Sobolev scalar products in the construction of velocity models – smoothing the regional model of Bohemian Massif for ray tracing

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This paper was prepared for presentation during the 11th International Congress of the Brazilian Geophysical Society held in Salvador, Brazil, August 24-28, 2009.

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Abstract

Minimization of a Sobolev norm during linearized inversion of given data enables to control the model parameters unresolved by the data being fitted.

Even if a reasonably looking model can be obtained without minimizing the Sobolev norm, it may be too rough for some computational methods. We may construct models optimally smooth for given computational methods by minimizing the corresponding Sobolev norm during the inversion.

Probably the smoothest models are required by the ray methods. The efficiency of ray tracing can be evaluated in terms of the "average Lyapunov exponent" for the model. The "average Lyapunov exponent" may be approximated by the square root of the corresponding Sobolev norm of the model, which allows models optimum for ray tracing to be constructed.

Sobolev scalar products and Sobolev norms

We define the Sobolev scalar product $((f,g))$ (see, e.g., Tarantola 1987) of real-valued functions f and g as a linear combination of the L_2 Lebesgue scalar products of the zero, first, second or higher derivatives of the functions. The linear combination should preserve the properties required of scalar products (symmetry and positiveness).

The Sobolev norm of a velocity model may, in general, be composed of the zero, first, second or higher derivatives of the functions describing the model. The model is very often described by cubic, bicubic or tricubic splines. Since the third homogeneous partial derivatives of the splines are discontinuous, the Sobolev scalar product composed of the fourth and higher derivatives could be infinite, and thus should be avoided. Simultaneously, minimization of the (zero derivatives of the) model functions usually has no meaning. We thus usually consider only Sobolev norms composed of the first, second and third derivatives. Since studies of the third derivatives have not yet been finished, we will concentrate on the first and second derivatives.

Including the Sobolev norm in the objective function for inversion of seismic data

Minimization of the Sobolev norm during linearized inversion of given data enables to control the model parameters unresolved by the data being fitted. For example, if we are fitting discrete values of a material parameter with a smooth function, the properties of interpolation between the discrete values may be controlled by means of minimizing the square of the selected Sobolev norm of the function.

Even if a reasonably looking model can be obtained without minimizing the Sobolev norm, it may be too rough for some computational methods. The errors of many computational methods may be approximated by a function dependent on the Sobolev norm of functions describing the model. We may thus construct models optimally smooth for given computational methods.

In the case of ray methods, the situation is more complicated. The feasibility and efficiency of ray tracing imposes relatively strong requirements on the smoothness of the velocity models, but, unfortunately, we still have no quantitative criteria of applicability and accuracy of ray methods or their extensions. On the other hand, the numerical efficiency of ray tracing can be evaluated in terms of the "average Lyapunov exponent", introduced by Klimeš (2002a) for 2-D models without structural interfaces.

Fitting discrete values and regularizing an ill-conditioned inversion

Assume that we are fitting discrete values u^α at points $\mathbf{x}^\alpha$ inside the model volume by minimizing objective function

$$y = \sum \left[\tilde{u}(\mathbf{x}^\alpha) - u^\alpha\right]^2 \left[\Delta u^\alpha\right]^{-2} + S^2((\tilde{u},\tilde{u})), \quad (1)$$

where Δu^α is a given standard deviation of model $\tilde{u}(\mathbf{x})$ from values u^α at points $\mathbf{x} = \mathbf{x}^\alpha$. The properties of interpolation between discrete values by means of minimizing the squares of different Sobolev norms have been investigated by Klimeš (2000). He showed that the Sobolev norm composed of second derivatives of model functions is suitable for fitting discrete values.

Assume that we are fitting a structural interface with a smooth function. In such a case the data for the function are available only in that part of the model, where the interface is located. The behaviour of the function in other parts of the model is not given by the data. We might

control the function by adding new artificial points to the data for the interface, but this manual intervention would probably slightly distort the modelled interface. It is much more convenient to control the function by increasing the weighting factor S of the Sobolev norm. A small increment of S will not influence the function in the region with the data, but will regularize its behaviour in the remaining parts of the model, see, e.g., Bulant (2002).

Constructing seismic models of optimal smoothness for a selected numerical method

The Sobolev norm is usually included in the inversion of seismic dataif the numerical method to be applied to the model requires a smooth model. The Sobolev norm included in the objective function acts as a low--pass wavenumber filter (Klimeš 2000).

We assume the objective function for the inversion of slowness or velocity $u(\mathbf{x})$ in the form of

$$y = \left[\int \mathrm{d}\mathbf{x}\right]^{-1} \int \mathrm{d}\mathbf{x}\, \tilde{C}(\mathbf{x},\mathbf{x}) + S^2((\tilde{u},\tilde{u})) , \quad (2)$$

where

$$\tilde{C}(\mathbf{x},\mathbf{x}') = \left\langle [\tilde{u}(\mathbf{x}) - u(\mathbf{x})][\tilde{u}(\mathbf{x}') - u(\mathbf{x}')] \right\rangle , \quad (3)$$

is the covariance function describing the deviation of model $\tilde{u}(\mathbf{x})$ from geological structure $u(\mathbf{x})$ (Klimeš 2002b), and S is the weighting factor of the Sobolev norm $((\tilde{u},\tilde{u}))$ of the functions describing the model.

The first summand in (3) is the mean squared difference between the model and the geological structure. It is then advantageous if $S((\tilde{u},\tilde{u}))^{-2}$ approximates the r.m.s. difference of the results of the numerical method under consideration from the exact solution in the model, expressed in the units of $u(\mathbf{x})$. Objective function (3) then gains the clear physical meaning of the squared deviation between the model and the geological structure plus the squared deviation between the exact and the numerical results in the model.

The selection of the Sobolev scalar products for particular numerical methods has been demonstrated by Klimeš (2000) on examples of centred finite differences of the second order, network shortest-path ray tracing, 2-D first-order grid travel time tracing, and 2-D second-order grid travel time tracing.

Constructing seismic models for ray tracing and Kirchhoff migrations

In complex models, the behavior of rays becomes chaotic and the geometrical spreading, number of arrivals and density of caustic surfaces increase exponentially with travel time. The exponential increment may be quantitatively described in terms of the "average Lyapunov exponent" for the model. The "average Lyapunov exponent" may roughly be approximated by the square root of the Sobolev norm composed of the second partial derivatives of the functions describing the slowness or velocity field in the model (Klimeš 2000). This enables us to determine the quantitative criteria on models, so as to be suitable for ray tracing, in terms of Sobolev scalar products. We choose the maximum "average Lyapunov exponent" for the model, we estimate the maximum Sobolev norm of the model, and we estimate the value of weighting factor S. We then perform the inversion, compute the values of the "average Lyapunov exponent" and the Sobolev norm in the inverted model, re-estimate the value of S, and continue iteratively with the inversion until we obtain the model of the required smoothness.

Numerical example - smoothing the regional model of Bohemian Massif for ray tracing

Routines for application of the above described minimization of the Sobolev norm during the seismic inversion have been coded in Fortran 77. All the computations are carried out on a PC Pentium with 64 Mbytes of memory. The software used for the computations is available at http://sw3d.cz.

Series of seismic refraction experiments CELEBRATION 2000, ALP 2002 and SUDETES 2003 (Guterch et al., 2003a ,2003b; Brückl et al., 2003; Grad et al., 2003) were performed in Central Europe in years 2000 to 2003. Data from 8 profiles (CEL09, CEL10, Alp01, S01, S02, S03, S04 and S05) were processed by Růžek *et.al.* (2007). As a result, 2-D depth-velocity cross-sections of P-waves along the profiles were obtained. The depth range of the velocity sections is 35 – 70 km, the velocity varies in the range of 3.5 – 9.2 km/s, MOHO is detected in the depths from 30 to 45 km.

The 2-D velocity sections calculated by Růžek *et.al.* (2007) are suitable for calculation of travel times along the selected profiles. For the purposes of locations of earthquake hypocenters, calculations of Green's functions for moment-tensor inversions, etc., a single regional model is needed. As the amount of data is not sufficient for creation of a complex three-dimensional model, we decided to create a one-dimensional model. As the model should reflect the variations in the MOHO depth across the region, a smooth 1-D model was chosen. The model should be smooth enough to be suitable for calculations using the ray method.

In this paper, we thus use the P-wave velocities calculated by Růžek *et.al.* (2007) as input data for linearized inversion, and we construct a smooth 1-D regional model of Bohemian Massif. Only the data relevant for the Bohemian Massif, i.e. the values corresponding to the latitude from 49 to 51 degree and longitude from 12 to 18 degree were used, see Figure 1.

The goal of this work was to construct seismic velocity model yielding travel times consistent with observed regional data. In the Bohemian Massif, two types of regional seismic P-waves are usually observed. These are the Pg waves propagating through the Earth's crust and arriving usually at the epicentral distances shorter than approximately 140 km, and the Pn waves propagating through the upper mantle and arriving at the

epicentral distances longer than 140 km. We thus wish to construct a model smooth enough to provide mostly a single arrival.

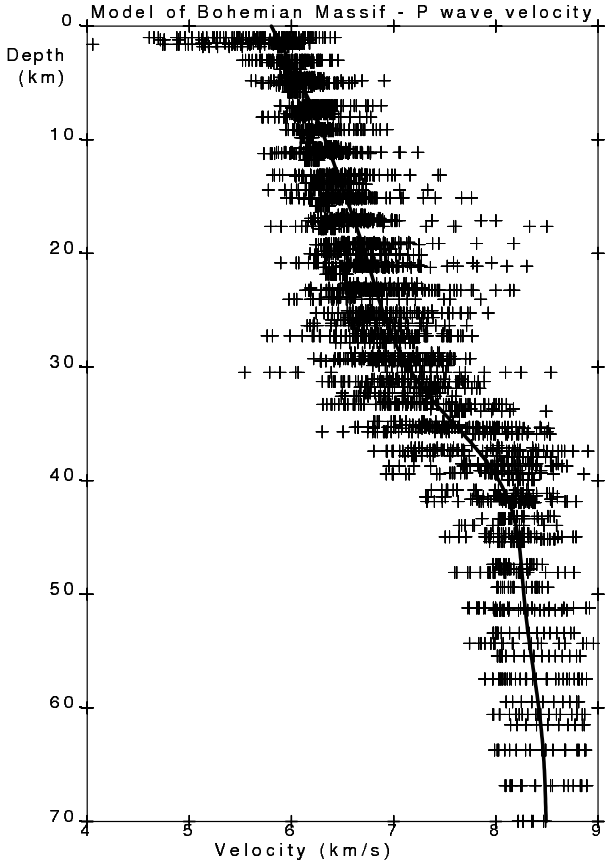

Figure 1: The values of P-wave velocity used as the data for construction of the Bohemian Massif velocity model are plotted by crosses. The resulting smooth velocity model is plotted by the thick line.

We choose the maximum average Lyapunov exponent $\bar{\lambda}$ of the model according to Eq. (11) of Bulant (2002), which reads $\bar{\lambda} = 2/T_{\max}$, where $T_{\max}$ is the maximum travel time in the model. If we consider the maximum ray path about 400 km and the average velocity in the model about 7,3 km/s, we arrive at estimation of $T_{\max} \cong 55\,\mathrm{s}$ and $\bar{\lambda} \cong 0.036\,\mathrm{s}^{-1}$. We choose the weighting factor S of the Sobolev norm of the model similarly as in Eq. (20) of Bulant (2002), and we enter the value of $S = 300000\,\mathrm{kms}$ for the inversion. The resulting smooth velocity model is plotted by thick line in Figure 1. The average Lyapunov exponent calculated in the resulting model is $0.043\,\mathrm{s}^{-1}$, which is slightly higher value than estimated. Smoothness of the model is thus further tested by ray tracing.

We performed several tests of ray tracing in the smooth model obtained by the inversion. Figure 2 displays the rays calculated from the source located in the depth of 5 km, which may be considered as a typical depth for small earthquakes in Western Bohemia. The behavior of rays is quite reasonable, providing a single arrival for most of the epicentral distances. We can see the arrivals of a direct wave for the distances up to 180 km, and the appearance of the wave after the triplication starting at the distance of 155 km. These two waves appear to be a reasonable approximation of the observed Pg and Pn waves.

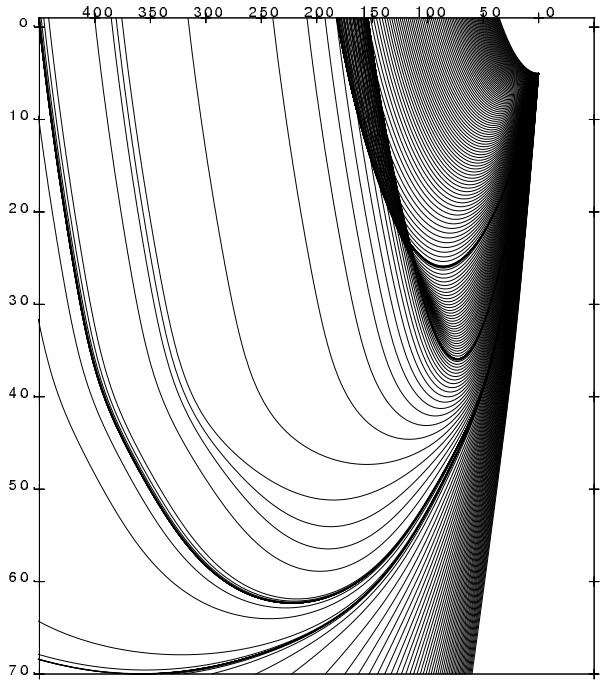

Figure 2: The x-z cross-section through the resulting smooth model displays the rays calculated from the source at the depth of 5 km. The rays arriving at the surface with epicentral distances lower than 180 km may be understood as an approximation of the observed Pg waves, the rays with epicentral distances higher than 155 km approximate the Pn wave.

Conclusions

The minimization of the Sobolev norm of the model during linearized inversion enables us to control the model parameters unresolved by the data being fitted, while the parameters determined by the data remain almost unchanged. It enables us to minimize not only the difference between the data and the model, but, for many computational methods, also the difference between the exact solution and the results obtained by the considered approximate numerical method to be used for calculating wavefields in the model.

The smoothness of the velocity field of the models constructed for the ray methods may be quantitatively described in terms of the "average Lyapunov exponent"

for the model. The "average Lyapunov exponent" may be approximated by the square root of the corresponding Sobolev norm of the model. This enables us to determine the quantitative criteria on the models, so as to be suitable for ray tracing, in terms of Sobolev scalar products. The criteria then allow us to construct the optimum models of geological structures for ray tracing, either by smoothing given rough models, or by inversion of seismic data.

The numerical example demonstrates the application of Sobolev norm in smoothing real velocity data by linearised inversion. The resulting smooth regional velocity model is suitable for ray tracing, and it mimics the basic features of the structure, i.e. the Pg and Pn waves observed in the data. In the future, the model will enable more precise locations of earthquake hypocenters, calculations of Green's functions for moment-tensor inversions, and many other useful applications.

Acknowledgments

This research has been supported by the Grant Agency of the Czech Republic under Contracts 205/07/0032, by the Ministry of Education of the Czech Republic within research Project MSM0021620860, and by the members of the consortium "Seismic Waves in Complex 3-D Structures" (see http://sw3d.cz).

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