Interpolation within ray cells

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Summary

3-D ray tracing followed by interpolation of the computed quantities between the rays is a powerful tool for computation of ray–theory travel times, amplitudes and other quantities at the gridpoints of dense rectangular grids. Several methods based on the decomposition of the model volume into ray cells, and on further interpolation within the individual ray cells, have recently been introduced. We propose and mutually compare bilinear and bicubic interpolation schemes, which should be applicable in any modelling method based on interpolation within ray cells. The bicubic interpolation is designed for travel times and the bilinear interpolation for other quantities.

1 Introduction

3-D ray modelling, especially the computation of ray-theory travel times at nodes of dense grids of points becomes to be quite important in recent years. The possibility to compute the travel times is of principal importance in imaging of depth structures, seismic tomography or hypocenter determination techniques. Several methods based on decomposition of the model volume into ray cells, and on further interpolation within individual cells, were introduced. Let us mention, e.g., the *wavefront construction method* by Vinje *et al.* (1996a, 1996b), similar method in smooth models by Lucio *et al.* (1996), the *recursive method* by Moser and Pajchel (1997), and the *controlled initial–value ray tracing method* by Bulant (1997).

Whereas the above mentioned methods differ in the manner of the decomposition of the model volume into ray cells, they all contain some procedure of interpolation of computed quantities to the receivers (usually nodes of a receiver grid) located within a single ray cell. We propose bilinear and bicubic interpolation schemes. The bilinear interpolation is easy and robust. It incorporates both the decision, whether a receiver lies in the ray cell, and the interpolation of quantities computed along the rays to the receiver. The scheme uses only the values of quantities at the vertices of the ray cell. Thus the scheme should be applicable in any modelling method based on interpolation within ray cells. If the partial derivatives are known in addition to the functional values at the vertices of the ray cell, the bicubic interpolation scheme may be used to increase the accuracy of the interpolation. The bicubic interpolation is designed for travel times and the bilinear interpolation for other quantities.

2 Determination of a ray cell

Each of three lateral edges of a roughly prismatic ray cell is determined by two points on the ray corresponding to the edge. Both the bottom and the top of the ray cell are formed by triangles defined by three points on the rays, mostly corresponding to the same travel–time level. In the vicinity of a structural interface or at a point source, some vertices of a ray cell may coincide, making the cell degenerate.

By determination of a ray cell we understand the decision, whether an arbitrary point lies within the ray cell, or not. Assume a ray cell defined by six points $\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3, \mathbf{C}_1, \mathbf{C}_2, \mathbf{C}_3$ on three rays, where points $\mathbf{B}_i = ({B_i}^1, {B_i}^2, {B_i}^3)^T$ are at the bottom, and points $\mathbf{C}_i$ at the top of the ray cell, see figure 1. We introduce a local barycentric coordinate system parametrized by parameters w_1, w_2, w_B. The coordinates of arbitrary point $\mathbf{X}$ are then given as

$$\mathbf{X} = \mathbf{A}_3 + w_1(\mathbf{A}_1 - \mathbf{A}_3) + w_2(\mathbf{A}_2 - \mathbf{A}_3) \quad , \tag{1}$$

where

$$\mathbf{A}_i = \mathbf{C}_i + w_B(\mathbf{B}_i - \mathbf{C}_i), \; i = 1, 2, 3. \tag{2}$$

Equations (1) and (2) may also be expressed as

$$\mathbf{X} = w_B \sum_{i=1}^{3} w_i \mathbf{B}_i + w_C \sum_{i=1}^{3} w_i \mathbf{C}_i \tag{3}$$

with $w_C = 1 - w_B, \quad w_3 = 1 - w_1 - w_2$. Equation (1) is the parametric description of planar triangle $\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3$. Equations (2) are parametric descriptions of straight-line segments $\mathbf{B}_1\mathbf{C}_1$, $\mathbf{B}_2\mathbf{C}_2$ and $\mathbf{B}_3\mathbf{C}_3$, approximating the ray segments. Note that only in the homogeneous parts of the model the straight segments lie on the exact rays.

If point $\mathbf{X}$ lies within the ray cell, all values w_1, w_2, w_3, w_B, w_C are within range $< 0, 1 >$. Thus, for each receiver, which might be located within the selected ray cell, we compute its local coordinates w_1, w_2, w_B and quantities

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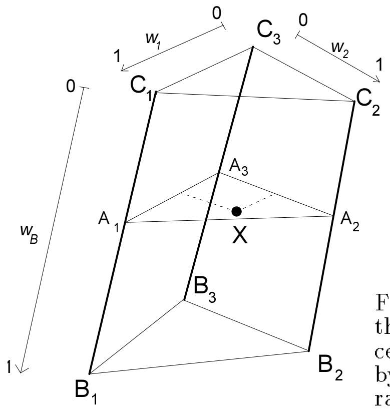

Figure 1: A ray cell is defined by six points $\mathbf{B}_1$, $\mathbf{B}_2$, $\mathbf{B}_3$, $\mathbf{C}_1$, $\mathbf{C}_2$, $\mathbf{C}_3$, situated on three rays (bold lines). Point $\mathbf{X}$ is examined, whether it is located within the ray cell. Its position may be expressed using the local coordinate system parametrized by parameters w_1, w_2, w_B. If all of the quantities w_1,w_2,w_B,$1-w_1-w_2$ are in range $<0,1>$, point $\mathbf{X}$ lies within the ray cell.

w_3, w_C. If all of them display values from range $<0,1>$, point $\mathbf{X}$ is within the ray cell, and the values of all required quantities will be determined by interpolation within the ray cell.

First the value of local coordinate w_B is determined by numerically solving the corresponding cubic equation (Bulant & Klimeš 1998). For known w_B, and thus known points (2), coordinates w_1 and w_2 are the solution of linear equations (1).

If the ray cell touches a caustic, two or three (in 3-D) different sets of values w_1, w_2, w_B may be found within range $<0,1>$ for the same point $\mathbf{X}$. Different coordinates w_1, w_2, w_B of the same point $\mathbf{X}$ correspond to different ray coordinates $\gamma_1, \gamma_2, \gamma_3$, and the interpolation should be performed for each set of coordinates w_1, w_2, w_B.

3 Interpolation within a ray cell

3.1 Bilinear interpolation

The simplest interpolation to fit the functional values at six points $\mathbf{B}_1$, $\mathbf{B}_2$, $\mathbf{B}_3$, $\mathbf{C}_1$, $\mathbf{C}_2$, $\mathbf{C}_3$ is the bilinear interpolation with respect to w_B and (w_1, w_2). Linear interpolation

$$\tau^{\mathbf{A}_k} = w_B \tau^{\mathbf{B}_k} + w_C \tau^{\mathbf{C}_k} \quad , \quad w_C = 1 - w_B \quad , \tag{4}$$

with respect to w_B is used to obtain the values $\tau^{\mathbf{A}_k}$ of an interpolated quantity τ at points $\mathbf{A}_k$ from known values $\tau^{\mathbf{B}_k}$ and $\tau^{\mathbf{C}_k}$ at the vertices of the ray cell. Linear interpolation

$$\tau^{\mathbf{X}} = w_1 \tau^{\mathbf{A}_1} + w_2 \tau^{\mathbf{A}_2} + w_3 \tau^{\mathbf{A}_3} \tag{5}$$

with respect to (w_1, w_2) is then used to obtain the values at point $\mathbf{X}$. This interpolation fits the functional values at the six vertices of the ray cell. Since the values at each cell face are dependent on the values at the vertices of the face only, the interpolated values are continuous across the cell faces. This interpolation seems convenient in interpolating all the quantities resulting from ray tracing, except for the travel time which should be interpolated more accurately, see figures 2a,b,c,d.

3.2 Bicubic interpolation

Assume that partial derivatives

$$p^i = \frac{\partial \tau}{\partial x^i} \tag{6}$$

are known in addition to the functional values of τ at six vertices $\mathbf{B}_1$, $\mathbf{B}_2$, $\mathbf{B}_3$, $\mathbf{C}_1$, $\mathbf{C}_2$, $\mathbf{C}_3$ of the ray cell. We may then wish to use the bicubic interpolation with respect to w_B and (w_1, w_2) to fit both the functional values and the first partial derivatives at the six vertices. To fit the functional values, we define cubic function

$$a(w) = w^2(3 - 2w) \tag{7}$$

with functional values $a(0) = 0$, $a(1) = 1$ and first derivatives $a'(0) = 0$, $a'(1) = 0$ at the endpoints of interval $<0,1>$. To fit the derivatives, we define cubic functions

$$b(w_B) = (w_B)^2 \sum_{i=1}^{3} p^i_{\mathbf{B}} [A^i(w_B) - B^i] \tag{8}$$

with functional values $b(0) = 0$, $b(1) = 0$ and derivatives $b'(0) = 0$, $b'(1) = \sum_{i=1}^{3} p^i_{\mathbf{B}} [B^i - C^i]$ at the endpoints of interval $<0,1>$. Here

$$\mathbf{A}(w_B) = w_B \mathbf{B} + (1 - w_B) \mathbf{C} \quad , \tag{9}$$

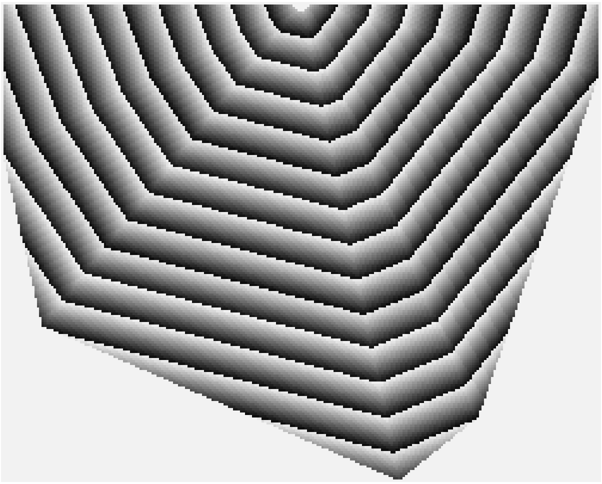

Figure 2a: Travel times in a homogeneous model interpolated bilinearly within wide 3-D ray tubes using equation (5). One grey–scale circle corresponds to 0.2 s.

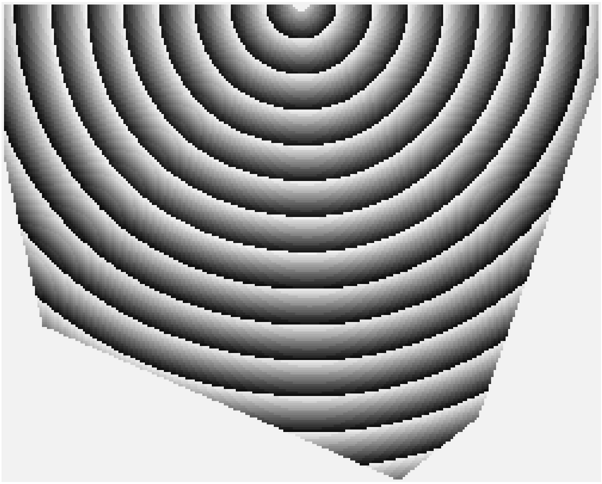

Figure 2b: Travel times in a homogeneous model interpolated within the same ray tubes using bicubic interpolation (13). The maximum travel time is about 2.7 s.

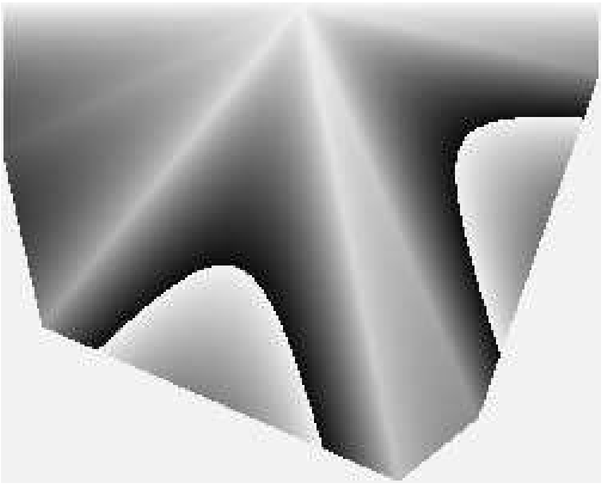

Figure 2c: Differences of the bilinearly interpolated travel times from the exact solution. The maximum difference is about 0.308 s.

Figure 2d: Differences of the bicubically interpolated travel times from the exact solution. The maximum difference is about 0.017 s.

see (2).

Analogously for functions

$$b(w_C) = (w_C)^2 \sum_{i=1}^{3} p^i_{\mathbf{C}}[A^i(w_C) - C^i] \quad . \tag{10}$$

Cubic interpolations

$$\boxed{\tau^{\mathbf{A}_k} = a(w_B)\tau^{\mathbf{B}_k} + a(w_C)\tau^{\mathbf{C}_k} + (w_B)^2 \sum_{i=1}^{3} p^i_{\mathbf{B}_k}[\mathbf{A}^i_k - \mathbf{B}^i_k] + (w_C)^2 \sum_{i=1}^{3} p^i_{\mathbf{C}_k}[\mathbf{A}^i_k - \mathbf{C}^i_k]} \tag{11}$$

and

$$\boxed{p^i_{\mathbf{A}_k} = a(w_B)p^i_{\mathbf{B}_k} + a(w_C)p^i_{\mathbf{C}_k} \quad , \quad i = 1,2,3,} \tag{12}$$

along the edges of the ray cell, parametrized by w_B, are used to obtain functional values $\tau^{\mathbf{A}_k}$ at points $\mathbf{A}_k$ together with their first derivatives $p^i_{\mathbf{A}_k}$. The derivatives will be used for the interpolation within triangle $\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3$.

Cubic interpolation within triangle $\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3$

$$\tau^{\mathbf{X}} = \sum_{k=1}^{3} a(w_k)\tau^{\mathbf{A}_k} + \sum_{k=1}^{3}(w_k)^2 \sum_{i=1}^{3} p^i_{\mathbf{A}_k}[X^i - \mathbf{A}^i_k] + w_1 w_2 w_3 \xi \tag{13}$$

with respect to (w_1, w_2) is then used to obtain the values at point $\mathbf{X}$. The three functional values and six first partial derivatives at the vertices of planar triangle $\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3$ are fitted by the cubic function of w_1 and w_2. Since a cubic function of two variables has 10 coefficients, there is a free coefficient ξ in the last equation. Coefficient ξ corresponds to the basis function $w_1 w_2 w_3$ which has both the functional values and the first derivatives at the vertices $\mathbf{A}_k$ equal to zero.

Coefficient ξ must depend linearly on the given interpolated quantities, and the linear combination must be symmetric with respect to the permutation of points $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3$. To interpolate linear and quadratic functions exactly, we have to take (Bulant & Klimeš 1998)

$$\xi = 2\sum_{k=1}^{3} \tau^{\mathbf{A}_k} + \frac{1}{2}\sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{i=1}^{3} p^i_{\mathbf{A}_k}[\mathbf{A}^i_l - \mathbf{A}^i_k] \quad . \tag{14}$$

Here the first term is necessary to interpolate linear functions exactly, and the second term is added to interpolate also quadratic functions exactly.

If we check the interpolation of cubic functions, we find that the scheme interpolates three of the four independent cubic functions exactly, but the last independent cubic function is not fitted. There is no way of removing this inaccuracy in the interpolation of cubic functions without additional information.

The interpolation fits the values and the first spatial partial derivatives at the six vertices of the ray cell. The functional values at each cell face depend on the functional values and derivatives at the vertices of the face only. The interpolated values are thus continuous across the cell faces. The gradient of the interpolated function is continuous at the triangular cell faces and along the cell edges, but may be discontinuous at the tetragonal cell faces separating ray tubes. The interpolation fits linear and quadratic functions exactly.

4 Conclusions

We have proposed bilinear and bicubic schemes for interpolation of ray-theory travel times and other quantities to the gridpoints of dense rectangular grids. Using the schemes, all the quantities computed along rays may be interpolated. The schemes require the values of the interpolated quantities only at the vertices of the corresponding ray cell. The schemes are applicable also to degenerate ray cells. The bilinear scheme incorporates both the decision, whether a receiver lies in the ray cell and the interpolation to the receiver. The bicubic interpolation scheme uses also the values of partial derivatives of the interpolated quantities at the vertices of the corresponding ray cell, which considerably improves the accuracy of interpolation

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