a})_{\Sigma^R}$ and $(\hat{\mathbf{P}}^{(x)a})_{\Sigma^R}$ are given by (7.5), and $\hat{\mathbf{Z}}(R_\gamma)$ is a $3\times 2$ transformation matrix from local Cartesian coordinates $z_I$ to the general Cartesian coordinates $x_i$, $Z_{kL}=\partial x_k/\partial z_L=\partial z_L/\partial x_k$. See also (7.13). Finally, $\bar{\mathbf{F}}(R_\gamma)$ a $2\times 2$ matrix, expressed in terms of the $2\times 2$ matrix $\mathbf{M}$ using (4.20). It represents the matrix of second derivatives of the travel time field with respect to $z_I, z_J$, taken along the plane tangent to the target surface $\Sigma^R$ at $R_\gamma$. As the choice of $\mathbf{M}(R_\gamma)$ is free, also $\bar{\mathbf{F}}(R_\gamma)$ may be chosen arbitrarily, without considering (4.20). Actually, the only difference between $\mathbf{M}(R_\gamma)$ and $\bar{\mathbf{F}}(R_\gamma)$ is that $\mathbf{M}(R_\gamma)$ is related to the plane tangent to the wavefront at $R_\gamma$, but $\bar{\mathbf{F}}(R_\gamma)$ is related to the plane tangent to the target surface $\Sigma^R$ at $R_\gamma$. The $3\times 2$ matrix $\hat{\mathbf{Z}}$ merely specifies the frame in which the matrix $\bar{\mathbf{F}}(R_\gamma)$ is expressed. Matrix $\bar{\mathbf{F}}(R_\gamma)$ may be chosen real-valued (summation of paraxial ray approximations), or complex-valued, with a positive-definite imaginary part (summation of paraxial Gaussian beams). For the choice $\bar{\mathbf{F}}(R_\gamma)=\mathbf{0}$, equations (7.9)–(7.11) yield the standard Maslov-Chapman integral (7.3), with $\boldsymbol{\mathcal{N}}(R_\gamma)$ given by (7.8).

We remind the reader that the coordinate system $z_i$ has its origin at $R_\gamma$. Denote the basis vectors of this coordinate system $\vec{i}_1^{(z)},\vec{i}_2^{(z)},\vec{i}_3^{(z)}$. The unit basis vector $\vec{i}_3^{(z)}$ coincides with the normal $\vec{n}(R_\gamma)$ to the target surface $\Sigma^R$ at $R_\gamma$, and $\vec{i}_1^{(z)},\vec{i}_2^{(z)}$ are mutually perpendicular and tangent to the target surface $\Sigma^R$ at $R_\gamma$. Otherwise, they can be chosen arbitrarily. We can choose them, e.g., by following relations:

$$\vec{i}_2^{(z)} = \frac{\vec{n}\times\vec{p}}{|\vec{p}\times\vec{n}|}\ , \quad \vec{i}_1^{(z)} = \vec{i}_2^{(z)}\times\vec{n}\ . \tag{7.12}$$

Then $\vec{i}_2^{(z)}$ is perpendicular to the plane specified by $\vec{n}$ and $\vec{p}$, and $\vec{i}_1^{(z)}$ is situated in this plane. The $3\times 2$ matrix $\hat{\mathbf{Z}}$ is then given by the relation:

$$\hat{\mathbf{Z}} = \begin{pmatrix} i_{11}^{(z)} & i_{21}^{(z)} \\ i_{12}^{(z)} & i_{22}^{(z)} \\ i_{13}^{(z)} & i_{23}^{(z)} \end{pmatrix}\ , \tag{7.13}$$

where $i_{Ik}^{(z)}$ denotes the $k$-th general Cartesian component of the unit vectors $\vec{i}_I^{(z)}$, $I = 1, 2$.

# 8 Alternative versions of the superposition integral

The general superposition integral (5.1) can be expressed in many alternative forms. For example, $U_i^{ray}(R_\gamma)$ can be replaced by $U_i^{(x)}(R_\gamma)$ using (4.2), or by the spreading-free amplitude $U_i^{\Omega}(R_\gamma)$ using (2.2) and (2.3). Here we shall present one important alternative of the superposition integral.

Using (2.8), we express $\boldsymbol{\mathcal{M}}(R_\gamma)$, given by (4.7), in the following form:

$$\begin{aligned}\boldsymbol{\mathcal{M}}(R_\gamma) &= [\mathbf{P}_1+\mathbf{P}_2\mathbf{M}(S_\gamma)][\mathbf{Q}_1+\mathbf{Q}_2\mathbf{M}(S_\gamma)]^{-1} \\ &- [\mathbf{P}_1+\mathbf{P}_2\mathbf{M}^a(S_\gamma)][\mathbf{Q}_1+\mathbf{Q}_2\mathbf{M}^a(S_\gamma)]^{-1}\ .\end{aligned}$$

Here $\mathbf{P}_1, \mathbf{P}_2$, $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are the minors of the ray propagator matrix $\mathbf{\Pi}(R_\gamma, S_\gamma)$ for anisotropic media, see (1.11). As $\mathbf{M}(R_\gamma)$ is symmetric, $\mathbf{M}(R_\gamma) = \mathbf{M}^T(R_\gamma)$, and the above equation yields

$$\begin{aligned}\boldsymbol{\mathcal{M}}(R_\gamma) &= (\mathbf{Q}_1+\mathbf{Q}_2\mathbf{M}(S_\gamma))^{-1T}[(\mathbf{P}_1+\mathbf{P}_2\mathbf{M}(S_\gamma))^T(\mathbf{Q}_1+\mathbf{Q}_2\mathbf{M}^a(S_\gamma)) \\ &- (\mathbf{Q}_1+\mathbf{Q}_2\mathbf{M}(S_\gamma))^T(\mathbf{P}_1+\mathbf{P}_2\mathbf{M}^a(S_\gamma))](\mathbf{Q}_1+\mathbf{Q}_2\mathbf{M}^a(S_\gamma))^{-1}\ .\end{aligned}$$

Using the symplectic properties of the ray propagator matrix, $\mathbf{Q}_1^T\mathbf{P}_1-\mathbf{P}_1^T\mathbf{Q}_1 = \mathbf{Q}_2^T\mathbf{P}_2-\mathbf{P}_2^T\mathbf{Q}_2 = \mathbf{0}$, $\mathbf{P}_2^T\mathbf{Q}_1-\mathbf{Q}_2^T\mathbf{P}_1 = \mathbf{Q}_1^T\mathbf{P}_2-\mathbf{P}_1^T\mathbf{Q}_2 = \mathbf{I}$, valid both for isotropic and anisotropic media, we can simplify the expression in the square brackets considerably; it reads $\mathbf{M}(S_\gamma)-\mathbf{M}^a(S_\gamma)$. Using also the first equation in (2.8), we obtain:

$$\boldsymbol{\mathcal{M}}(R_\gamma) = (\mathbf{Q}(S_\gamma)\mathbf{Q}^{-1}(R_\gamma))^T\boldsymbol{\mathcal{M}}(S_\gamma)(\mathbf{Q}^a(S_\gamma)\mathbf{Q}^{a-1}(R_\gamma))\ . \tag{8.1}$$

Here $\boldsymbol{\mathcal{M}}(S_\gamma)$ is again given by (4.7). Equation (8.1) represents the continuation relation for $\boldsymbol{\mathcal{M}}$ along ray $\Omega$. If we use (4.13), we obtain

$$\Phi(\gamma_1,\gamma_2) = (\omega/2\pi)[-\det\boldsymbol{\mathcal{M}}(S_\gamma)]^{1/2}|\det\mathbf{Q}^a(S_\gamma)|\Psi_0(\gamma_1,\gamma_2)\ , \tag{8.2}$$

where $\Psi_0(\gamma_1,\gamma_2)$ is given by (4.3). If we use (4.5), Eq. (8.2) yields a simple expression for the weighting function $\Psi(\gamma_1,\gamma_2)$:

$$\Psi(\gamma_1,\gamma_2) = (\omega/2\pi)[-\det\boldsymbol{\mathcal{M}}(S_\gamma)]^{1/2}|\det\mathbf{Q}^a(S_\gamma)|\ . \tag{8.3}$$

The superposition integral (5.1) then reads

$$u_i^{(x)}(R,\omega) = \frac{\omega}{2\pi}\iint_{\mathcal{D}} U_i^{(x)}(R_\gamma)[-\det \boldsymbol{\mathcal{M}}(S_\gamma)]^{1/2}|\det \mathbf{Q}^a(S_\gamma)|\exp[\mathrm{i}\omega T(R,R_\gamma)]\mathrm{d}\gamma_1\mathrm{d}\gamma_2 \ , \tag{8.4}$$

where $\boldsymbol{\mathcal{M}}(S_\gamma)$ is again given by (5.2) with (5.3), only it is taken at the initial point $S_\gamma$. The travel time function $T(R,R_\gamma)$ is the same as in (5.4).

It should be reminded that $U_i^{(x)}(R_\gamma)$ in the superposition integral (8.4) represents the amplitude of a paraxial ray approximation or of a paraxial Gaussian beam, see (2.2), not the ray amplitude. Actually, (8.4) represents the original superposition integral (4.1).

The superposition integral (8.4) is more transparent than (5.1) if we wish to specify $\mathbf{M}$ at $S_\gamma$. The determination of $U_i^{(x)}(R_\gamma)$ is, however, slightly more involved than the determination of $U_i^{ray}(R_\gamma)$, and requires the computation of the whole ray propagator matrix along ray $\Omega$. Another advantage of the superposition integral (5.1) is that the computer routines for the determination of $U_i^{ray}(R_\gamma)$ are broadly available in various program packages, but not the routines for $U_i^{(x)}$.

# 9 Summation in 2-D models

We shall now consider a 2-D model, in which the structural parameters do not depend on one Cartesian coordinate, say $x_2$, but may depend on remaining coordinates $x_1, x_3$. We shall further consider a one-parameteric system of planar rays $\Omega(\gamma_1)$, situated in plane $\Sigma^{\|}$, and assume that the ray plane $\Sigma^{\|}$ coincides with plane $x_2 = 0$. In isotropic media, such a system of planar rays $\Omega(\gamma_1)$ in a 2-D model always exists; it is only necessary to consider the initial slowness vectors $\vec{p}_0$ in such a way that $p_{20} = 0$. In anisotropic media, however, the rays with $p_{20} = 0$ form a system of planar rays $\Omega(\gamma_1)$ situated in $\Sigma^{\|}$ only exceptionally, in case of some simpler types of anisotropy. In general, the rays in anisotropic media deviate from the plane $\Sigma^{\|}$ even if $p_{20} = 0$. See Červený (2000), Section 3.6.5.

We further assume that all $2\times 2$ matrices $\mathbf{Q}$, $\mathbf{P}$, $\mathbf{M}$, $\mathbf{Q}^a$, $\mathbf{P}^a$ and $\mathbf{M}^a$ can be diagonalized along rays $\Omega(\gamma_1)$, so that $\det \mathbf{Q} = Q^{\|}Q^{\perp}$, $\det \mathbf{Q}^a = Q^{\|a}Q^{\perp a}$, $\det \mathbf{P} = P^{\|}P^{\perp}$, etc. For isotropic media, such a diagonalization is always possible. For anisotropic media, this is not the general case, but we shall consider here only such simple models in which this is possible. We can then introduce 2-D paraxial ray approximations and 2-D paraxial Gaussian beams by putting $Q^{\perp} = Q^{\perp a}$, $P^{\perp} = P^{\perp a}$, $M^{\perp} = M^{\perp a}$. Thus, $Q^{\perp}$, $P^{\perp}$ and $M^{\perp}$ are real-valued even for paraxial Gaussian beams. Quantities $Q^{\|}$, $P^{\|}$ and $M^{\|}$, however, are real-valued for paraxial ray approximations, but complex-valued for paraxial Gaussian beams. The system of paraxial ray approximations connected with a selected ray $\Omega(\gamma_1)$ is then one-parameteric, with one free parameter $M^{\|}$. Similarly, the system of paraxial Gaussian beams connected with a selected ray $\Omega(\gamma_1)$ is two-parameteric, with two free parameters $\mathrm{Re}M^{\|}$ and $\mathrm{Im}M^{\|} > 0$. These parameters may be specified at any point of ray $\Omega(\gamma_1)$.

We shall now assume that the receiver point $R$ is situated in plane $\Sigma^{\|}$. The superposition integral can be derived in the same way as in Section 4. Instead of (4.9), however, we shall exploit an analogous single integral, see (A.10),

$$\int_{-\infty}^{\infty} \exp[\tfrac{1}{2}\mathrm{i}\omega x^2 W]\mathrm{d}x = (2\pi/\omega)^{1/2}[-\mathrm{i}W]^{-1/2} \ . \tag{9.1}$$

Here $W$ is a real-valued or a complex-valued constant different from zero. If $W$ is complex-valued, its imaginary part must be positive, $\mathrm{Im}W > 0$. The argument of $[-\mathrm{i}W]^{1/2}$ is given by the following relations, see (A.11),

$$\begin{aligned} &\mathrm{Re}[-\mathrm{i}W]^{1/2} > 0 && \text{for } \mathrm{Im}W > 0 \ , \\ &[-\mathrm{i}W]^{1/2} = |W|^{1/2}\exp[-\mathrm{i}\tfrac{\pi}{4}\mathrm{sgn}W] && \text{for } \mathrm{Im}W = 0 \ . \end{aligned} \tag{9.2}$$

The 2-D superposition integral, analogous to (5.1), then reads

$$u_i^{(x)}(R,\omega) = (\frac{\omega}{2\pi})^{1/2}\int_{\mathcal{D}} U_i^{ray}(R_\gamma)[-\mathrm{i}\mathcal{M}^{\parallel}]^{1/2}|Q^{\parallel a}(R_\gamma)|\exp[\mathrm{i}\omega T(R,R_\gamma)]\mathrm{d}\gamma_1 \ . \tag{9.3}$$

Here

$$\mathcal{M}^{\parallel}(R_\gamma) = M^{\parallel}(R_\gamma) - M^{\parallel a}(R_\gamma) \ , \tag{9.4}$$

and the argument of $[-\mathrm{i}\mathcal{M}^{\parallel}]^{1/2}$ is given by (9.2), for $W = \mathcal{M}^{\parallel}(R_\gamma)$. The travel time function $T(R,R_\gamma)$, expressed in Cartesian coordinates, follows immediately from (5.4) for $x_2(R) - x_2(R_\gamma) = 0$, and from appropriately simplified expression for $\hat{\mathbf{M}}^{(x)}(R_\gamma)$, given by (5.5) and (5.6).

The discussion of the superposition integral (9.3) remains practically the same as the discussion of (5.1). The superposition integral (9.3) may be used both for the summation of paraxial ray approximations ($\mathrm{Im}M^{\parallel}(R_\gamma) = 0$) and for the summation of paraxial Gaussian beams ($\mathrm{Im}M^{\parallel}(R_\gamma) > 0$). The superposition of paraxial ray approximations fails at pseudocaustic points $R$, at which $M^{\parallel}(R) = M^{\parallel a}(R)$. The summation of paraxial Gaussian beams, however, remains regular even at pseudocaustic points, as $\mathrm{Im}M^{\parallel}(R) > 0$ and $\mathrm{Im}M^{\parallel a}(R) = 0$ in this case.

The superposition integral (9.3) may be used both for strictly 2-D case (a line source perpendicular to $\Sigma^{\parallel}$ in a 2-D model) and for 2.5-D case (point source in the plane $\Sigma^{\parallel}$ in a 2-D model). The integral remains exactly the same in both cases, only the geometrical spreading in the expression for the ray amplitudes $U_i^{ray}(R_\gamma)$ are different.

Commonly, the summation in (9.3) is performed along a target line $C^R$, at which the receiver $R$ and the end-points $R_\gamma$ of rays $\Omega(\gamma_1)$ are situated. The specification of $M^{\parallel}$ is also fully analogous to the specification of matrix $\mathbf{M}$, discussed in Section 4.4. We only replace the $2\times 2$ matrices in (4.23), (4.24) and (4.22) by their upper diagonal elements.

Superposition integral (9.3) can be easily simplified for 1-D models, in which the structural parameters depend on Cartesian coordinate $x_3$ only (say depth), and in which the ray plane $\Sigma^{\parallel}$ again coincides with plane $x_2 = 0$. Let us consider $\mathrm{Im}M^{\parallel}(R_\gamma) = 0$, i.e., the summation of paraxial ray approximations. The resulting superposition integrals are then usually called the WKBJ integrals. It is common to express these integrals directly in the time domain. The WKBJ integrals were first derived by Chapman (1976, 1978), using the transform methods. Chapman also proposed that the resulting synthetic seismograms be called WKBJ seismograms. For more details, see also Dey-Sarkar and Chapman (1978), Chapman and Drummond (1982), Chapman and Orcutt (1985), Chapman (1985), Garmany (1988). An efficient and general computer program to compute the WKBJ seismograms in vertically inhomogeneous isotropic layered structures is described by Chapman, Chu and Lyness (1988). For WKBJ method in anisotropic media see Singh and Chapman (1988), and for surface waves see Wang and Dahlen (1994).


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# Appendix A
# A

## Computation of integral $\mathcal{K}$

In this Appendix, we shall compute the double integral $\mathcal{K}$ given by the relation:

$$\mathcal{K} = \iint_{-\infty}^{\infty} \exp\left[\tfrac{1}{2}\mathrm{i}\omega \mathbf{x}^T \mathbf{W}\mathbf{x}\right] \mathrm{d}x_1 \mathrm{d}x_2 \ . \tag{A.1}$$

Here $\omega$ is a real-valued positive constant (frequency), $x_1$ and $x_2$ are Cartesian coordinates, $\mathbf{x} = (x_1, x_2)^T$, and $\mathbf{W}$ is a $2 \times 2$ complex valued symmetrical matrix with a positive-definite imaginary part,

$$\mathbf{W} = \mathrm{Re}\mathbf{W} + \mathrm{i}\ \mathrm{Im}\mathbf{W} \ . \tag{A.2}$$

The method we shall use to compute (A.1) is based on the simultaneous diagonalization of the $2 \times 2$ matrices Re$\mathbf{W}$ and Im$\mathbf{W}$, and on the consequent transformation of the double integral $\mathcal{K}$ into a product of two relevant single integrals, see Červený (1982). Alternatively, also other approaches can be used to obtain the same result.

## A.1 Auxiliary computation. Single integral

Before we compute the double integral $\mathcal{K}$, we shall discuss the analogous single integral $\mathcal{I}$,

$$\mathcal{I} = \int_{-\infty}^{\infty} \exp\left[\tfrac{1}{2}\mathrm{i}\omega x^2 W\right] \mathrm{d}x \ , \tag{A.3}$$

where $\omega$ is again a real-valued positive constant, $x$ is a real-valued variable and $W$ is a complex-valued constant with a non-negative imaginary part:

$$W = \mathrm{Re}W + \mathrm{i}\ \mathrm{Im}W \ , \qquad \mathrm{Im}W \geq 0 \ . \tag{A.4}$$

Similar single integrals are well-known from mathematical textbooks:

$$\int_{-\infty}^{\infty} \exp[-ax^2]\mathrm{d}x = \sqrt{\pi/a} \ , \qquad \text{for } \mathrm{Re}(a) > 0 \ . \tag{A.5}$$

The branch of the square root $\sqrt{a}$ is specified as follows:

$$\mathrm{Re}\sqrt{a} > 0 \; . \tag{A.6}$$

We can express (A.5) in an alternative form, if we use $a = |a| \exp[\mathrm{i} \arg(a)]$,

$$\int_{-\infty}^{\infty} \exp[-ax^2]\mathrm{d}x = \sqrt{\pi/|a|}\exp\left[-\tfrac{1}{2}\mathrm{i}\arg(a)\right] , \qquad -\tfrac{\pi}{2} < \arg(a) < \tfrac{\pi}{2} \; . \tag{A.7}$$

As a limiting case of (A.7), we can also consider $\mathrm{Re}(a) = 0$.

Using (A.7) with $a = -\frac{1}{2}\mathrm{i}\omega W = \frac{1}{2}\omega(\mathrm{Im}W - \mathrm{i}\,\mathrm{Re}W)$, we obtain the following result for $\mathcal{I}$:

$$\mathcal{I} = \int_{-\infty}^{\infty} \exp\left[\tfrac{1}{2}\mathrm{i}\omega x^2 W\right]\mathrm{d}x = \sqrt{2\pi/\omega|W|}\exp\left[\tfrac{1}{2}\mathrm{i}\arg(\mathrm{Im}W - \mathrm{i}\,\mathrm{Re}W)\right] . \tag{A.8}$$

For the limiting case of $\mathrm{Im}W = 0$, we obtain $\arg(\mathrm{Im}W - \mathrm{i}\mathrm{Re}W) = \arg(-\mathrm{i}\mathrm{Re}W) = -\frac{\pi}{2}\mathrm{sgn}W$, and Eq. (A.8) yields the well-known Poisson integral

$$\mathcal{I} = \int_{-\infty}^{\infty} \exp[\tfrac{1}{2}\mathrm{i}\omega x^2 W]\mathrm{d}x = \sqrt{2\pi/\omega|W|}\exp[\mathrm{i}\tfrac{\pi}{4}\mathrm{sgn}W] \; . \tag{A.9}$$

Consequently, we can express the integral $\mathcal{I}$ in the following general form, valid for any $W$ with $\mathrm{Im}W \geq 0$:

$$\mathcal{I} = \int_{-\infty}^{\infty} \exp[\tfrac{1}{2}\mathrm{i}\omega x^2 W]\mathrm{d}x = \sqrt{2\pi/\omega}\,\frac{1}{\sqrt{-\mathrm{i}W}} \; . \tag{A.10}$$

Here the branch of $\sqrt{-\mathrm{i}W}$ is specified by relations; see (A.6) with $a = -\mathrm{i}W$,

$$\begin{aligned} \mathrm{Re}\sqrt{-\mathrm{i}W} &> 0 \qquad \text{for } \mathrm{Im}W > 0 \; , \\ \sqrt{-\mathrm{i}W} &= \sqrt{|W|}\exp[-\mathrm{i}\tfrac{\pi}{4}\mathrm{sgn}W] \quad \text{for } \mathrm{Im}W = 0 \; . \end{aligned} \tag{A.11}$$

Equation (A.10) with (A.11) represents the final result of this section. The integral exists for any complex-valued $W$ with $\mathrm{Im}W \geq 0$, and the argument of $\mathcal{I}$ satisfies the inequality:

$$-\tfrac{\pi}{4} \leq \arg\mathcal{I} \leq \tfrac{\pi}{4} \; . \tag{A.12}$$

## A.2 Computation of $\mathcal{K}$ for real-valued matrix W

The treatment of $\mathcal{K}$ for real-valued matrix $\mathbf{W}$ is simple; we can diagonalize $\mathbf{W}$ and use (A.9) to compute the two consequent single integrals.

We diagonalize matrix $\mathbf{W}$ by the linear transformation $\mathbf{x} = \mathbf{C}\mathbf{z}$, where $\mathbf{z} = (z_1, z_2)^T$ are Cartesian coordinates and $\mathbf{C}$ is a $2\times 2$ real-valued constant matrix, with $\det\mathbf{C} > 0$,

$$\mathbf{x}^T\mathbf{W}\mathbf{x} = \mathbf{z}^T\mathbf{C}^T\mathbf{W}\mathbf{C}\mathbf{z} \; , \tag{A.13}$$

where

$$\mathbf{C}^T\mathbf{W}\mathbf{C} = \begin{pmatrix} \nu_1 & 0 \\ 0 & \nu_2 \end{pmatrix} . \tag{A.14}$$

As $\mathbf{W}$ is symmetric, the eigenvalues $\nu_1$ and $\nu_2$ are real-valued. The transformation matrix $\mathbf{C}$ is formed by eigenvectors corresponding to $\nu_1$ and $\nu_2$. Then we obtain from (A.1):

$$\begin{aligned}\mathcal{K} &= \iint_{-\infty}^{\infty} \exp\left[\tfrac{1}{2}\mathrm{i}\omega\mathbf{z}^T\mathbf{C}^T\mathbf{W}\mathbf{C}\mathbf{z}\right]\det\mathbf{C}\mathrm{d}z_1\mathrm{d}z_2 \\ &= \det\mathbf{C}\int_{-\infty}^{\infty}\exp\left[\tfrac{1}{2}\mathrm{i}\omega\nu_1 z_1^2\right]\mathrm{d}z_1\int_{-\infty}^{\infty}\exp\left[\tfrac{1}{2}\mathrm{i}\omega\nu_2 z_2^2\right]\mathrm{d}z_2 \; .\end{aligned}$$

Both integrals can be calculated using (A.9). This yields

$$\mathcal{K} = \frac{2\pi}{\omega}\,\frac{\det\mathbf{C}}{\sqrt{|\nu_1\nu_2|}}\exp\left[\tfrac{1}{4}\mathrm{i}\pi(\mathrm{sgn}\nu_1 + \mathrm{sgn}\nu_2)\right] .$$

We now use the relations

$$\begin{aligned}|\nu_1\nu_2| &= |\det(\mathbf{C}^T\mathbf{W}\mathbf{C})| = (\det\mathbf{C})^2|\det\mathbf{W}| \; , \\ \mathrm{sgn}\nu_1 + \mathrm{sgn}\nu_2 &= \mathrm{Sgn}(\mathbf{C}^T\mathbf{W}\mathbf{C}) = \mathrm{Sgn}\mathbf{W} \; .\end{aligned}$$

Here $\mathrm{Sgn}\mathbf{W}$ denotes the signature of matrix $\mathbf{W}$. It equals the number of positive eigenvalues minus the number of negative eigenvalues of $\mathbf{W}$, i.e. 2, 0 or -2. In the second equation, we have used the Sylvester law of inertia, which states that the signature of matrix $\mathbf{C}^T\mathbf{W}\mathbf{C}$ equals the signature of matrix $\mathbf{W}$, if $\mathbf{C}$ is a regular matrix. Then the final equation for $\mathcal{K}$, for $\mathrm{Im}\mathbf{W} = \mathbf{0}$, is as follows:

$$\mathcal{K} = \frac{2\pi}{\omega}\,\frac{1}{|\det\mathbf{W}|^{1/2}}\exp\left[\tfrac{1}{4}\mathrm{i}\pi\mathrm{Sgn}\mathbf{W}\right] . \tag{A.15}$$

Consequently, $\arg\mathcal{K} = \frac{\pi}{2}$, 0 or $-\frac{\pi}{2}$.

## A.3 Computation of $\mathcal{K}$ for complex-valued W

If we wished to use to approach of the preceeding section to calculate $\mathcal{K}$ for a complex-valued $\mathbf{W}$, it would be necessary to diagonalize simultaneously two $2\times 2$ matrices, $\mathrm{Re}\mathbf{W}$ and $\mathrm{Im}\mathbf{W}$, by one linear transformation. This is possible if $\mathrm{Im}\mathbf{W}$ is positive definite. In mathematical textbooks, the relevant procedure is usually called the "generalized problem of eigenvalues". See Korn & Korn (1961), §13.5-5 and §14.8-7.

We shall describe here the procedure only briefly. We emphasize that $\mathrm{Im}\mathbf{W}$ is pressumably positive-definite in our case. Let us denote the relevant $2 \times 2$ constant transformation matrix by $\mathbf{B}$, with $\mathbf{x} = \mathbf{B}\mathbf{z}$, $\mathbf{z} = (z_1, z_2)^T$. Then

$$\mathbf{x}^T\mathbf{W}\mathbf{x} = \mathbf{z}^T\mathbf{B}^T\mathbf{W}\mathbf{B}\mathbf{z} = \mathbf{z}^T\begin{pmatrix}\mu_1 & 0\\ 0 & \mu_2\end{pmatrix}\mathbf{z} + \mathrm{i}\mathbf{z}^T\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\mathbf{z} \; . \tag{A.16}$$

Note only briefly that $\mu_1$ and $\mu_2$ in (A.16) can be computed as eigenvalues of the $2 \times 2$ matrix $(\mathrm{Im}\mathbf{W})^{-1}\mathrm{Re}\mathbf{W}$, and the transformation matrix $\mathbf{B}$ is formed by relevant eigenvectors. Eigenvalues $\mu_1, \mu_2$ are real-valued, as $(\mathrm{Im}\mathbf{W})^{-1}\mathrm{Re}\mathbf{W}$ is symmetric.

Using (A.1) and (A.16) yields:

$$\begin{aligned}\mathcal{K} &= \iint_{-\infty}^{\infty}\exp\left[\tfrac{1}{2}\mathrm{i}\omega(\mathbf{z}^T\mathbf{B}^T(\mathrm{Re}\mathbf{W})\mathbf{B}\mathbf{z} + \mathrm{i}\mathbf{z}^T\mathbf{B}^T(\mathrm{Im}\mathbf{W})\mathbf{B}\mathbf{z})\right]\det\mathbf{B}\mathrm{d}z_1\mathrm{d}z_2 \\ &= \det\mathbf{B}\int_{-\infty}^{\infty}\exp\left[\tfrac{1}{2}\mathrm{i}\omega z_1^2(\mu_1+\mathrm{i})\right]\mathrm{d}z_1\int_{-\infty}^{\infty}\exp\left[\tfrac{1}{2}\mathrm{i}\omega z_2^2(\mu_2+\mathrm{i})\right]\mathrm{d}z_2 \; .\end{aligned}$$

The two single integrals in the above equation can be calculated using (A.10):

$$\mathcal{K} = \frac{2\pi}{\omega} \det \mathbf{B} \frac{1}{\sqrt{1 - \mathrm{i}\mu_1}\sqrt{1 - \mathrm{i}\mu_2}} \ , \tag{A.17}$$

with

$$\mathrm{Re}\sqrt{1 - \mathrm{i}\mu_1} > 0 \ , \quad \mathrm{Re}\sqrt{1 - \mathrm{i}\mu_2} > 0 \ . \tag{A.18}$$

Expression (A.17) can be simplified if we use matrix $\mathbf{W}$ instead of $\mu_1$ and $\mu_2$. Using (A.16), we obtain

$$\det(\mathbf{B}^T \mathbf{W} \mathbf{B}) = (\mu_1 + \mathrm{i})(\mu_2 + \mathrm{i}) = -(1 - \mathrm{i}\mu_1)(1 - \mathrm{i}\mu_2) \ .$$

This yields

$$(1 - \mathrm{i}\mu_1)(1 - \mathrm{i}\mu_2) = -\det(\mathbf{B}^T \mathbf{W} \mathbf{B}) = -(\det \mathbf{B})^2 \det \mathbf{W} \ .$$

Consequently,

$$\mathcal{K} = \frac{2\pi}{\omega} \frac{1}{[-\det \mathbf{W}]^{1/2}} \ . \tag{A.19}$$

It remains to determine the correct sign of the square root $[-\det \mathbf{W}]^{1/2}$ for $\mathrm{Im}\mathbf{W} \neq \mathbf{0}$. Consider two complex members $a = |a| \exp[i\varphi_a] = \sqrt{1 - \mathrm{i}\mu_1}$, and $b = |b| \exp[i\varphi_b] = \sqrt{1 - \mathrm{i}\mu_2}$. The arguments $\varphi_a$ and $\varphi_b$ of $a$ and $b$ satisfy the conditions $-\frac{\pi}{4} < \varphi_a < \frac{\pi}{4}$, $-\frac{\pi}{4} < \varphi_b < \frac{\pi}{4}$, see (A.12). Then the argument of the product of $a$ and $b$, $c = ab = |c| \exp[\mathrm{i}\varphi_c]$, satisfies the condition $-\frac{\pi}{2} < \varphi_c < \frac{\pi}{2}$, i.e. $\mathrm{Re}(c) > 0$. Consequently, the final condition for $\mathrm{Im}\mathbf{W} \neq \mathbf{0}$ is as follows:

$$\mathrm{Re}[-\det \mathbf{W}]^{1/2} > 0 \ . \tag{A.20}$$

Note that the method of computation of $\mathcal{K}$, based on the simultaneous diagonalization of $\mathrm{Re}\mathbf{W}$ and $\mathrm{Im}\mathbf{W}$, was proposed already by Červený (1982).

## A.4 Final equations

Using (A.19), (A.20) and (A.15), we obtain the final expressions for the integral $\mathcal{K}$:

$$\mathcal{K} = \iint_{-\infty}^{\infty} \exp\left[\tfrac{1}{2}\mathrm{i}\omega \mathbf{x}^T \mathbf{W} \mathbf{x}\right] \mathrm{d}x_1 \mathrm{d}x_2 = \frac{2\pi}{\omega} \frac{1}{[-\det \mathbf{W}]^{1/2}} \ . \tag{A.21}$$

The branch of $[-\det \mathbf{W}]^{1/2}$ is specified by relations

$$\begin{array}{ll} \mathrm{Re}[-\det \mathbf{W}]^{1/2} > 0 & \text{for } \mathrm{Im}\mathbf{W} \neq \mathbf{0} \ , \\ [-\det \mathbf{W}]^{1/2} = |\det \mathbf{W}|^{1/2} \exp[-\mathrm{i}\frac{\pi}{4}\mathrm{Sgn}\mathbf{W}] & \text{for } \mathrm{Im}\mathbf{W} = \mathbf{0} \ . \end{array} \tag{A.22}$$

The integral $\mathcal{K}$ exists for any real-valued matrix $\mathbf{W}$ and for any complex-valued matrix $\mathbf{W}$ with a positive-definite $\mathrm{Im}\mathbf{W}$. Similarly as in (A.12) for the single integral $\mathcal{I}$, we obtain

$$-\frac{\pi}{2} \leq \arg\mathcal{K} \leq \frac{\pi}{2} \ . \tag{A.23}$$