Calculation of geometrical spreading from gridded slowness vectors in 2-D

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Summary

The paper is devoted to the numerical calculation of the point–source geometrical spreading from gridded travel times or slowness vectors in 2-D. The geometrical spreading is obtained by finite differencing the slowness vectors corresponding to two different positions of a point source. The slowness vectors may be calculated directly, or as the finite–difference gradients of gridded travel times.

Keywords

Paraxial ray approximation, travel times, geometrical spreading, amplitudes, Green function.

1 Introduction

There are some algorithms or computer programs reliably and accurately calculating travel times and their gradients (slowness vectors) on a dense rectangular grid of points. If we are interested also in amplitudes or geometrical spreading, although the computer program used is not supplemented with the calculation of geometrical spreading, the geometrical spreading can be, under certain circumstances, calculated by finite differencing the gradients of travel times corresponding to different positions of a point source.

Assume that the travel–time field, corresponding to a point source, is smooth and the travel times coincide with the ray–theory travel times, i.e., that the travel times correspond to rays starting at the point source. Then the travel time field can be calculated for two or more positions of the point source. The geometrical spreading, and even the whole paraxial ray propagator matrix, can then be obtained by finite differencing the travel times and their gradients with respect to the position of a point source. At least two different positions of a point source are required for the point–source geometrical spreading in 2-D, three different positions of a point source are required for the matrix of point–source geometrical spreading in 3-D, and more positions are required for the whole paraxial ray propagator matrix.

The relevant equations can be obtained from the equation for the second–order paraxial approximation of the two–point eikonal, derived, e.g., by Arnaud (1973, eq. 11.1). In this paper, we show the calculation of the 2-D point–source geometrical spreading from gridded travel times as an example. We start with the second–order paraxial approximation of the two–point eikonal in the notation of Červený, Klimeš & Pšenčík (1984), who suggested its application in geophysics.

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2 Paraxial approximation of the two–point eikonal

Second–order Taylor expansion of travel time $\tau(\mathbf{x},\mathbf{x}^0)$ between source point $\mathbf{x}^0$ and point $\mathbf{x}$ reads

$$\tau(\mathbf{x}+\Delta\mathbf{x},\mathbf{x}^0+\Delta\mathbf{x}^0)=\tau(\mathbf{x},\mathbf{x}^0)+\frac{\partial\tau}{\partial x_i}(\mathbf{x},\mathbf{x}^0)\Delta x_i+\frac{\partial\tau}{\partial x_i^0}(\mathbf{x},\mathbf{x}^0)\Delta x_i^0$$

$$+\frac{1}{2}\frac{\partial^2\tau}{\partial x_i\partial x_j}(\mathbf{x},\mathbf{x}^0)\Delta x_i\Delta x_j+\frac{\partial^2\tau}{\partial x_i\partial x_j^0}(\mathbf{x},\mathbf{x}^0)\Delta x_i\Delta x_j^0$$

$$+\frac{1}{2}\frac{\partial^2\tau}{\partial x_i^0\partial x_j^0}(\mathbf{x},\mathbf{x}^0)\Delta x_i^0\Delta x_j^0+O(|\Delta\mathbf{x}|^3)+O(|\Delta\mathbf{x}^0|^3)\quad, \tag{1}$$

where

$$|\Delta\mathbf{x}|=\sqrt{\Delta x_i\Delta x_i}\quad,\qquad|\Delta\mathbf{x}^0|=\sqrt{\Delta x_i^0\Delta x_i^0}\quad. \tag{2}$$

Lower–case subscripts take values $i=1,2,3$ in 3-D and, e.g., values $i=1,3$ in 2-D.

The first travel–time derivatives correspond to the slowness vectors at points $\mathbf{x}$ and $\mathbf{x}^0$, the second travel–time derivatives are dependent on the submatrices of the paraxial ray propagator matrix. For more details refer to equation (22) of Červený, Klimeš & Pšenčík (1984). Here we shall concentrate on the mixed second derivatives, which are dependent on the right upper submatrix

$$\mathbf{Q}=\mathbf{\Pi}_{12}(\mathbf{x},\mathbf{x}^0) \tag{3}$$

of the paraxial ray propagator matrix

$$\mathbf{\Pi}(\mathbf{x},\mathbf{x}^0)=\begin{pmatrix}\mathbf{\Pi}_{11}(\mathbf{x},\mathbf{x}^0) & \mathbf{\Pi}_{12}(\mathbf{x},\mathbf{x}^0)\\ \mathbf{\Pi}_{21}(\mathbf{x},\mathbf{x}^0) & \mathbf{\Pi}_{22}(\mathbf{x},\mathbf{x}^0)\end{pmatrix}\quad. \tag{4}$$

Matrix $\mathbf{Q}$ describes the geometrical spreading corresponding to a point source.

If paraxial ray propagator matrix (4) is expressed in ray–centred coordinates q_i, introduced by Luneburg (1944) and Popov & Pšenčík (1978), then $Q_{i3}=0$, $Q_{3i}=0$ (Klimeš 1994) and

$$\frac{\partial^2\tau}{\partial x_i\partial x_j^0}(\mathbf{x},\mathbf{x}^0)=-H_{iK}Q^{-1}_{KL}H^0_{jL} \tag{5}$$

(Červený, Klimeš & Pšenčík 1984), where Q^{-1}_{KL} is inverse matrix to 2×2 submatrix Q_{KL} and

$$H_{ik}=\frac{\partial x_i}{\partial q_k}\quad,\qquad H^0_{ik}=\frac{\partial x^0_i}{\partial q^0_k} \tag{6}$$

are the transformation matrices between the coordinate systems. Upper–case subscripts take values $I=1,2$ in 3-D and, e.g., value $I=1$ in 2-D.

Slowness vector

$$p_i(\mathbf{x},\mathbf{x}^0)=\frac{\partial\tau}{\partial x_i}(\mathbf{x},\mathbf{x}^0) \tag{7}$$

corresponding to point source at $\mathbf{x}^0$ is obtained by differentiating (1) with respect to Δx_i,

$$p_i(\mathbf{x}+\Delta\mathbf{x},\mathbf{x}^0+\Delta\mathbf{x}^0) = \frac{\partial\tau}{\partial x_i}(\mathbf{x},\mathbf{x}^0) + \frac{\partial^2\tau}{\partial x_i\partial x_j}(\mathbf{x},\mathbf{x}^0)\Delta x_j + \frac{\partial^2\tau}{\partial x_i\partial x_j^0}(\mathbf{x},\mathbf{x}^0)\Delta x_j^0$$
$$+O(|\Delta\mathbf{x}|^2)+O(|\Delta\mathbf{x}^0|^2) \quad . \tag{8}$$

Difference

$$\Delta p_i = p_i(\mathbf{x},\mathbf{x}^0+\tfrac{1}{2}\Delta\mathbf{x}^0) - p_i(\mathbf{x},\mathbf{x}^0-\tfrac{1}{2}\Delta\mathbf{x}^0) \tag{9}$$

of the slowness vectors corresponding to a point source shifted by $\pm\frac{1}{2}\Delta\mathbf{x}$ is

$$\Delta p_i = \frac{\partial^2\tau}{\partial x_i\partial x_j^0}(\mathbf{x},\mathbf{x}^0)\Delta x_j^0 + O(|\Delta\mathbf{x}^0|^3) \quad . \tag{10}$$

3 Calculation of point–source geometrical spreading in 2-D

Equation (10) with (5) in 2-D reads

$$\Delta p_i = -H_{i1}Q_{11}^{-1}H_{j1}^0\Delta x_j^0 + O(|\Delta\mathbf{x}^0|^3) \quad , \tag{11}$$

where Q_{11} is the in–plane component of matrix (3). Unit basis vector H_{i1} of the ray–centred coordinate system is perpendicular to the slowness vector,

$$(H_{11},H_{31}) = v\ (-p_3,p_1) \quad , \tag{12a}$$

$$(H_{11}^0,H_{31}^0) = v^0\ (-p_3^0,p_1^0) \quad . \tag{12b}$$

Slowness vector $\mathbf{p}$ at point $\mathbf{x}$ is known or may be approximated by

$$p_i(\mathbf{x},\mathbf{x}^0) = \tfrac{1}{2}\left[p_i(\mathbf{x},\mathbf{x}^0+\tfrac{1}{2}\Delta\mathbf{x}) + p_i(\mathbf{x},\mathbf{x}^0-\tfrac{1}{2}\Delta\mathbf{x})\right] + O(|\Delta\mathbf{x}^0|^2) \quad . \tag{13}$$

On the other hand, initial slowness vector $\mathbf{p^0}$ at point $\mathbf{x}^0$ is unknown and should be estimated from the travel time difference

$$\Delta\tau = \tau(\mathbf{x},\mathbf{x}^0+\tfrac{1}{2}\Delta\mathbf{x}) - \tau(\mathbf{x},\mathbf{x}^0-\tfrac{1}{2}\Delta\mathbf{x}) \quad . \tag{14}$$

and eikonal equation

$$p_i^0p_i^0 = (v^0)^2 \tag{15}$$

at the initial point. Equation (1) yields

$$\Delta\tau = \frac{\partial\tau}{\partial x_i^0}(\mathbf{x},\mathbf{x}^0)\Delta x_i^0 + O(|\Delta\mathbf{x}^0|^3) = -p_j^0\Delta x_j^0 + O(|\Delta\mathbf{x}^0|^3) \quad . \tag{16}$$

The initial slowness vector may be decomposed into the unit vector $\Delta x_j^0/|\Delta\mathbf{x}^0|$ in the direction of the source–point shift and the perpendicular vector

$$(E_1,E_3) = (-\Delta x_3^0,\Delta x_1^0)/|\Delta\mathbf{x}^0| \quad , \tag{17}$$

$$p_j^0 = a\ \Delta x_j^0/|\Delta\mathbf{x}^0| + b\ E_j \quad . \tag{18}$$

Inserting (18) into (16), we get coefficient

$$a = -\Delta\tau/|\Delta\mathbf{x}^0| + O(|\Delta\mathbf{x}^0|^2) \quad . \tag{19}$$

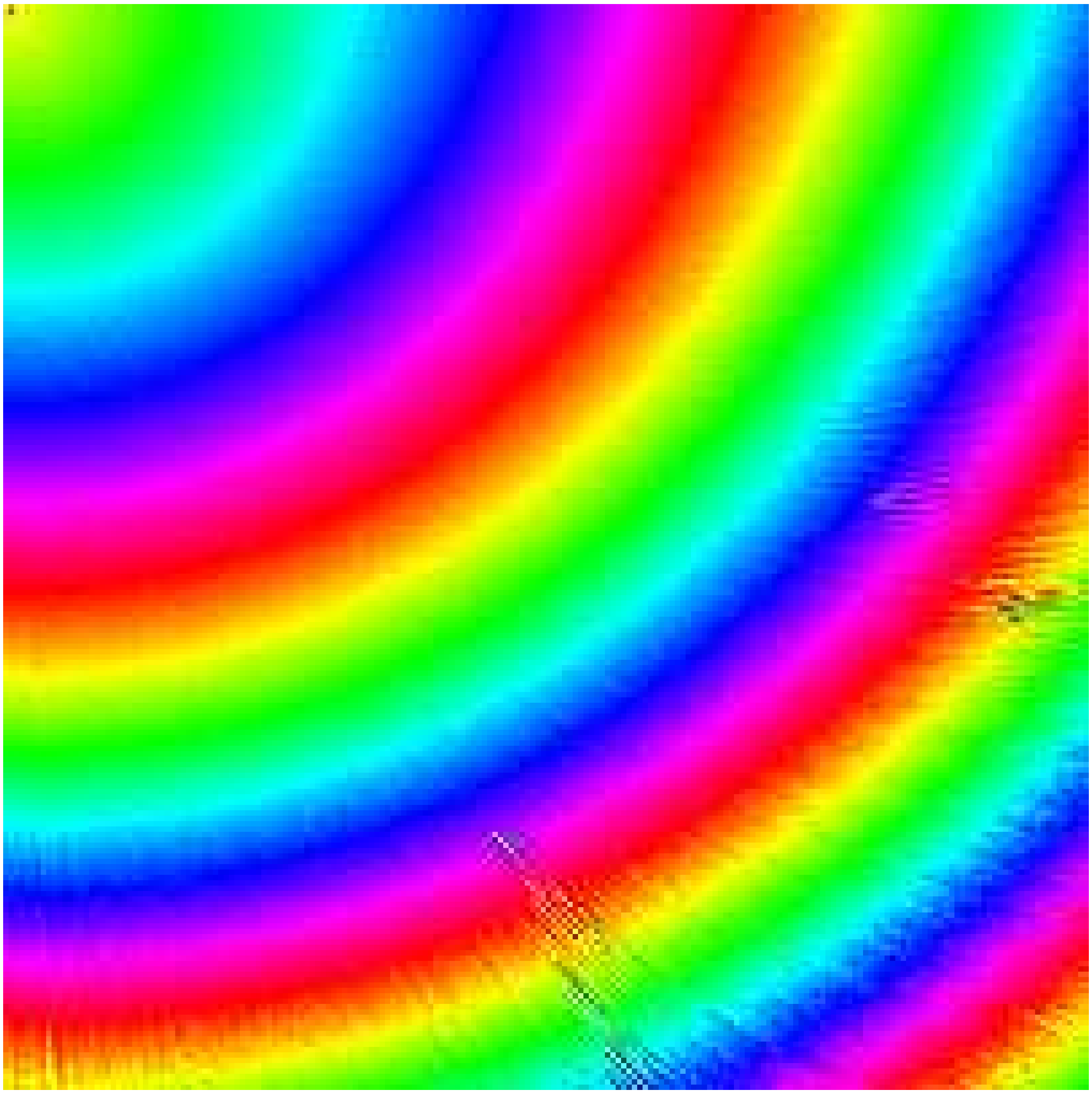

Figure 1. Inverse geometrical spreading numerically calculated from gridded slowness vectors computed by program "net.for". Yellow colour corresponds to values $Q_{11} = 0, 1, 2, 3$.

Absolute value of coefficient b can be calculated from eikonal equation (15),

$$|b| = \sqrt{(v^0)^{-2} - a^2} \quad . \tag{20}$$

Outer product of initial slowness vector p_i^0 with equation (18) yields

$$0 = a\, p_i^0 \varepsilon_{ij} \Delta x_j^0 + b\, p_i^0 \Delta x_i^0 \quad , \tag{21}$$

see (17). Here

$$\varepsilon_{ij} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad . \tag{22}$$

Using (16) and (19), equation (21) reads

$$0 = -p_i^0 \varepsilon_{ij} \Delta x_j^0 / |\Delta \mathbf{x}^0| - b \quad , \tag{23}$$

Figure 2. Inverse geometrical spreading interpolated within ray cells by program "mtt.for". Yellow colour corresponds to values $Q_{11} = 0, 1, 2, 3$. The appearance of the figure is influenced by sampling Q_{11} into just 255 levels.

which may be used to determine the sign of b,

$$b = -\sqrt{\left(\frac{1}{v^0}\right)^2 - \left(\frac{\Delta\tau}{|\Delta\mathbf{x}^0|}\right)^2}\ \mathrm{sgn}(p_i^0 \varepsilon_{ij} \Delta x_j^0) \quad . \tag{24}$$

After insertion of (18), equation (12b) reads

$$H_{j1}^0 = v^0\ (a\ E_j - b\ \Delta x_j^0 / |\Delta\mathbf{x}^0|) \tag{25}$$

and thus

$$H_{j1}^0 \Delta x_j^0 = -b\ v^0\ |\Delta\mathbf{x}^0| \quad . \tag{26}$$

Outer product of slowness vector p_i with (11) reads, after insertion of (12a) and (26),

$$p_i\varepsilon_{ij}\Delta p_j = v^{-1}Q_{11}^{-1}b\,v^0\,|\Delta\mathbf{x}^0| + O(|\Delta\mathbf{x}^0|^3) \quad . \tag{27}$$

The final estimation of the 2-D geometrical spreading is thus

$$Q_{11}^{-1} = \frac{v\,p_i\varepsilon_{ij}\Delta p_j}{b\;v^0\;|\Delta\mathbf{x}^0|} + O(|\Delta\mathbf{x}^0|^2) \tag{28}$$

with b given by equation (24). Quantities $\Delta\tau$ in (24) and Δp_j in (28) are defined by (14) and (9), respectively. Slowness vector p_i in (28) may be approximated by (13), using the gridded slowness vectors for two shifted initial points. The gridded slowness vectors may be calculated from gridded travel times by the centred finite differences.

The main problem in application of equation (28) is the choice of proper size of shift $\Delta\mathbf{x}^0$. For too small $\Delta\mathbf{x}^0$, we get too small $\Delta\mathbf{p}$ in comparison with its accuracy. For too large $\Delta\mathbf{x}^0$, we get too large error caused by neglecting $O(|\Delta\mathbf{x}^0|^2$ in (28).

4 Numerical example

Calculation of Q_{11} from gridded slowness vectors according to equation (28) has been coded in the form of command file "q2.cal" for program "grdcal.for".

The chosen constant velocity gradient model is located in a unit vertical square, the velocity increases linearly with depth, from 1 unit at the top to 3 units at the bottom. The grid interval is 0.002. The sources are shifted from the left–hand top corner of the square by 1 grid interval right and 1 interval upwards, and by 1 interval left and 1 interval downwards, respectively.

Travel times and slowness vectors in a constant velocity gradient model have been calculated by the second–order grid travel–time tracing method by Klimeš (1996) using program "net.for". Resulting values of Q_{11}, displayed in Figure 1, may be compared with the values of Q_{11} interpolated within ray cells by program "mtt.for", see Figure 2. The data for the numerical example are located in subdirectory "data/vgr" on compact disk SW3D-CD-4.

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