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# Energy flux in viscoelastic anisotropic media

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## SUMMARY

We study properties of the energy-flux vector and other related energy quantities of homogeneous and inhomogeneous time-harmonic *P* and *S* plane waves, propagating in unbounded viscoelastic anisotropic media, both analytically and numerically. We propose an algorithm for the computation of the energy-flux vector, which can be used for media of unrestricted anisotropy and viscoelasticity, and for arbitrary homogeneous or inhomogeneous plane waves. Basic part of the algorithm is determination of the slowness vector of a homogeneous or inhomogeneous wave, which satisfies certain constraints following from the equation of motion. Approaches for determination of a slowness vector commonly used in viscoelastic isotropic media are usually difficult to use in viscoelastic anisotropic media. Sometimes they may even lead to non-physical solutions. To avoid these problems, we use the so-called mixed specification of the slowness vector, which requires, in a general case, solution of a complex-valued algebraic equation of the sixth degree. For simpler cases, as for *SH* waves propagating in symmetry planes, the algorithm yields simple analytic solutions. Once the slowness vector is known, determination of energy flux and of other energy quantities is easy. We present numerical examples illustrating the behaviour of the energy-flux vector and other energy quantities, for homogeneous and inhomogeneous plane *P*, *SV* and *SH* waves.

**Key words:** attenuation vector, energy flux, energy-velocity vector, inhomogeneous plane waves, propagation vector, viscoelastic anisotropic media.



## 1 INTRODUCTION

In this paper, we study the properties of the energy flux of homogeneous and inhomogeneous time-harmonic plane waves propagating in an unbounded viscoelastic anisotropic medium. The complex-valued energy-flux vector **F** (also called the Poynting or Umov–Poynting vector) is used as the quantity which fully characterizes the energy flux and the dissipation of energy of the wave. The real part of the vector **F** represents the time-averaged energy flux **S**. Both real, Re **F** = **S**, and imaginary, Im **F**, parts are closely related to the density of the time-averaged dissipated energy $W_d$: $W_d$ equals twice the scalar product of $-$Im **F** and propagation vector **P** (real part of the slowness vector, perpendicular to the wave front) or twice the scalar product of Re **F** and attenuation vector **A** (imaginary part of the slowness vector, perpendicular to the plane of constant amplitude). The aim of this paper is to present a concise derivation of the expression for the complex-valued energy-flux vector **F** of homogeneous and inhomogeneous plane waves propagating in unbounded media of unrestricted anisotropy and viscoelasticity, to propose an algorithm for its calculation, and to perform analytical and numerical studies necessary for correct understanding of energy propagation and dissipation of the relevant plane waves.

Considerable attention in seismology has been devoted to the energy flux of plane waves propagating in elastic anisotropic and in viscoelastic isotropic media. For elastic anisotropic media, see, for example, Fedorov (1968), Auld (1973), Burridge (1976), Gajewski & Pšenčík (1987), Helbig (1994), Carcione (2001), Červený (2001a), where many other references can be found. The energy flux of inhomogeneous plane waves in elastic anisotropic media was discussed by Hayes (1980), Boulanger & Hayes (2000) and Deschamps & Poncelet (2002). For viscoelastic isotropic media, refer to Buchen (1971), Borcherdt (1973), Aki & Richards (1980), Krebes (1983), Richards (1984), Borcherdt & Wennerberg (1985), Mann *et al.* (1987), Leroy *et al.* (1988), Deschamps (1990), Caviglia & Morro (1992, 1999), Ainslie & Burns (1995), Deschamps *et al.* (1997), Boulanger (1998), Brokešová & Červený (1998) and Carcione (2001). In many parts of the Earth's interior, however, both anisotropy and viscoelasticity appear together and thus viscoelastic anisotropic media must be considered. Certain useful relations for the energy flux of plane waves in viscoelastic anisotropic media can already be found in the classical book by Auld (1973). A systematic study of both homogeneous and inhomogeneous plane waves in viscoelastic anisotropic media can be found in a series of papers by Carcione, see Carcione (1994, 1995, 1997a,b, 2001), and Carcione & Cavallini (1993, 1995). See also Krebes & Le (1994), Deschamps & Assouline (2000) and Červený (2001b). An excellent treatment of the Poynting vector, dissipated energy, and related energy quantities can be found particularly in the paper by Carcione & Cavallini (1993), see also Carcione (2001). Certain equations presented here were derived already in that paper.

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For viscoporoelastic anisotropic media, see Carcione *et al.* (2003) or Hanyga (1999) and Hanyga & Seredyńska (1999). A valuable review of papers related to inhomogeneous plane waves propagating in various types of media can be found in Declercq *et al.* (2005).

The algorithms of the computation of the energy-flux vector in viscoelastic anisotropic media depend on the complexity of anisotropy of the medium. For *SH* homogeneous and inhomogeneous plane waves propagating in a plane of symmetry of a monoclinic (orthorhombic, hexagonal) viscoelastic medium, analytical solutions can be found. See Krebes & Le (1994) and Carcione & Cavallini (1995), where certain interesting properties of energy flux in viscoelastic anisotropic media were described. Another interesting study offering numerical and approximate analytic results for *P* and *S* waves is presented by Deschamps & Assouline (2000). Several results presented here closely relate to those derived in the above-mentioned paper. The general algorithms for unrestricted anisotropy and viscoelasticity, however, have been proposed only recently. The basic role in these algorithms is played by the method of determination of the complex-valued slowness vector $\mathbf{p} = \mathbf{P} + \mathrm{i}\mathbf{A}$, satisfying the equation of motion. Several alternative, but mutually related, methods have been proposed to compute $\mathbf{p}$. For a detailed description of these methods, see Červený & Pšenčík (2005a). One of them is based on the generalization of the so-called Stroh formalism for viscoelastic media. It was used by Caviglia & Morro (1999) to discuss theoretically the reflection/transmission of plane waves at a plane interface between two viscoelastic anisotropic media. Considerable attention in that paper was also devoted to the theoretical treatment of the Poynting vector. The Stroh formalism has also been applied to studying theoretically homogeneous and inhomogeneous plane waves in perfectly elastic, viscoelastic and thermoviscoelastic anisotropic media, including the energy flux, by Shuvalov & Scott (1999, 2000) and Shuvalov (2001). A method close to the Stroh formalism has also been used traditionally in seismology for perfectly elastic media to compute the velocity-stress propagators in 1-D anisotropic structures, see Frazer & Fryer (1989), Thomson (1996a,b), etc. This theory can also be generalized for viscoelastic anisotropic media, as shown in the presented references. Here we shall use the method proposed by Červený (2004) and Červený & Pšenčík (2005a), based on the so-called mixed specification of the slowness vector, see Section 3.1. In the mixed specification, the slowness vector is expressed in terms of a complex-valued constant $\sigma$, which can be determined as a root of an algebraic equation of the sixth degree. The algebraic equation for $\sigma$ is supplemented by a system of linear equations for the determination of the amplitude vector $\mathbf{U}$. These quantities are sufficient to determine the complex-valued energy-flux vector $\mathbf{F}$ and all related energy quantities. A more detailed description and comparison of the individual approaches for the specification of the slowness vector can be found in Červený & Pšenčík (2005a).

In Section 2, energy balance equations for time-harmonic waves propagating in an inhomogeneous viscoelastic anisotropic medium are derived. Important energy quantities, such as the complex-valued energy-flux vector $\mathbf{F}$, real-valued kinetic energy $K$, strain energy $U$, total energy $E = K + U$, dissipated energy $W_d$ and rate of dissipated energy $P_d$, are introduced. Actually, these quantities represent the densities of the relevant time-averaged quantities, but the words 'density' and 'time-averaged' are often omitted in this paper. Moreover, the energy flux $\mathbf{S}$ is introduced as the real part of $\mathbf{F}$, and the energy velocity vector as $\boldsymbol{\mathcal{U}} = \mathbf{S}/E$.

In Section 3, all energy quantities are specified for time-harmonic, homogeneous and inhomogeneous plane waves. In Section 3.1, the algorithms for the computation of the complex-valued slowness vector $\mathbf{p} = \mathbf{P} + \mathrm{i}\mathbf{A}$ and for complex-valued amplitude vector $\mathbf{U}$ are explained. Section 3.2 presents equations for the complex-valued energy-flux vector $\mathbf{F}$ and for other energy quantities, for homogeneous and inhomogeneous plane waves. Various relations between the energy quantities and propagation vector $\mathbf{P}$ and attenuation vector $\mathbf{A}$ are derived and discussed.

In Section 4, the significant directions of plane waves propagating in viscoelastic anisotropic media, and the angles between the real and imaginary parts of certain complex-valued vectors, are studied. All the equations presented in Sections 2, 3 and 4 are valid quite generally, for inhomogeneous and homogeneous *P*, *S1* and *S2* plane waves, propagating in media with unrestricted anisotropy and viscoelasticity.

In Section 5, *SH* plane waves, propagating in a plane of symmetry of monoclinic (orthorhombic, hexagonal) anisotropic viscoelastic media, are studied. In this case, the algebraic equation of the sixth degree for the determination of the slowness vector reduces to a quadratic equation. Consequently, all equations for the complex-valued energy-flux vector and other energy quantities may, in this case, be expressed analytically. These analytical solutions offer a simple physical insight into the energy flux quantities in anisotropic viscoelastic media, and into their properties and mutual relations. The analytical procedures of Section 5 are supplemented by numerical examples in Section 6. Finally, the properties of the complex-valued energy-flux vector and of other energy quantities of homogeneous and inhomogeneous plane *P* and *S* waves are investigated numerically in Section 7.

We use Cartesian coordinates $x_i$, and denote time $t$, the Cartesian components of the displacement vector $u_i(x_k, t)$, of the particle velocity $v_i(x_k, t) = \dot{u}_i(x_k, t)$, of the stress tensor $\tau_{ij}(x_k, t)$, and of the infinitesimal strain tensor $e_{ij}(x_k, t)$. Both tensors $\tau_{ij}$ and $e_{ij}$ are symmetric, $\tau_{ij} = \tau_{ji}$ and $e_{ij} = e_{ji}$. The infinitesimal strain tensor $e_{ij}$ is expressed in terms of the displacement vector as follows:

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}). \tag{1}$$

The dot above the letter denotes the partial derivative with respect to time ($\dot{u}_i = \partial u_i/\partial t$), the index following the comma in the subscript indicates the partial derivative with respect to the relevant Cartesian coordinate ($u_{i,j} = \partial u_i/\partial x_j$). The asterisk used as a superscript denotes complex conjugacy. The Einstein summation convention over repeated indices is used throughout the paper. For time-harmonic wave fields, we consider the exponential time factor $\exp(-\mathrm{i}\omega t)$, where $\omega$ is a fixed real, positive circular frequency, $\omega > 0$. We also use the concept of homogeneous and inhomogeneous vectors for complex-valued vectors. Let us consider an arbitrary complex-valued vector $\mathbf{B} = \mathbf{C} + \mathrm{i}\mathbf{D}$, where $\mathbf{C}$ and $\mathbf{D}$ are real-valued vectors. We call $\mathbf{B}$ a homogeneous complex-valued vector if $\mathbf{C}$ and $\mathbf{D}$ are parallel, and an inhomogeneous complex-valued vector if $\mathbf{C}$ and $\mathbf{D}$ are not parallel.

## 2 ENERGY BALANCE EQUATIONS FOR TIME-HARMONIC WAVES

We consider a wave field in an inhomogeneous viscoelastic anisotropic medium, satisfying the equation of motion,

$$\tau_{ij,j} + f_i = \rho \dot{v}_i, \tag{2}$$

where $\rho = \rho(x_k)$ is the density, and $f_i(x_l, t)$ are Cartesian components of the body force vector $\mathbf{f}(x_l, t)$ (force per unit volume).

The generalized Hooke's law for a viscoelastic anisotropic solid can be simply derived from Hooke's law for perfectly elastic media

using the correspondence principle, see Carcione (2001, p. 102). It is sufficient to replace the elastic moduli in the standard Hooke's law by viscoelastic ones. The generalized Hooke's law then reads

$$\tau_{ij} = c_{ijkl} e_{kl}, \tag{3}$$

where $c_{ijkl}(x_n, \omega)$ are complex-valued frequency-dependent viscoelastic moduli. As $\omega$ is fixed, we do not explicitly indicate the frequency-dependence of $c_{ijkl}$. We assume that the viscoelastic moduli $c_{ijkl}$ satisfy the symmetry relations:

$$c_{ijkl} = c_{jikl} = c_{ijlk} = c_{klij}, \tag{4}$$

so that the number of independent viscoelastic moduli reduces from 81 to 21. Note that the relation $\mathrm{Im}c_{ijkl} = \mathrm{Im}c_{klij}$ has not yet been proved to hold generally, see discussion of this matter in e.g. Scott (1997) or Carcione (2001, Section 2.1.1). Alternatively to the viscoelastic moduli $c_{ijkl}$, we also use the density-normalized viscoelastic moduli $a_{ijkl} = c_{ijkl}/\rho$. We assume that the density-normalized viscoelastic moduli also satisfy the symmetry relations (4).

The Voigt notation has often been used for $\tau_{ij}$, $e_{kl}$ and $c_{ijkl}$, in which two indices $i, j$ ($i, j = 1, 2, 3$) are replaced by one index $\alpha$ ($\alpha = 1, 2, \ldots, 6$), and similarly two indices $k, l$ ($k, l = 1, 2, 3$) by one index $\beta$ ($\beta = 1, 2, \ldots, 6$). Then (3) can be expressed in the form $\tau_\alpha = C_{\alpha\beta} e_\beta$, where $C_{\alpha\beta}$ represents a $6 \times 6$ symmetric matrix. In this case, the symmetry relations (4) reduce to one symmetry relation $C_{\alpha\beta} = C_{\beta\alpha}$. See Fedorov (1968), Auld (1973), Carcione (1994, 1995, 1997a,b, 2001), and Carcione & Cavallini (1993, 1995). Analogously to $C_{\alpha\beta}$, we also introduce the Voigt notation $A_{\alpha\beta}$ for the density-normalized viscoelastic moduli.

We shall now derive the energy-balance equations for time-harmonic waves, propagating in inhomogeneous anisotropic media. See also Carcione & Cavallini (1993) and Carcione (2001).

Multiplying eq. (2) by $-v_i^*$, and the complex conjugate of the time derivative of eq. (1) by $-\tau_{ij}$, and adding both, we obtain

$$\left(-v_i^* \tau_{ij}\right)_{,j} = -\rho v_i^* \dot{v}_i - \tau_{ij} \dot{e}_{ij}^* + v_i^* f_i. \tag{5}$$

For time-harmonic waves, with the exponential time factor $\exp(-\mathrm{i}\,\omega t)$, eq. (5) yields,

$$\left(-v_i^* \tau_{ij}\right)_{,j} = \mathrm{i}\omega\left(\rho v_i v_i^* - \tau_{ij} e_{ij}^*\right) + v_i^* f_i. \tag{6}$$

Eqs (5) and (6) are exact, valid for elastic or viscoelastic, isotropic or anisotropic media. It is possible to express eq. (6) in the following form:

$$\mathrm{div}\mathbf{F} = 2\mathrm{i}\omega(K - U) - P_d + P_s, \tag{7}$$

where

$$K = \frac{1}{4}\rho v_i v_i^*, \quad U = \frac{1}{4}\mathrm{Re}\left(\tau_{ij} e_{ij}^*\right),$$

$$P_d = \omega W_d, \qquad W_d = -\frac{1}{2}\mathrm{Im}\left(\tau_{ij} e_{ij}^*\right), \tag{8}$$

$$F_j = -\frac{1}{2} v_i^* \tau_{ij}, \qquad P_s = \frac{1}{2} f_i v_i^*.$$

Eq. (7) is called the energy balance equation for time-harmonic waves for an inhomogeneous anisotropic viscoelastic medium. Note that $K$, $U$, $W_d$ and $P_d$ are real-valued scalars, $P_s$ is a complex-valued scalar, and $\mathbf{F}$ is a complex-valued vector. All quantities $K$, $U$, $P_d$, $W_d$, $P_s$ and $\mathbf{F}$ are time independent.

The individual terms in eq. (7) can be interpreted as follows, see, for example Carcione & Cavallini (1993), Carcione (2001): $K$ and $U$ are time-averaged kinetic and strain energy densities, respectively, $P_d$ is the time-averaged rate of dissipated energy density, $W_d$ is the time-averaged dissipated energy density, $P_s$ is the time-averaged complex-valued energy density supplied by body forces, and $\mathbf{F}$ is the time-averaged complex-valued energy-flux vector. We also introduce the real-valued time-averaged total energy density $E$ by the relation

$$E = U + K. \tag{9}$$

The time averaging is understood over one period $T = 2\pi/\omega$. For the sake of brevity, we omit specifications 'density' and 'time-averaged' in most of the following text.

We emphasize again that the complex-valued energy-flux vector $\mathbf{F}$ used here is a time-averaged quantity. The real part of the vector $\mathbf{F}$ is also called active part and the imaginary part is called reactive part, see Mann *et al.* (1987). We denote the active part of the vector $\mathbf{F}$ by $\mathbf{S}$:

$$\mathbf{S} = \mathrm{Re}\mathbf{F}, \tag{10}$$

and call it the energy-flux vector, or time-averaged energy-flux vector. In order to distinguish the real-valued $\mathbf{S}$ from complex-valued $\mathbf{F}$, we call $\mathbf{F}$ systematically the complex-valued energy-flux vector.

Let us now briefly discuss the physical meaning of the energy balance eq. (7). Separating real and imaginary parts of eq. (7), we obtain two real-valued energy balance equations:

$$\mathrm{div}\,\mathrm{Re}\mathbf{F} = -P_d + \mathrm{Re}P_s, \tag{11}$$

$$\mathrm{div}\,\mathrm{Im}\mathbf{F} = 2\omega(K - U) + \mathrm{Im}P_s. \tag{12}$$

Eq. (11) shows that the energy supplied by internal sources $\mathrm{Re}P_s$ decreased by the rate of the dissipated energy $P_d$ controls the energy flux $\mathbf{S}$. For $\mathrm{Re}P_s > P_d$, the energy flux is outward, in the opposite case inward. For $\mathrm{Re}P_s = P_d$, the energy fluxes in and out are equal.

The energy balance eq. (11) for time-averaged harmonic waves follows directly from the well-known energy conservation law (see, e.g. Auld 1973)

$$\mathrm{div}\mathbf{S}^t + \frac{\partial}{\partial t}(K^t + U^t) + P_d^t = P_s^t, \tag{13}$$

valid for real-valued time-dependent quantities, specifically densities of energy-flux vector $\mathbf{S}^t$, of kinetic energy $K^t$, of strain energy $U^t$, of rate of dissipated energy $P_d^t$ and of energy $P_s^t$ supplied by internal sources. It is only necessary to express each time-harmonic quantity in eq. (13) in a real-valued form (e.g. $v_i = \frac{1}{2}[V_i \exp(-\mathrm{i}\omega t) + V_i^* \exp(\mathrm{i}\omega t)]$), and perform time averaging over one period. As a result, time-averaged $K^t$ and $U^t$ are time independent, and the term $\frac{\partial}{\partial t}(K^t + U^t)$ in eq. (13) vanishes. Then the energy conservation law (eq. 13) gives eq. (11).

Let us also briefly discuss the energy balance eq. (12) and let us first consider the case $\mathrm{Im}\,P_s = 0$. Then the reactive part of the complex-valued energy-flux vector, $\mathrm{Im}\mathbf{F}$, is fully controlled by the difference between $K$ and $U$. For $U = K$, the reactive part of the energy flux $\mathrm{Im}\,\mathbf{F}$ vanishes. This holds in perfectly elastic media. In viscoelastic media, however, $U \neq K$, and the balance eq. (12) plays an important role. The explanation of eq. (12) for $\mathrm{Im}\,P_s \neq 0$ is straightforward.

Using $\mathbf{S}$, we can introduce an important concept of time-averaged energy velocity vector $\boldsymbol{\mathcal{U}}$

$$\boldsymbol{\mathcal{U}} = \mathbf{S}/E, \tag{14}$$

and energy velocity $\mathcal{U} = |\boldsymbol{\mathcal{U}}|$.

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We can insert the generalized Hooke's law (eq. 3) into equations (8). The expressions for $U$, $W_d$ and $F_j$ are then as follows:

$$U = \frac{1}{4}\mathrm{Re}\left(c_{ijkl}e_{kl}e^*_{ij}\right), \quad W_d = -\frac{1}{2}\mathrm{Im}\left(c_{ijkl}e_{kl}e^*_{ij}\right).$$
$$F_j = -\frac{1}{2}v^*_i c_{ijkl}e_{kl}. \tag{15}$$

We use the Voigt notation $C_{\alpha\beta}$ for $c_{ijkl}$ and prove that $U$ is positive if Re $C_{\alpha\beta}$ is positive definite, and $W_d$ is non-negative if $-$Im $C_{\alpha\beta}$ is positive definite or zero. We can write

$$c_{ijkl}e_{kl}e^*_{ij} = C_{\alpha\beta}e^*_\alpha e_\beta = C_{\alpha\beta}\left(e^R_\alpha e^R_\beta + e^I_\alpha e^I_\beta\right),$$

where $e^R_\alpha = \mathrm{Re}\,(e^\alpha)$, $e^I_\alpha = \mathrm{Im}(e_\alpha)$, and similarly for $e_\beta$. Quantities $e^R_\alpha$ and $e^I_\alpha$ represent the real-valued components of the $6 \times 1$ real-valued vectors ($\alpha, \beta = 1, 2, \ldots, 6$). The strain energy is positive, $U > 0$, if Re $C_{\alpha\beta}$ is positive definite. Similarly, $W_d$ is positive or equal zero if $-$Im$C_{\alpha\beta}$ is positive definite or zero. Throughout the paper, we assume Re $C_{\alpha\beta}$ to be positive definite and $-$Im$C_{\alpha\beta}$ positive definite or zero.

The introduced quantities are measured in the following units: stress $\tau_{ij}$, the viscoelastic moduli $c_{ijkl}$ or $C_{\alpha\beta}$ and the energy quantities $U$, $K$, $E$ and $W_d$ in pascals (Pa ; kg m$^{-1}$ s$^{-2}$), the components of body forces $f_i$ in newtons per cubic metre (N/m$^3$; kg m$^{-2}$ s$^{-2}$), density $\rho$ in kilograms per cubic metre (kg m$^{-3}$), displacement components $u_i$ in metres (m), the particle velocity components $v_i$ and the energy velocity vector $\boldsymbol{\mathcal{U}}$ in m s$^{-1}$, the density-normalized elastic moduli $a_{ijkl}$ and $A_{\alpha\beta}$ in m$^2$ s$^{-2}$, the quantities $P_d$ and $P_s$ in Pascal per second (Pa s$^{-1}$ ; kg m$^{-1}$ s$^{-3}$), the vector $\mathbf{F}$ in watts per square metre (W m$^{-2}$ ; kg s$^{-3}$).

## 3 ENERGY FLUX OF A TIME-HARMONIC PLANE WAVE

We consider a homogeneous unbounded viscoelastic anisotropic medium, and a time-harmonic plane wave, propagating in the medium,

$$u_i(x_j, t) = U_i \exp[-\mathrm{i}\omega(t - p_n x_n)]. \tag{16}$$

Here $\mathbf{p}$ is a complex-valued slowness vector, and $\mathbf{U}$ a complex-valued amplitude vector. Eq. (16) represents a plane wave only if $\mathbf{U}$ and $\mathbf{p}$ are chosen in such a way that eq. (16) satisfies the equation of motion, see eqs (2) and (3). This requirement yields a system of linear equations for $U_1$, $U_2$, $U_3$:

$$a_{ijkl}p_j p_l U_k = U_i, \quad i = 1, 2, 3. \tag{17}$$

The condition of solvability of system (17),

$$\det[a_{ijkl}p_j p_l - \delta_{ik}] = 0, \tag{18}$$

is the constraint relation for slowness vector $\mathbf{p}$. Eq. (17) is the constraint relation for amplitude vector $\mathbf{U}$. Amplitude vector $\mathbf{U}$ may, of course, be suitably normalized.

### 3.1 Determination of slowness vectors

The determination of the slowness vector $\mathbf{p}$, which satisfies the constraint relation (18), is not straightforward. For this purpose, it is useful to specify the slowness vector in the following way,

$$\mathbf{p} = \sigma\mathbf{n} + \mathrm{i}D\mathbf{m}, \quad \text{with} \quad \mathbf{m}\cdot\mathbf{n} = 0. \tag{19}$$

Here $\mathbf{n}$ and $\mathbf{m}$ are two arbitrary, mutually perpendicular real-valued unit vectors; they are perpendicular and tangent, respectively, to wave front $\Sigma$. Quantity $D$ is a scalar real-valued quantity called the inhomogeneity parameter, $D \in (-\infty, \infty)$. Unit vectors $\mathbf{n}$, $\mathbf{m}$ and inhomogeneity parameter $D$ can be arbitrarily chosen, and the complex-valued quantity $\sigma$ is then determined from

$$\det[a_{ijkl}(\sigma n_j + \mathrm{i}Dm_j)(\sigma n_l + \mathrm{i}Dm_l) - \delta_{ik}] = 0, \tag{20}$$

which results from eq. (18). This is an algebraic equation of the sixth degree for $\sigma$. It has six complex-valued roots $\sigma$, corresponding to *P*, *S1* and *S2* waves, propagating in opposite directions.

Once quantity $\sigma$ has been found, we can simply determine the complex-valued slowness vector,

$$\mathbf{p} = \mathbf{P} + \mathrm{i}\mathbf{A}, \tag{21}$$

so that

$$\mathbf{P} = \mathbf{n}\,\mathrm{Re}\sigma, \quad \mathbf{A} = \mathbf{n}\,\mathrm{Im}\sigma + D\mathbf{m}. \tag{22}$$

Here $\mathbf{P}$ is called the propagation vector. It is perpendicular to wave front $\Sigma$, and oriented in the direction of propagation of the wave front. Vector $\mathbf{A}$ is called the attenuation vector, perpendicular to the plane of constant amplitude, oriented in the direction of the maximum decay of amplitudes. We can further compute the lengths of vectors $\mathbf{P}$ and $\mathbf{A}$, $|\mathbf{P}|$ and $|\mathbf{A}|$, unit vectors $\mathbf{N}$ and $\mathbf{M}$ along $\mathbf{P}$ and $\mathbf{A}$, phase velocity $\mathcal{C}$ and attenuation angle $\gamma$ ($\cos\gamma = \mathbf{N}\cdot\mathbf{M}$):

$$|\mathbf{P}| = |\mathrm{Re}\sigma|, \qquad |\mathbf{A}| = [(\mathrm{Im}\sigma)^2 + D^2]^{1/2},$$
$$\mathbf{N} = \mathbf{P}/|\mathbf{P}| = \epsilon\mathbf{n}, \quad \mathbf{M} = \mathbf{A}/|\mathbf{A}|, \tag{23}$$
$$\mathcal{C} = 1/|\mathrm{Re}\sigma|, \qquad \cos\gamma = \epsilon(\mathrm{Im}\sigma)/[(\mathrm{Im}\sigma)^2 + D^2]^{1/2},$$

where $\epsilon = \mathrm{Re}\sigma/|\mathrm{Re}\sigma| = \pm 1$. For more details, see Červený & Pšenčík (2005a).

Note that the plane wave with real-valued slowness vector $\mathbf{p}$ ($\mathbf{A} = \mathbf{0}$) is called the bulk plane wave, the plane wave with $\mathbf{A} \neq \mathbf{0}$ and $D = 0$ the homogeneous plane wave, and the plane wave with $\mathbf{A} \neq \mathbf{0}$ and $D \neq 0$ the inhomogeneous plane wave. For homogeneous plane waves, $\mathbf{P}$ and $\mathbf{A}$ are parallel, and for inhomogeneous plane waves, they are not. The plane specified by $\mathbf{P}$ and $\mathbf{A}$ (or by $\mathbf{n}$ and $\mathbf{m}$) is called the propagation–attenuation plane $\Sigma^{\|}$.

### 3.2 Complex-valued energy-flux vector of a time-harmonic plane wave

Once plane wave (eq. 16), including $\mathbf{p}$ and $\mathbf{U}$, has been determined, we can find an explicit equation for the complex-valued energy-flux vector $\mathbf{F}$ from eq. (8). For $\dot{u}_i$ and $\tau_{ij}$, we obtain

$$\dot{u}_i = -\mathrm{i}\omega U_i \exp[-\mathrm{i}\omega(t - p_n x_n)],$$
$$\tau_{ij} = \mathrm{i}\omega\, a_{ijkl}U_k p_l \exp[-\mathrm{i}\omega(t - p_n x_n)].$$

Vector $\mathbf{F}$ then reads

$$F_i = \frac{1}{2}\rho\omega^2 a_{ijkl}U^*_j U_k p_l \exp(-2\omega A_n x_n). \tag{24}$$

Here $p_l$ must satisfy eq. (18), and $U_k$ (equations 17).

The energy flux $\mathbf{S}$ is then given by the equation:

$$S_i = \mathrm{Re}F_i = \frac{1}{2}\rho\omega^2\mathrm{Re}\left(a_{ijkl}U_k U^*_j p_l\right)\exp(-2\omega A_n x_n). \tag{25}$$

Analogously, we obtain explicit expressions for other important energy quantities: the kinetic energy $K$, the strain energy $U$, and the

dissipated energy $W_d$, see equations (8):

$$K = \frac{1}{4}\rho\omega^2 U_i U_i^* \exp(-2\omega A_n x_n),$$
$$U = \frac{1}{4}\rho\omega^2 \mathrm{Re}\left(a_{ijkl} U_k U_i^* p_l p_j^*\right) \exp(-2\omega A_n x_n),$$
$$W_d = -\frac{1}{2}\rho\omega^2 \mathrm{Im}\left(a_{ijkl} U_k U_i^* p_l p_j^*\right) \exp(-2\omega A_n x_n). \quad (26)$$

From eq. (14), we get the expression for the energy velocity vector $\mathcal{U}$:

$$\mathcal{U}_i = \frac{S_i}{E} = \frac{2\mathrm{Re}\left(a_{ijkl} U_k U_j^* p_l\right)}{U_k U_k^* + \mathrm{Re}\left(a_{ijkl} U_i^* U_k p_l p_j^*\right)}. \quad (27)$$

Using eqs (24)–(27) and constraint relation (17), we can obtain certain useful relations. Multiplying eq. (17) by $U_i^*$ yields

$$a_{ijkl} p_j p_l U_k U_i^* = U_i U_i^*. \quad (28)$$

Using eqs (24) and (28), we then obtain

$$\mathbf{p}\cdot\mathbf{F} = 2K, \quad \mathbf{p}^*\cdot\mathbf{F} = 2U - \mathrm{i}W_d. \quad (29)$$

Equations (29) yield the scalar products of the complex-valued energy-flux vector **F** with propagation vector **P** and attenuation vector **A**, respectively:

$$\mathbf{P}\cdot\mathbf{F} = E - \frac{1}{2}\mathrm{i}W_d, \quad \mathbf{A}\cdot\mathbf{F} = \frac{1}{2}W_d - \mathrm{i}(K - U). \quad (30)$$

Equations (30) may be expressed in terms of Re **F** = **S** and Im**F** as follows (see also Carcione & Cavallini 1993):

$$\begin{aligned} &\mathbf{P}\cdot\mathbf{S} = E, && \mathbf{A}\cdot\mathbf{S} = \frac{1}{2}W_d, \\ &\mathbf{P}\cdot\mathrm{Im}\mathbf{F} = -\frac{1}{2}W_d, && \mathbf{A}\cdot\mathrm{Im}\mathbf{F} = U - K. \end{aligned} \quad (31)$$

All the equations given above are valid generally, for homogeneous, inhomogeneous and bulk plane waves, for isotropic and anisotropic, elastic and viscoelastic media. They have many important consequences. We list several of them.

(1) **The total energy** $E$. The total energy $E = K + U$ can be simply computed using eq. (26). It can, however, be alternatively expressed in terms of **P** and **S**, see equations (31),

$$E = K + U = \mathbf{P}\cdot\mathbf{S}$$
$$= \frac{1}{2}\rho\omega^2 \mathrm{Re}(p_j)\mathrm{Re}(a_{ijkl} U_k U_i^* p_l) \exp(-2\omega A_n x_n). \quad (32)$$

Thus, $E$ is represented by the scalar product of propagation vector **P** and the time-averaged energy flux vector **S**. For **P** perpendicular to **S**, $E$ vanishes.

(2) **The dissipated energy** $W_d = 2\,\mathbf{A}\cdot\mathbf{S} = -2\,\mathbf{P}\cdot\mathrm{Im}\mathbf{F}$. It is represented by the scalar product of attenuation vector **A** and the energy flux vector **S** multiplied by 2. Alternatively, it is represented by the scalar product of propagation vector **P** and the imaginary part of the complex-valued energy-flux vector, **F**, multiplied by $-2$ . Let us note that a quantity proportional to the scalar product $\mathbf{A}\cdot\mathbf{S}$ plays a key role in the study of 'intrinsic attenuation' of Deschamps & Assouline (2000).

(3) **Energy angle** $i_{en}$. It is introduced here as the angle between propagation vector **P** (normal to the wave front) and the energy flux **S**,

$$\cos i_{en} = \mathbf{P}\cdot\mathbf{S}/|\mathbf{P}||\mathbf{S}| = E/|\mathbf{P}||\mathbf{S}| \geq 0, \quad (33)$$

see equations (31). Consequently, angle $i_{en}$ is always less or equal to 90°. Moreover, the total energy $E$ is proportional to $\cos i_{en}$.

(4) **Energy-attenuation angle** $\gamma_{en}$. It is introduced here as the angle between attenuation vector **A** and the energy flux **S**,

$$\cos\gamma_{en} = \mathbf{A}\cdot\mathbf{S}/|\mathbf{A}||\mathbf{S}| = \frac{1}{2}W_d/|\mathbf{A}||\mathbf{S}| \geq 0, \quad (34)$$

see equations (31). Consequently, angle $\gamma_{en}$ does not exceed 90°. Moreover, the time-averaged dissipated energy $W_d$ is proportional to $\cos\gamma_{en}$. The inhomogeneous plane wave, for which $\gamma_{en} = 0$, is called the intrinsic inhomogeneous plane wave, or intrinsic attenuated plane wave, see Deschamps & Assouline (2000).

(5) **The energy-velocity vector** $\mathcal{U}$. It is given by eq. (27), but can also be expressed in a simpler form using eq. (32):

$$\mathcal{U}_i = \frac{S_i}{E} = \frac{S_i}{\mathbf{P}\cdot\mathbf{S}} = \frac{\mathrm{Re}\left(a_{ijkl} U_k U_j^* p_l\right)}{\mathrm{Re}\left(a_{ijkl} U_k U_i^* p_l\right)\mathrm{Re}(p_j)}. \quad (35)$$

(6) **Expression for** $P_i\mathcal{U}_i$. It immediately follows from eq. (35) that

$$\mathbf{P}\cdot\mathcal{U} = 1. \quad (36)$$

This equation represents a generalization of the important relation $\mathbf{p}\cdot\mathcal{U} = 1$ known from perfectly elastic anisotropic media, in which $\mathcal{U}$ is the group velocity vector.

(7) **Quality factor** $Q$. **Loss factor** $\mathcal{L}$. In isotropic viscoelastic media, the quality factor has mostly been introduced by the relation $Q = -\mathrm{Re}\,(M)/\mathrm{Im}\,(M)$, where $M$ is the relevant complex-valued viscoelastic modulus. In anisotropic viscoelastic media, however, such a definition fails, see Krebes & Le (1994). It is more suitable to define the quality factor $Q$ in terms of time-averaged energy quantities. Such a definition was also introduced by Buchen (1971) for isotropic viscoelastic media. For example, the definition valid both for isotropic and anisotropic viscoelastic media, given by Carcione (2001, p. 138), is as follows

$$Q = 2U/W_d. \quad (37)$$

Using equations (31), we can rewrite eq. (37) it in an alternative way,

$$Q = \frac{U}{A_i S_i} = \frac{1}{2}\frac{U_i U_i^*}{\mathrm{Re}\left(a_{ijkl} U_k U_j^* p_l\right)\mathrm{Im}(p_i)}. \quad (38)$$

We further introduce the loss factor $\mathcal{L}$ as the inverse of $Q$,

$$\mathcal{L} = Q^{-1} = 2\mathrm{Re}\left(a_{ijkl} U_k U_j^* p_l\right)\mathrm{Im}(p_i)/\left(U_i U_i^*\right). \quad (39)$$

(8) **Perfectly elastic anisotropic medium.** In a perfectly elastic anisotropic medium ($a_{ijkl}$ real valued), the quantity $a_{ijkl} p_j p_l^* U_i U_k^*$, appearing in the expressions for $U$ and $W_d$, see equations (26), is real valued, for arbitrary complex-valued **p** and **U**. This indicates that the dissipated energy vanishes, $W_d = 0$, see equations (26). From equations (31), we then obtain

$$\mathbf{A}\cdot\mathbf{S} = 0, \quad \mathbf{P}\cdot\mathrm{Im}\mathbf{F} = 0. \quad (40)$$

In perfectly elastic isotropic media, the energy flux **S** is always parallel to propagation vector **P**. It follows from eq. (40) that both vectors **P** and **S** are perpendicular to attenuation vector **A** and to the imaginary part of the vector **F**.

## 4 SIGNIFICANT DIRECTIONS IN PLANE WAVE PROPAGATION

In the propagation of scalar plane waves in non-dissipative media, there is only one significant direction of propagation, corresponding to the direction of (real-valued) slowness vector **p**. For vectorial

plane waves, the number of distinct directions is larger. It increases with the increasing complexity of the model (isotropic, anisotropic, perfectly elastic, viscoelastic).

Let us now briefly summarize the significant directions of vectorial plane waves propagating in viscoelastic anisotropic media:

(a) Slowness vector **p**. It is complex valued, $\mathbf{p} = \mathbf{P} + \mathrm{i}\mathbf{A}$, with **P** and **A** oriented in generally different directions: propagation vector **P** in the direction of propagation (perpendicular to the wave front), and attenuation vector **A** in the direction perpendicular to the plane of constant amplitudes.

(b) Amplitude vector **U**. In viscoelastic media, the relevant amplitude vector is complex valued, $\mathbf{U} = \mathrm{Re}\,\mathbf{U} + \mathrm{i}\,\mathrm{Im}\,\mathbf{U}$. Consequently, the amplitude vector is specified by two real valued, generally different directions, along Re**U** and Im**U** responsible for elliptical polarization.

(c) Complex-valued energy-flux vector **F**. It is specified by two real valued, generally different directions, along Re**F** and Im **F**. Vector Re**F** represents the energy-flux vector **S** and points in the direction of the energy-velocity vector $\mathcal{U}$, see eq. (27). Both **S** and $\mathcal{U}$ are real valued.

If the above listed vectors are complex valued, they may be either *homogeneous or inhomogeneous*, see the explanation of the concept of homogeneous and inhomogeneous vectors in Introduction. An example where this terminology has often been applied is slowness vector $\mathbf{p} = \mathbf{P} + \mathrm{i}\mathbf{A}$. This terminology has also been extended to plane waves, see Section 3.1. The actual inhomogeneity of the plane wave is controlled by inhomogeneity strength $|D|$. The plane wave is homogeneous for $|D| = 0$, weakly inhomogeneous for small $|D|$, and strongly inhomogeneous for large $|D|$.

The term homogeneous and inhomogeneous can also be used for other complex-valued vectors. For example, the amplitude vectors of *P*, *S1*, and *S2* waves are, in general, inhomogeneous, but the amplitude vector of an *SH* plane wave propagating in the plane of symmetry $\Sigma^S$ is homogeneous, as its real and imaginary parts are parallel (perpendicular to the plane of symmetry $\Sigma^S$). Similarly, the complex-valued energy-flux vector **F** of plane waves is, in general, inhomogeneous. It may, however, be homogeneous for certain small nonzero values of *D*.

We will now briefly discuss an artificially constructed complex-valued vector $\mathbf{s} = \mathbf{S} + \mathrm{i}g\mathbf{A}$. A real-valued positive unit constant $g$ is introduced to obtain the same dimension of $g\mathbf{A}$ and **S**. We call the complex-valued vector **s** the energy-attenuation vector. Its real part is the time-averaged energy flux, and imaginary part has the direction of the attenuation vector.

The energy-attenuation vector $\mathbf{s} = \mathbf{S} + \mathrm{i}g\mathbf{A}$ has certain interesting properties, which differ from properties of slowness vector $\mathbf{p} = \mathbf{P} + \mathrm{i}\mathbf{A}$. The differences are mainly between attenuation angle $\gamma$ (which is the angle between **P** and **A**) and energy-attenuation angle $\gamma_{en}$ (which is the angle between **S** and **A**):

$$\cos\gamma = \mathbf{P}\cdot\mathbf{A}/|\mathbf{P}||\mathbf{A}|, \quad \cos\gamma_{en} = \mathbf{S}\cdot\mathbf{A}/|\mathbf{S}||\mathbf{A}|. \tag{41}$$

It is well known that attenuation angle $\gamma$ behaves simply in isotropic viscoelastic media, where its magnitude is always less than 90°. In anisotropic viscoelastic media, however, its behaviour is more complicated. The maximum attenuation angle $\gamma^*$ is not necessarily 90°, but may be less, equal or larger than 90°. See the detailed investigation of $\gamma$ in Červený & Pšenčík (2005a), Červený & Pšenčík (2005b). Contrary to it, the behaviour of the energy-attenuation angle $\gamma_{en}$ is simple; it is always less or equal to 90°, for unrestricted anisotropy and viscoelasticity, see eq. (34). It does not depend on $g$. We remind the reader that the inhomogeneous plane wave for which $\gamma_{en} = 0$ is called the intrinsic inhomogeneous plane wave, see Section 3.2/4.

## 5 ENERGY FLUX OF *SH* PLANE WAVES

Eq. (24) for the complex-valued energy-flux vector of plane waves can be used for unrestricted anisotropy and viscoelasticity, and for an unrestricted value of inhomogeneity parameter $D$. It generally requires solving an algebraic equation of the sixth degree (eq. 20). In this section we consider only a special case, corresponding to *SH* plane waves propagating in the plane of symmetry of a monoclinic (orthorhombic, hexagonal) viscoelastic anisotropic medium. In this case, the algebraic equation of the sixth degree (eq. 20) factorizes and yields a quadratic equation, which may be simply solved analytically. Consequently, certain properties of the energy flux of *SH* waves can be simply studied analytically, and discussed in detail.

### 5.1 Slowness vector of *SH* plane waves

We consider a plane of symmetry $\Sigma^S$ in a monoclinic anisotropic viscoelastic medium, and choose the Cartesian coordinate system $x_i$ in such a way that the plane of symmetry $\Sigma^S$ corresponds to coordinate plane $x_1x_3$. We further assume that propagation–attenuation plane $\Sigma^{\parallel}$ coincides with $\Sigma^S$, that is, $p_2 = 0$. The propagation of *SH* plane waves is then controlled by three Voigt density-normalized viscoelastic moduli $A_{44}$, $A_{66}$ and $A_{46}$. We introduce the $2 \times 2$ complex-valued matrix

$$\mathbf{A_{SH}} = \begin{pmatrix} A_{66} & A_{46} \\ A_{46} & A_{44} \end{pmatrix}, \tag{42}$$

and assume that Re $\mathbf{A}_{\mathrm{SH}}$ is positive definite, and $-$Im $\mathbf{A}_{\mathrm{SH}}$ positive definite or zero.

The constraint relation (18) for the slowness vector **p** of *SH* waves yields:

$$A_{66}p_1^2 + A_{44}p_3^2 + 2A_{46}p_1p_3 = 1, \tag{43}$$

and the constraint relation (17) for amplitude vector **U** yields

$$\mathbf{U}(p_n) = (0, U_2(p_n), 0)^T. \tag{44}$$

As expected, the amplitude vector **U** of *SH* plane waves is perpendicular to symmetry plane $\Sigma^S$ and is homogeneous. Component $U_2$ may be complex valued. We also introduce the unit real-valued vectors **n** and **m** as follows:

$$\mathbf{n} = (n_1, 0, n_3)^T, \quad \mathbf{m} = (n_3, 0, -n_1)^T. \tag{45}$$

As we can see, **n** and **m**, given by eq. (45), are mutually perpendicular.

The expressions for the slowness vector components $p_1$ and $p_3$ can be obtained by inserting eq. (45) into eq. (19):

$$p_1 = \sigma n_1 + \mathrm{i}Dn_3, \quad p_3 = \sigma n_3 - \mathrm{i}Dn_1. \tag{46}$$

The quadratic eq. (43) for $\sigma$ then reads:

$$A_{66}(\sigma n_1 + \mathrm{i}Dn_3)^2 + A_{44}(\sigma n_3 - \mathrm{i}Dn_1)^2 + 2A_{46}(\sigma n_1 + \mathrm{i}Dn_3)(\sigma n_3 - \mathrm{i}Dn_1) = 1. \tag{47}$$

It has two roots, $\sigma_1$ and $\sigma_2$,

$$\sigma_{1,2} = -\mathrm{i}D\Lambda/\Gamma_{22} \pm \left[1/\Gamma_{22} + D^2\Delta/\Gamma_{22}^2\right]^{1/2}, \tag{48}$$

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where

$$\Gamma_{22} = A_{66}n_1^2 + A_{44}n_3^2 + 2A_{46}n_1n_3,$$
$$\Lambda = (A_{66} - A_{44})n_1n_3 + A_{46}(n_3^2 - n_1^2),$$
$$\Delta = A_{44}A_{66} - A_{46}^2. \tag{49}$$

Once quantity $\sigma$ has been found, we can determine $\mathbf{P}, \mathbf{A}, |\mathbf{P}|, |\mathbf{A}|, \mathbf{N}, \mathbf{M}, \mathcal{C}$ and $\gamma$ using the relations, see eqs (22) and (23),

$$P_1 = n_1\mathrm{Re}\sigma, \quad A_1 = n_1\mathrm{Im}\sigma + n_3D,$$
$$P_3 = n_3\mathrm{Re}\sigma, \quad A_3 = n_3\mathrm{Im}\sigma - n_1D,$$
$$|\mathbf{P}| = |\mathrm{Re}\sigma|, \quad |\mathbf{A}| = [(\mathrm{Im}\sigma)^2 + D^2]^{1/2},$$
$$N_1 = \epsilon n_1, \quad M_1 = [n_1\mathrm{Im}\sigma + n_3D]/[(\mathrm{Im}\sigma)^2 + D^2]^{1/2}, \tag{50}$$
$$N_3 = \epsilon n_3, \quad M_3 = [n_3\mathrm{Im}\sigma - n_1D]/[(\mathrm{Im}\sigma)^2 + D^2]^{1/2},$$
$$\mathcal{C} = 1/|\mathrm{Re}\sigma|, \ \cos\gamma = \epsilon\,\mathrm{Im}\sigma/[(\mathrm{Im}\sigma)^2 + D^2]^{1/2}.$$

We also get $P_2 = 0$, $A_2 = 0$, $N_2 = 0$, $M_2 = 0$.

### 5.2 Energy quantities of *SH* plane waves

Let us now discuss the complex-valued energy-flux vector $\mathbf{F}$ of *SH* plane waves. Using eq. (24), we obtain

$$F_i = \frac{1}{2}\rho\omega^2U_2U_2^*a_{i2k2}p_k\exp(-2\omega A_nx_n). \tag{51}$$

In the following, we shall consider normalized form $\mathbf{G}$ of the vector $\mathbf{F}$:

$$F_i = A^F G_i, \tag{52}$$

where $A^F$ is the normalization factor, given by the relation,

$$A^F = \frac{1}{2}\rho\omega^2U_2U_2^*\exp(-2\omega A_nx_n). \tag{53}$$

Normalization factor $A^F$ is always real valued and positive, $A^F > 0$. Consequently, the normalized vector $\mathbf{G}$ of *SH* waves is given by a simple expression

$$G_i = a_{i2k2}p_k. \tag{54}$$

If we use the Voigt notation for the density-normalized complex-valued viscoelastic moduli, $G_1$ and $G_3$ are given by relations

$$G_1 = A_{66}p_1 + A_{46}p_3, \quad G_3 = A_{46}p_1 + A_{44}p_3. \tag{55}$$

Equations (55) represent the final expressions for the normalized complex-valued energy-flux vector of *SH* waves. We again emphasize that $A_{44}$, $A_{66}$ and $A_{46}$ are complex valued, and that $p_1$, $p_3$ are also complex valued and given by eq. (46).

Using eqs (53) and (55), we can express also other energy quantities of *SH* plane waves in terms of $A^F$ and $\mathbf{G}$:

The kinetic energy $K$, the strain energy $U$, and the dissipated energy $W_d$ read:

$$K = \frac{1}{2}A^F, \quad U = \frac{1}{2}A^F\mathrm{Re}(\mathbf{G}\cdot\mathbf{p}^*), \quad W_d = -A^F\mathrm{Im}(\mathbf{G}\cdot\mathbf{p}^*). \tag{56}$$

The energy flux $\mathbf{S}$, and the energy velocity $\mathcal{U}$ are:

$$\mathbf{S} = A^F\mathrm{Re}(\mathbf{G}), \quad \mathcal{U} = 2\mathrm{Re}(\mathbf{G})/[1 + \mathrm{Re}(\mathbf{G}\cdot\mathbf{p}^*)]. \tag{57}$$

Quality factor $Q$ and loss factor $\mathcal{L}$ are:

$$Q = 1/(2\mathbf{A}\cdot\mathrm{Re}\mathbf{G}), \quad \mathcal{L} = 2\mathbf{A}\cdot\mathrm{Re}\mathbf{G}. \tag{58}$$

### 5.3 Significant directions in *SH* plane wave propagation

In this section, we shall pay attention mainly to energy-attenuation angle $\gamma_{en}$ and to the angle between real and imaginary part of the complex-valued energy-flux vector of plane *SH* waves.

For *SH* waves, $\sin\gamma_{en}$ can be computed analytically:

$$\sin\gamma_{en} = |A_3\mathrm{Re}G_1 - A_1\mathrm{Re}G_3|/|\mathbf{A}||\mathrm{Re}\mathbf{G}|. \tag{59}$$

This yields, see eqs (50) and (55),

$$\sin\gamma_{en} = |\mathrm{Re}\sigma[\mathrm{Im}\sigma\mathrm{Re}\Lambda - D\mathrm{Re}|\Gamma_{22}]|/|\mathbf{A}||\mathrm{Re}\mathbf{G}|, \tag{60}$$

where $\Lambda$ is given in eq. (49). Now consider weakly inhomogeneous plane waves (small $|D|$), and use approximately

$$\mathrm{Im}\sigma \doteq e - fD, \tag{61}$$

where

$$e = \mathrm{Im}(1/\Gamma_{22})^{1/2}, \quad f = \mathrm{Re}(\Lambda/\Gamma_{22}) \doteq \mathrm{Re}(\Lambda)/\mathrm{Re}(\Gamma_{22}). \tag{62}$$

see Červený & Pšenčík (2005a, eqs 66 and 67). Inserting eq. (61) into eq. (60) yields

$$\sin\gamma_{en} = |\mathrm{Re}\sigma[e\mathrm{Re}\Lambda - D(f\mathrm{Re}\Lambda + \mathrm{Re}\Gamma_{22})]|/|\mathbf{A}||\mathrm{Re}\mathbf{G}|. \tag{63}$$

The energy-attenuation vector $\mathbf{s} = \mathbf{S} + \mathrm{i}g\mathbf{A}$ is homogeneous for such $D$, for which eq. (63) vanishes. We denote such $D$ for which $\mathbf{s}$ is homogeneous by $D_{att}$, and obtain for it a simple approximate result:

$$D_{att} \doteq ef/(1 + f^2). \tag{64}$$

It was shown in Červený & Pšenčík (2005a, Section 3.4.7) that $D_{att}$ has a very interesting property: Attenuation $|\mathbf{A}|$, as a function of $D$, is minimum for such $D$. For this reason, Červený & Pšenčík (2005a, eq. 71) denoted it $D_{att}$.

The conclusion: For weakly inhomogeneous *SH* plane waves, attenuation $|\mathbf{A}|$, as a function of $D$ (for fixed direction of propagation), is minimum for approximately the inhomogeneity parameter $D = D_{att}$, for which the energy-attenuation vector $\mathbf{s} = \mathbf{S} + \mathrm{i}\,g\,\mathbf{A}$ is homogeneous. The direction of $\mathbf{A}$ is generally different from the propagation direction specified by $\mathbf{P}$, which means that the corresponding plane wave is inhomogeneous. Note again that the plane wave with the homogeneous energy-attenuation vector is called the intrinsic inhomogeneous plane wave, see Section 3.2/4. While $D_{att}$ is generally non-zero in viscoelastic anisotropic media, it is strictly zero in viscoelastic isotropic ones.

As shown in eq. (51), for *SH* plane waves, we can determine the vector $\mathbf{F}$ analytically. Thus, it is also possible to determine analytically the angle $\gamma_F$ between $\mathbf{S}$ and Im $\mathbf{F}$,

$$\sin\gamma_F = |\mathrm{Re}G_1\mathrm{Im}G_3 - \mathrm{Re}G_3\mathrm{Im}G_1|/|\mathrm{Re}\mathbf{G}||\mathrm{Im}\mathbf{G}|. \tag{65}$$

After simple, but lengthy computations we find that $\sin\gamma_F \doteq 0$ for $D = D^M$, where

$$D^M \doteq \pm b/2c, \tag{66}$$

and where

$$b = \mathrm{Im}(\Lambda/\Gamma_{22}), \quad c = \frac{1}{2}\mathrm{Re}\left(\Delta/\Gamma_{22}^{3/2}\right). \tag{67}$$

It was shown by Červený & Pšenčík (2005a, Section 3.4.7) that $D^M$ also represents the inhomogeneity parameter for which the phase velocity (as a function of $D$) is maximum.

The conclusion: For weakly inhomogeneous *SH* plane waves, phase velocity $\mathcal{C}$, as a function of $D$ (for fixed direction of propagation), is maximum for approximately inhomogeneity parameter $D = D^M$, for which the complex-valued energy-flux vector $\mathbf{F}$ is homogeneous.

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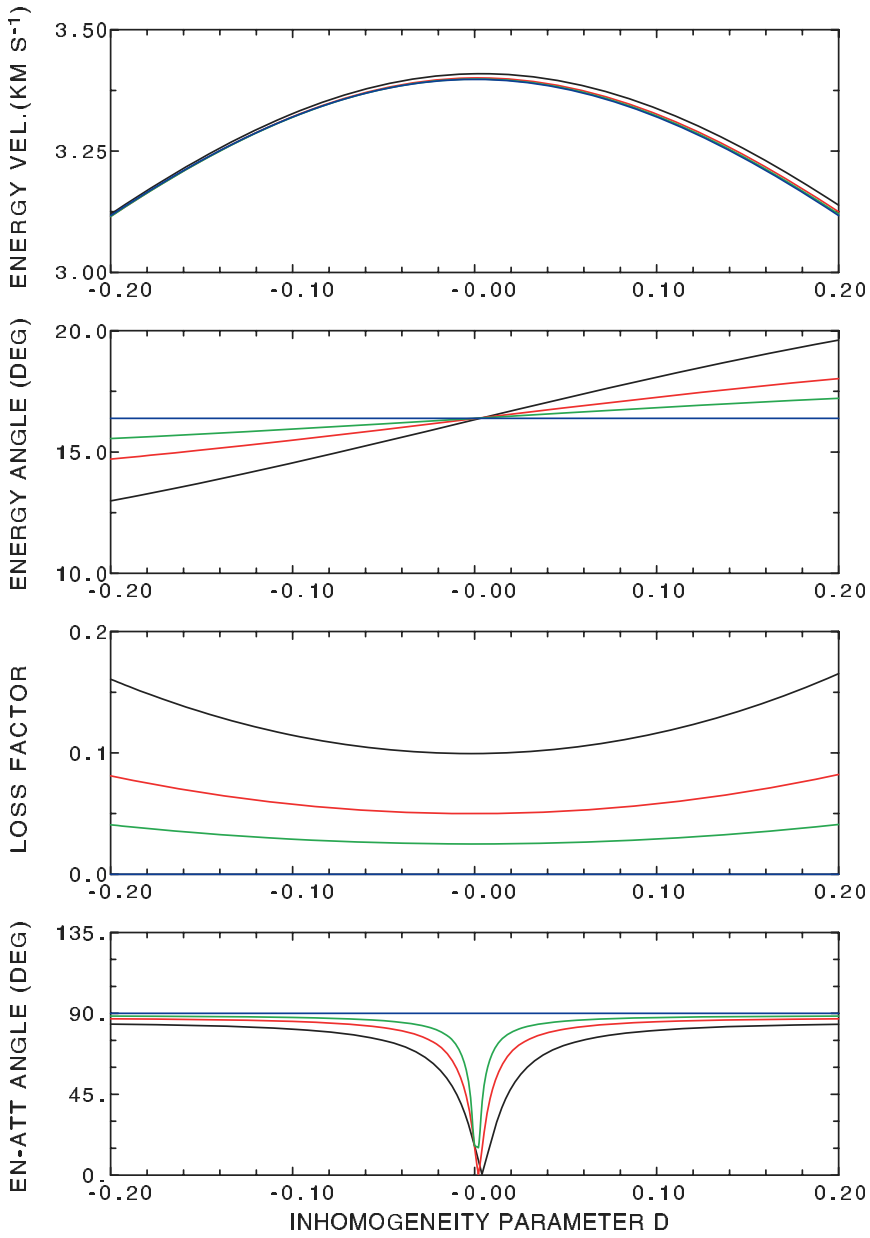


**Figure 1.** Variations with $D$ ($-0.2 \leq D \leq 0.2$) of energy velocity (km s$^{-1}$), energy angle $i_{en}$ (degrees), loss factor $\mathcal{L}$ and energy-attenuation angle $\gamma_{en}$ (degrees) of the *SH* inhomogeneous/homogeneous plane waves for Model 1 (black), Model 2 (red), Model 3 (green) and Model 4 (blue, perfectly elastic). Propagation angle $i = 45°$.

## 6 NUMERICAL EXAMPLES FOR *SH* WAVES

In this section, we present numerical results for the simplest possible case of homogeneous and inhomogeneous plane waves propagating in viscoelastic anisotropic media, namely the SH plane waves propagating in a plane of symmetry of an anisotropic monoclinic (orthorhombic, hexagonal) viscoelastic medium, see Section 5.

In this case, the viscoelastic anisotropic medium is fully specified by three complex-valued, density-normalized Voigt moduli $A_{44}$, $A_{66}$ and $A_{46}$. The properties of the complex-valued slowness vector $\mathbf{p}$ and of related real-valued quantities (propagation vector $\mathbf{P}$, attenuation vector $\mathbf{A}$, phase velocity $C$, attenuation $|\mathbf{A}|$, the value of Im $\sigma$, and attenuation angle $\gamma$) of inhomogeneous *SH* plane waves propagating in such a medium, were numerically investigated in Červený & Pšenčík (2005b, Section 3). We supplement here these studies by studies of the energy flux vector $\mathbf{S}$ and of some related quantities (energy velocity $\mathcal{U}$, loss factor $\mathcal{L}$, energy direction $i_{en}$, and energy-attenuation angle $\gamma_{en}$).

Model 1 is defined by the following moduli:

$$A_{44} = 5 - \mathrm{i}, \quad A_{66} = 11.25 - 1.125\mathrm{i}, \quad A_{46} = 2.5. \tag{68}$$

All quantities in eq. (68) are expressed in (km/s)$^2$. They fully correspond to those given in equation (23) of Červený & Pšenčík (2005b). They are also very close to those used in Carcione & Cavallini (1995). The dissipation parameters (i.e. the imaginary parts of $A_{44}$, $A_{66}$ and $A_{46}$) considered in eq. (68) are rather strong. We, therefore, consider also three other models, with decreasing dissipation parameters (similarly as in Červený & Pšenčík 2005b):

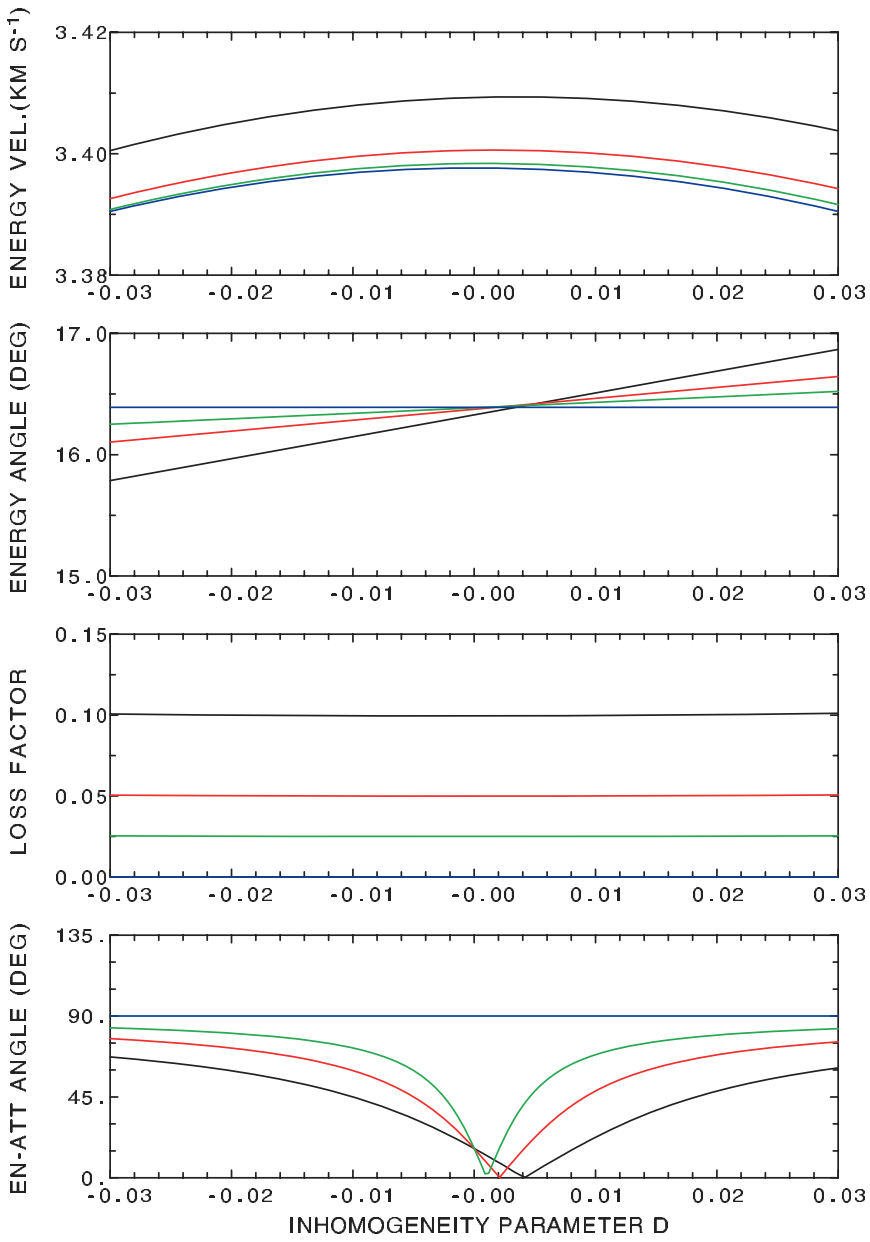


**Figure 2.** The same as in Fig. 1, but for $-0.03 \leq D \leq 0.03$. Certain effects of weakly inhomogeneous plane waves are more clearly expressed.

Model 2:

$$A_{44} = 5 - 0.5\mathrm{i}, \quad A_{66} = 11.25 - 0.5625\mathrm{i}, \quad A_{46} = 2.5, \tag{69}$$

Model 3:

$$A_{44} = 5 - 0.25\mathrm{i}, \quad A_{66} = 11.25 - 0.28125\mathrm{i}, \quad A_{46} = 2.5, \tag{70}$$

Model 4:

$$A_{44} = 5, \quad A_{66} = 11.25, \quad A_{46} = 2.5. \tag{71}$$

Model 4 represents a perfectly elastic anisotropic medium, with $\mathrm{Im}A_{44} = \mathrm{Im}A_{66} = \mathrm{Im}A_{46} = 0$.

Figs 1–6 display four plots each. From top to bottom, they show: (a) the energy velocity $\mathcal{U}$, given by eq. (57), (b) the energy angle $i_{en}$, given by eq. (33), representing the angle between the energy-flux vector **S** and propagation vector **P**, (c) the loss factor $\mathcal{L}$, given by eq. (58), (d) the energy-attenuation angle $\gamma_{en}$, given by eq. (60), and representing the angle between attenuation vector **A** and the energy-flux vector **S**.

The colour convention used in Figs 1–5 corresponds exactly to that used in Červený & Pšenčík (2005b): the black curves correspond to Model 1, red curves to Model 2, green curves to Model 3, and blue curves to Model 4 (perfectly elastic). Thus, it is simple to compare the plots related to slowness vector **p** from Červený & Pšenčík (2005b) with plots related to the energy flux vector **S** presented here.

The quantities under investigation are presented as a function of: a) inhomogeneity parameter $D$, $-0.2 < D < 0.2$. b) propagation angle $i$, $0 \leq i \leq 360°$, related to propagation vector **P**, and given by equations

$$n_1 = \sin i, \quad n_2 = 0, \quad n_3 = \cos i. \tag{72}$$

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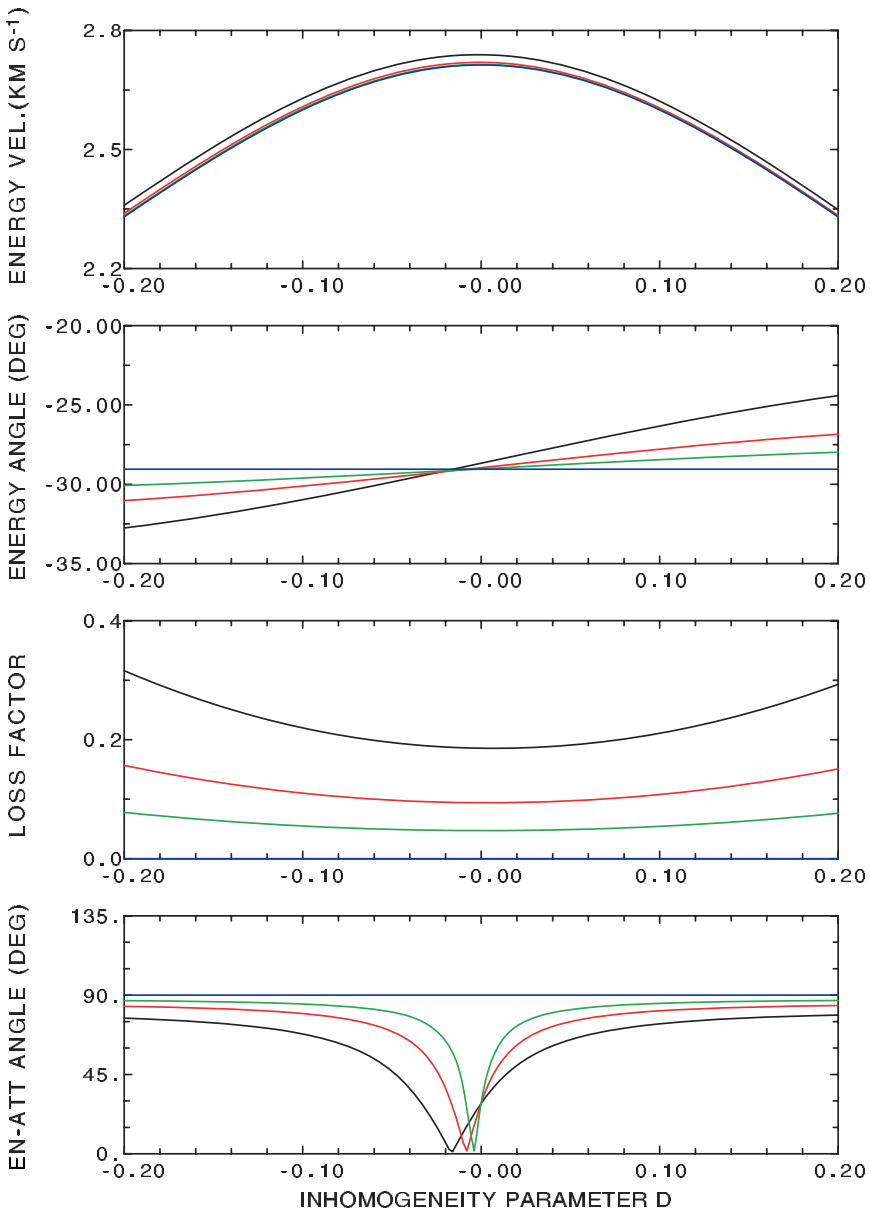


**Figure 3.** The same as in Fig. 1, for propagation angle $i = 135°$.

Consequently, two types of figures are presented: (a) For propagation angle $i$ fixed, and inhomogeneity parameter $D$ varying, see Section 6.1; (b) for inhomogeneity parameter $D$ fixed, but propagation angle $i$ varying (polar graphs), see Section 6.2. Let us emphasize that propagation angle $i$ is related to the direction of propagation of the wave front, not to the direction of energy propagation.

### 6.1 Dependence on inhomogeneity parameter $D$, propagation angle $i$ fixed

Similarly as in Červený & Pšenčík (2005b), we consider two propagation angles: $i = 45°$ (in Figs 1 and 2) and $i = 135°$ (in Figs 3 and 4).

In Fig. 1, the same input data are used as in Fig. 2 of Červený & Pšenčík (2005b), so that the computed quantities in both figures can be directly related. As we are mostly interested in weakly inhomogeneous plane waves ($D$ small), region $D \in (-0.2, 0.2)$ is quite sufficient for our studies. As energy angle $i_{en}$ and energy-attenuation angle $\gamma_{en}$ display certain interesting peculiarities for very small values of $D$, we also include Figs 2 and 4, which show details of the curves from Figs 1 and 3 for very small values of $D$, varying from $-0.03$ to 0.03. A common feature of all these plots is that they are non-symmetric with respect to $D = 0$.

**Energy velocity** $\mathcal{U}$ varies quite smoothly as a function of $D$. It has a maximum at $D = 0$ or very close to it, and decreases with increasing inhomogeneity strength $|D|$. The distinct shift of the maximum of the energy velocity from $D = 0$ to some non-zero $D$, observed for phase velocities $\mathcal{C}$, is observed also for energy velocity $\mathcal{U}$, but is only slight.

Let us now discuss **energy angle** $i_{en}$. For weakly dissipative media, the energy angle is practically constant, independent of $D$. It is close to the value of $i_{en}$, computed for a perfectly elastic anisotropic

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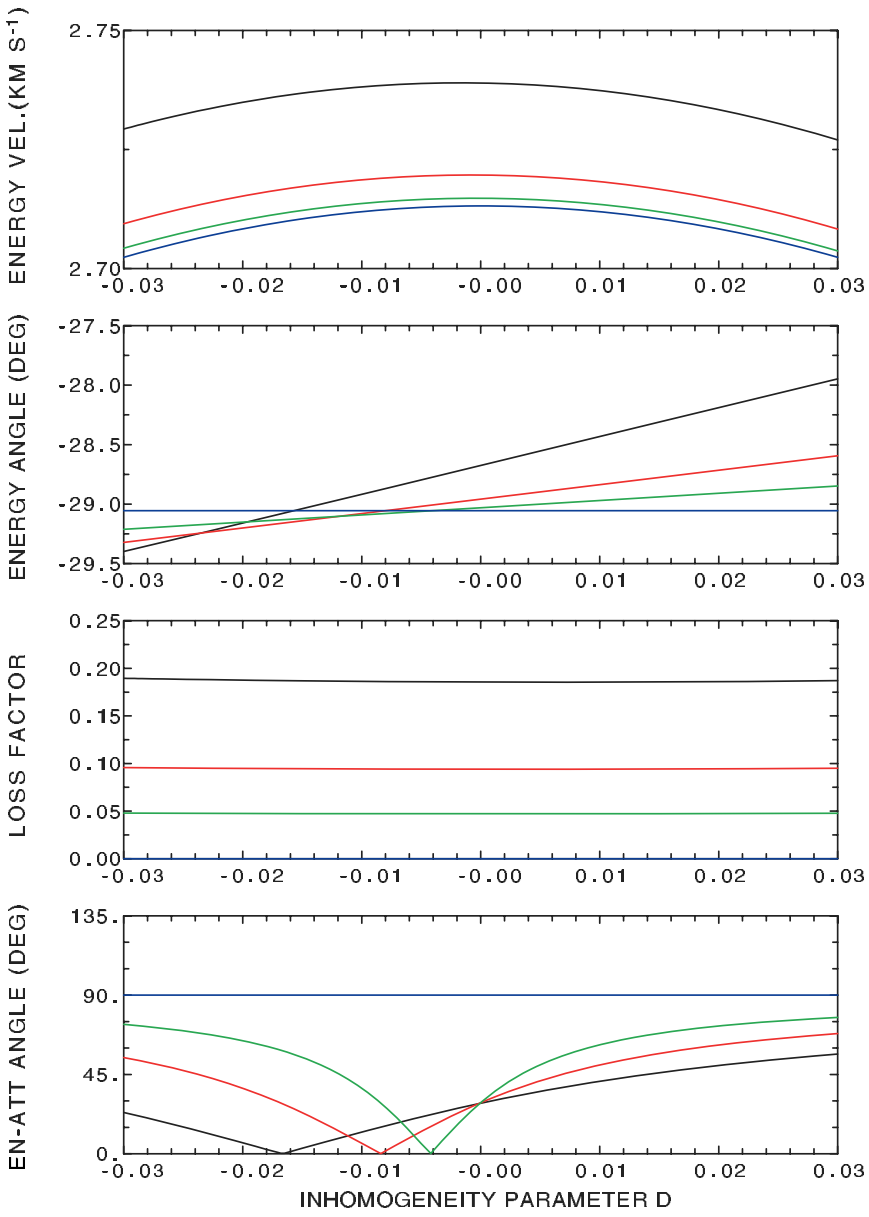


**Figure 4.** The same as in Fig. 2, for propagation angle $i = 135°$.

medium, $i_{en} \doteq 16.4°$. For viscoelastic media, $i_{en}$ varies approximately linearly with $D$. With increasing dissipation, the slope of the straight line $i_{en}(D)$ increases.

**Loss factor** $\mathcal{L}$ depends strongly on dissipation. It is zero for perfectly elastic media. In viscoelastic media, for $D$ given, it increases with increasing dissipation. For a given model, it is minimum for approximately $D = D_{att}$, and increases with increasing $|D|$. It can be taken as a measure of dissipation of the viscoelastic anisotropic medium for a given **n**, **m** and $D$. For $D = D_{att}$, we get an intrinsic loss factor (corresponding to the intrinsic inhomogeneous plane wave), which represents a measure of dissipation of the viscoelastic anisotropic medium for a given **n** and **m**.

**Energy-attenuation angle** $\gamma_{en}$ has different properties than attenuation angle $\gamma$. It is always less or equal 90°, whereas attenuation angle $\gamma$ may be larger, equal to or smaller than 90°. This means that the amplitudes never increase in the direction of energy flux vector **S**, as it may be the case in the direction of the propagation vector **P**, see Červený & Pšenčík (2005b). For perfectly elastic anisotropic media, $\gamma_{en}$ is independent of $D$ and equals 90°. The dominant feature of $\gamma_{en}$ for anisotropic viscoelastic media is that, for fixed propagation angle $i$ and $D$ varying, there is always a (small) $D = D_{att}$, for which energy-attenuation angle $\gamma_{en}$ equals zero. It was shown in Section 5.3 that for this $D_{att}$, attenuation $|\mathbf{A}|$ is minimum. The plane wave with $D = D_{att}$ is the intrinsic inhomogeneous plane wave, see Section 3.2/4. The value of $D_{att}$ can be approximated by eq. (64) with eq. (62).

Figs 3 and 4 correspond to the propagation angle $i = 135°$. All the conclusions obtained for $i = 45°$ remain valid also for $i = 135°$. Some effects are even more pronounced in Figs 3, 4 than in Figs 1, 2. For example, see the values of $\gamma_{en}$ and the relevant $D_{att}$ in Fig. 4.

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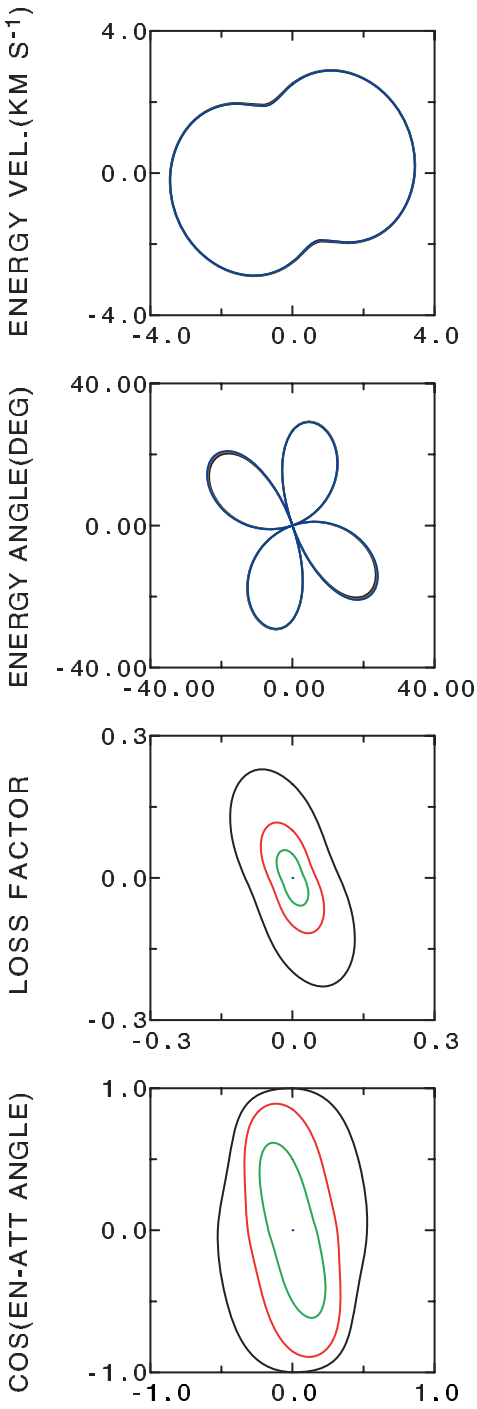


**Figure 5.** Polar diagrams of energy velocity $\mathcal{U}$ (km s$^{-1}$), energy angle $i_{en}$ (degrees), loss factor $\mathcal{L}$, and cosine of energy-attenuation angle $\gamma_{en}$ of the *SH* homogeneous/inhomogeneous plane waves, for $D = 0.02$ and for the same models as in Fig. 1. The meaning of colours is the same as in Fig. 1.

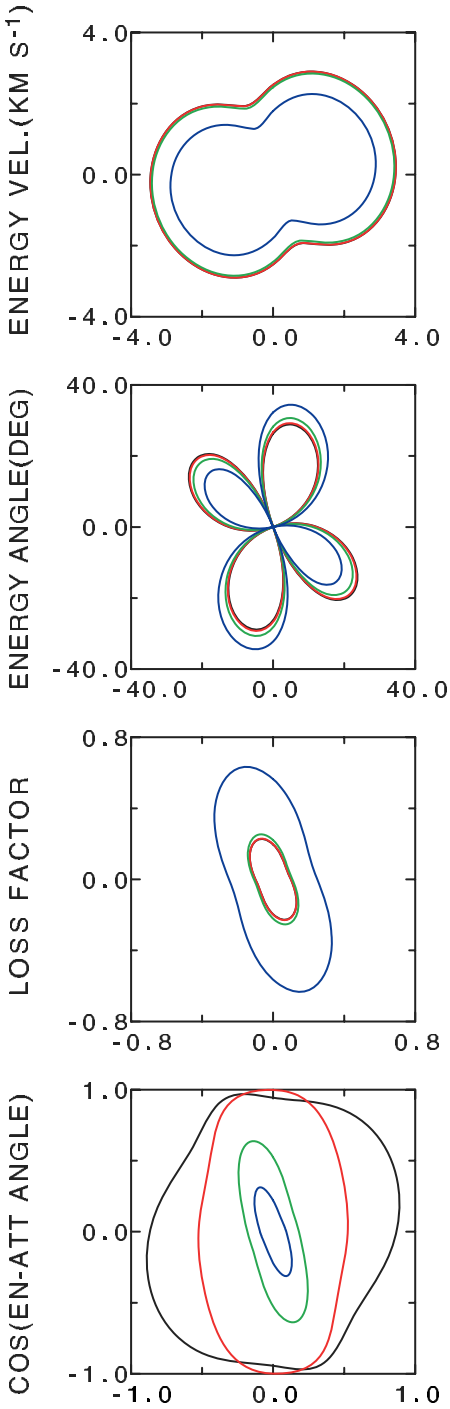


**Figure 6.** Polar diagrams of energy velocity $\mathcal{U}$ (km s$^{-1}$), energy angle $i_{en}$ (degrees), loss factor $\mathcal{L}$, and cosine of energy-attenuation angle $\gamma_{en}$ of *SH* plane homogeneous/inhomogeneous waves, for Model 1, and for $D = 0$ (black), $D = 0.001$ (red), $D = 0.005$ (green), $D = 0.01$ (blue).

### 6.2 Dependence on propagation angle $i$, inhomogeneity parameter $D$ fixed

Here we discuss the propagation of homogeneous and inhomogeneous plane *SH* waves in a symmetry plane of a monoclinic anisotropic medium, as a function of propagation angle $i$, for selected fixed values of inhomogeneity parameter $D$. We use polar graphs, with $i = 0°$ upwards, $i = 90°$ to the right, etc., see eq. (72). In the figures for energy angle $i_{en}$, we show the absolute value of $i_{en}$. The three other quantities ($\mathcal{U}$, $\mathcal{L}$, $\cos\gamma_{en}$) are always non-negative.

In Fig. 5, the four curves in the individual plots correspond to the four models (Model 1–Model 4), and to fixed inhomogeneity parameter $D = 0.02$. The same colour convention is used as in Figs 1–4: Blue corresponds to the perfectly elastic Model 4, black, red and green successively to Models 1–3.

The plots of energy velocity $\mathcal{U}$ and energy angle $i_{en}$ display a very small influence of dissipation on the result. The curves for all three models 1, 2, 3 show practically the same $\mathcal{U}$ and $i_{en}$ as those for perfectly elastic media (Model 4). The polar graphs for energy velocity $\mathcal{U}$ are quite smooth, and have a bean-like shape. The form and size of the energy–velocity plots is very similar to the form and size of the phase-velocity plots, see Fig. 4 of Červený & Pšenčík (2005b). The polar graphs for energy angle $i_{en}$ have four lobes. Energy angle $i_{en}$ changes sign approximately at directions corresponding to the axes of the bean-like polar graph of energy velocity $\mathcal{U}$.

The polar plot for the loss factor  is again of a bean-like shape, but is rotated by 90° with respect to the polar plot of $\mathcal{U}$. Consequently, the loss factor is maximum for propagation angles $i$, for which energy velocity  is minimum, and vice versa. The loss factor  depends strongly on the dissipation parameters. For a perfectly elastic medium, the loss factor is zero, and for increasing dissipation it increases.

The polar plots of $\cos(\gamma_{en})$ again show a smooth variation with angle $i$. For a perfectly elastic medium (Model 4), $\gamma_{en} = 90°$ for any $i$. For dissipative models, the values of $\cos(\gamma_{en})$ increase with increasing dissipation, being largest for, approximately, $i = 0°$ and $i = 180°$. For Model 1, $\gamma_{en}$ is close to zero in these directions.

Fig. 6 corresponds to the strongly dissipative Model 1, see eq. (68). The black curves correspond to $D = 0$ (homogeneous wave), the red curves to $D = 0.001$, the green curves to $D = 0.005$, and the blue curves to $D = 0.01$. The most important observation is that the shapes of the individual curves in Fig. 6 remain very similar to those in Fig. 5. Consequently, the behaviour of $\mathcal{U}$, $i_{en}$, $\mathcal{L}$ and $\gamma_{en}$ described in Section 6.1 has a more general character, at least for *SH* waves.

## 7 PLANE WAVES IN GENERAL VISCOELASTIC ANISOTROPIC MEDIA

In the preceding sections we considered the case of *SH* waves propagating in a symmetry plane of a viscoelastic anisotropic medium, which allows derivation of explicit analytic formulae and their simple analysis. In this section we present and discuss results based on the determination of the corresponding slowness vectors through the numerical solution of eq. (17). In this way, we can investigate homogeneous and inhomogeneous plane waves propagating in arbitrary isotropic or anisotropic, perfectly elastic or anelastic media. We concentrate on inhomogeneous and homogeneous plane waves propagating in model of Jakobsen *et al.* (2003) described and used by Červený & Pšenčík (2005b). We choose again the model corresponding to the frequency of approximately 35 Hz. For simplicity, we consider the density of 1000 kg m$^{-3}$. The matrix of complex valued, density-normalized viscoelastic moduli, measured in (km/s)$^2$ then reads:

$$\mathbf{A_C} = \mathbf{A_1} - \mathrm{i}\mathbf{A_2}, \tag{73}$$

where

$$\mathbf{A_1} = \begin{pmatrix} 46.631 & 5.983 & 4.278 & 0. & 0. & 0. \\ & 46.631 & 4.278 & 0. & 0. & 0. \\ & & 19.931 & 0. & 0. & 0. \\ & & & 13.444 & 0. & 0. \\ & & & & 13.444 & 0. \\ & & & & & 20.324 \end{pmatrix}$$

and

$$\mathbf{A_2} = \begin{pmatrix} 0.033 & 0.022 & 0.156 & 0. & 0. & 0. \\ & 0.033 & 0.156 & 0. & 0. & 0. \\ & & 1.312 & 0. & 0. & 0. \\ & & & 0.055 & 0. & 0. \\ & & & & 0.055 & 0. \\ & & & & & 0.005 \end{pmatrix}.$$

Note that both matrices $\mathbf{A_1}$ and $\mathbf{A_2}$ are positive definite. The matrix $\mathbf{A_1}$ corresponds to an anisotropic medium of hexagonal symmetry with vertical axis of symmetry, which represents the direction of the kiss singularity.

We calculate energy velocities $\mathcal{U}$, see eq. (27), energy angle $i_{en}$, see eq. (33), energy-attenuation angle $\gamma_{en}$, see eq. (34) and loss factor , see eq. (39) of a *P* wave and two *S* waves propagating in the plane of symmetry of the medium specified by the above set of moduli. Note that we could extend this study out of the symmetry plane without any problem. We concentrate on it because the behaviour of listed quantities can be studied better in the symmetry plane. We first study the energy velocity as a function of direction of propagation vector (perpendicular to the wave front), and as a function of direction of the energy flux vector, with $D$ fixed. Then we study all the above listed parameters as functions of the inhomogeneity parameter $D$, for fixed direction of propagation vector.

### 7.1 Dependence on directions of propagation vector and of energy flux vector, inhomogeneity parameter $D$ fixed

In Figs 7 and 8, we study the energy velocities of *P* and *S* waves for several selected values of inhomogeneity parameter $D$ in the symmetry plane of the model of viscoelastic anisotropic medium (eq. 73). The propagation–attenuation plane, that is a plane formed by vectors **n** and **m**, coincides with the symmetry plane. The colours distinguish the curves according to corresponding phase velocities. The red colour corresponds to the greatest, the black colour to intermediate and the blue colour to the smallest phase velocity. It was shown in Červený & Pšenčík (2005b) that the red colour in model (73) always corresponds to the *P* wave, the black colour corresponds mostly to the *SH* wave, only in a vicinity of the vertical axis it may sometimes correspond to the *SV* wave.

The polar graphs of **energy velocity** as a function of direction of propagation vector (perpendicular to the wave front) for six values of inhomogeneity parameter $D$ are shown in Fig. 7: $D = 0$ (homogeneous wave), $D = 0.02$, $D = 0.06$, $D = 0.08$, $D = 0.1$ and $D = 0.2$. Note that the plots resemble the phase-velocity plots in Figure 7 of Červený & Pšenčík (2005b). For $D = 0$, the energy–velocity curves are symmetric with respect to the vertical axis. The red energy–velocity curve has a bean-like shape and it is separated from the remaining surfaces. It corresponds to the *P* wave. The black curve corresponds to the *SH* wave, the blue one to the *SV* wave. Both *S*-wave energy velocities are equal along the vertical axis. With increasing $D$, the form of the surfaces becomes more complicated especially in the vicinity of the vertical axis. The *SV*-wave curve intersects the *SH*-wave curve. At the same time, the *P*- and *SV*-wave energy velocities approach each other and touch at a point slightly off the vertical axis. All the curves loose their symmetry with respect to the vertical axis.

Fig. 8 shows the same as Fig. 7 but now energy velocity is a function of the direction of energy flux vector. For $D = 0$, the energy–velocity curves of *S* waves have a form similar to those of Fig. 7. The *P*-wave curve, however, has an ellipsoidal rather than bean-like

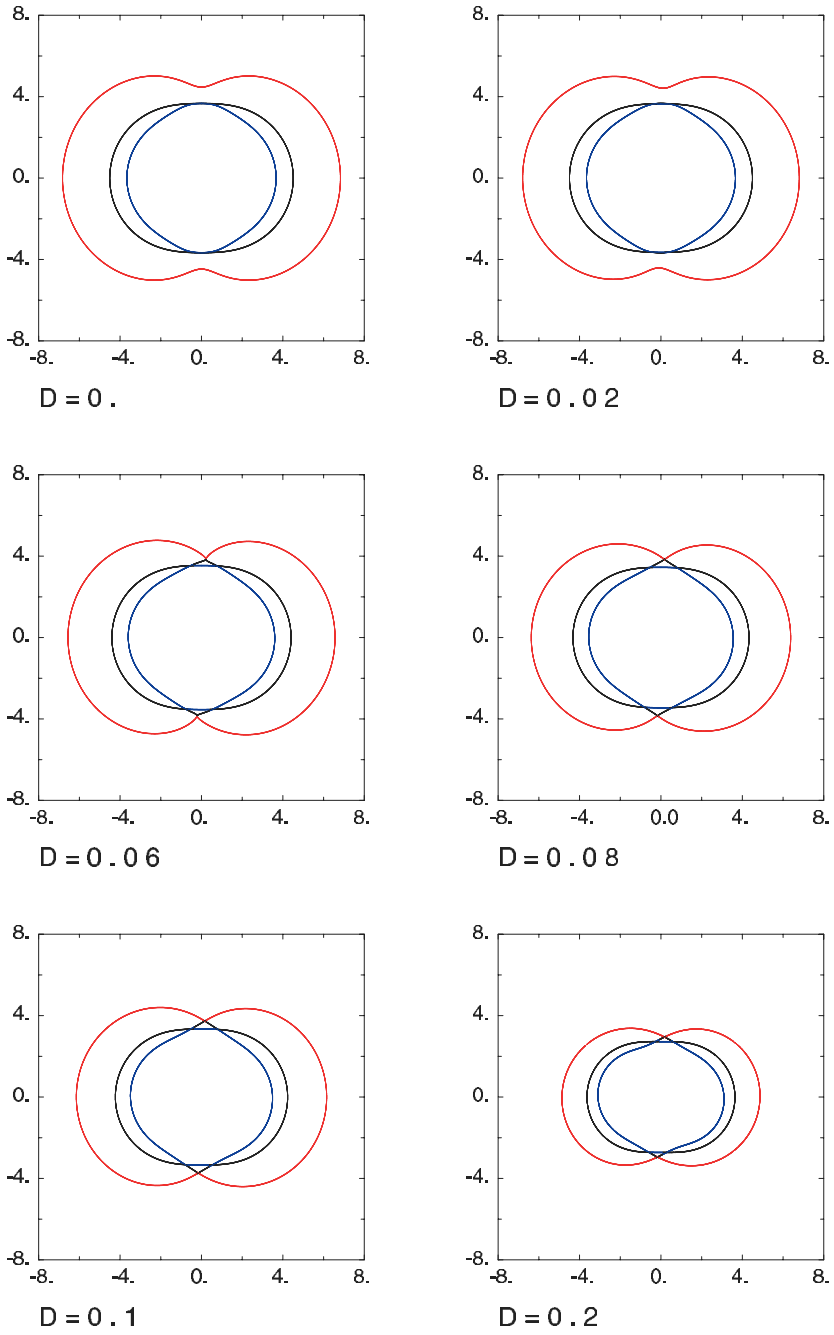


**Figure 7.** Polar diagrams of energy velocity $\mathcal{U}$ (km s$^{-1}$) of homogeneous/inhomogeneous plane waves in a symmetry plane of the medium (eq. 73) as a function of direction of propagation vector **P** (perpendicular to the wave front) for varying $D$: fastest wave (red; $P$ wave), intermediate wave (black; mostly $SH$ wave), slowest wave (blue; mostly $SV$ wave).

form. With increasing $D$, the form of the curves becomes again more complicated, especially in the vicinity of the vertical axis, and the curves of $P$ and $S$ waves approach each other; with increasing $D$, the energy velocities decrease and plots become non-symmetric with respect to the vertical axis. There are some features, which distinguish the energy–velocity plots from those in Fig. 7. We can see that with increasing $D$, the energy velocity of the $P$ wave does not point into directions close to the vertical axis, see the plots for $D = 0.06$ and greater. The energy–velocity curve of the $P$ wave continuously changes into the energy–velocity curve of the faster of the two $S$ waves. It is obvious that the behaviour of weakly inhomogeneous waves in the vicinity of the vertical axis deserves a more detailed study.

The character of the above plots remains very similar even for plane waves propagating in a perfectly elastic anisotropic medium specified by matrix $\mathbf{A}_1$ only, see eq. (73). The only remarkable difference is the preserved symmetry (even for non-zero $D$) with respect to the vertical axis in a perfectly elastic medium. This seems to indicate that for weakly dissipative media approximate approaches, in which the imaginary parts of the viscoelastic moduli, that is, elements of matrix $\mathbf{A}_2$, are considered as small perturbations, should work well.

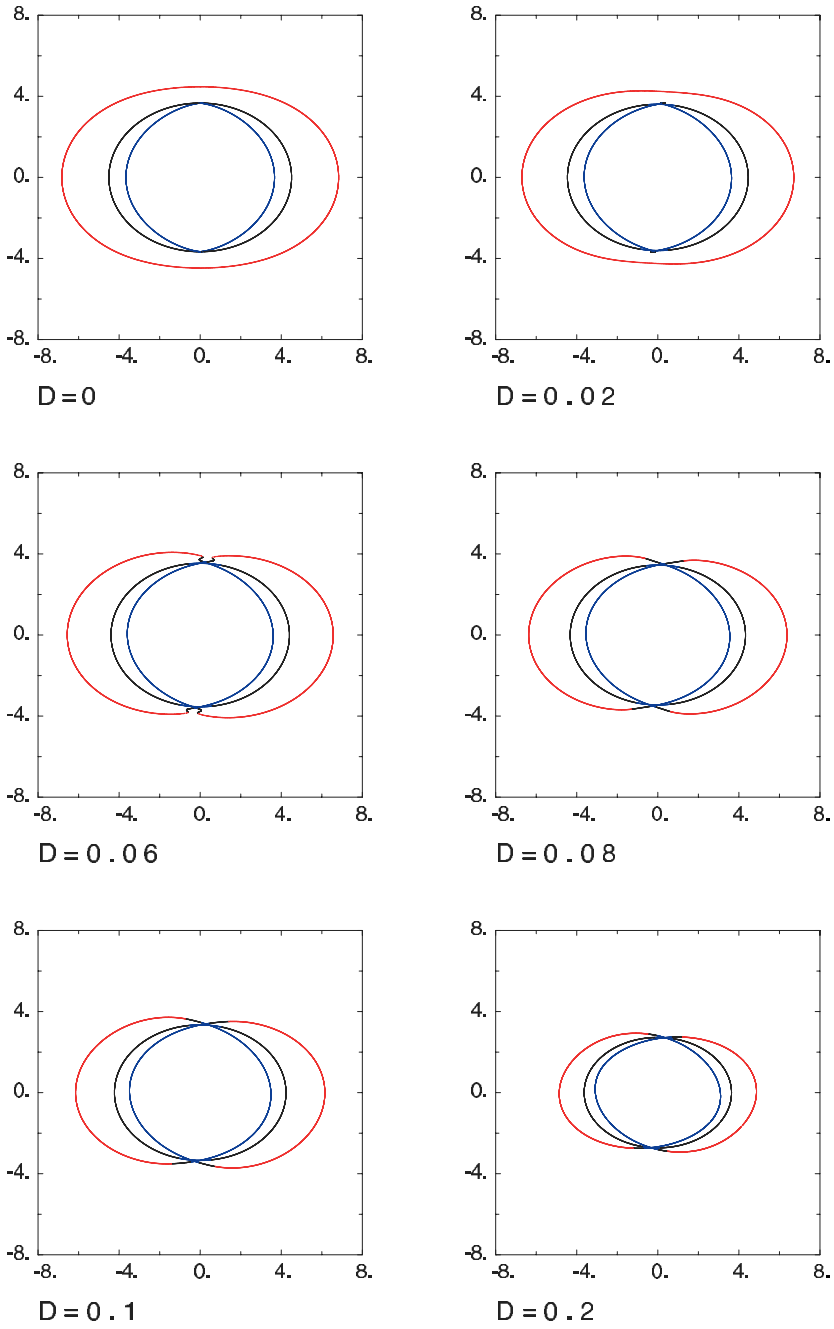


**Figure 8.** The same is in Fig. 7, but as a function of the direction of energy flux vector **S**.

In viscoelastic or perfectly elastic *isotropic* media, the sections of the energy–velocity surfaces of both $P$ and $S$ waves would be circular.

### 7.2 Dependence on inhomogeneity parameter $D$, propagation angle $i$ fixed

In Fig. 9, we can see the variation of the **energy velocity** in the plane of symmetry of the model of the viscoelastic anisotropic medium (eq. 73) as a function of inhomogeneity parameter $D$, for several propagation angles $i$. The propagation–attenuation plane coincides again with the symmetry plane. Four values of the propagation angle are considered: $i = 0°$, 5°, 10° and 45°, and inhomogeneity parameter $D$ varies from $-0.2$ to 0.2. The red colour again corresponds to the wave with greatest phase velocity, and according to Červený & Pšenčík (2005b), it is the $P$ wave. The black colour corresponds to the $S$ wave with greater phase velocity and the blue colour is reserved for the $S$ wave with smallest phase velocity. Fig. 9 confirms what we can see in Fig. 7. For propagation angles close to the vertical axis, which points into a singular direction in a perfectly elastic case, the behaviour of energy velocities as a function of $D$ is more complicated than, for example, for propagation angle $i = 45°$. Only for propagation angle $i = 45°$, the behaviour of the plots is similar to that of the $SH$ waves in Figs 1 and 3.

The first feature which we can observe in Fig. 9 is that the plots are non-symmetric with respect to $D = 0$ for propagation angles $i = 5°$,

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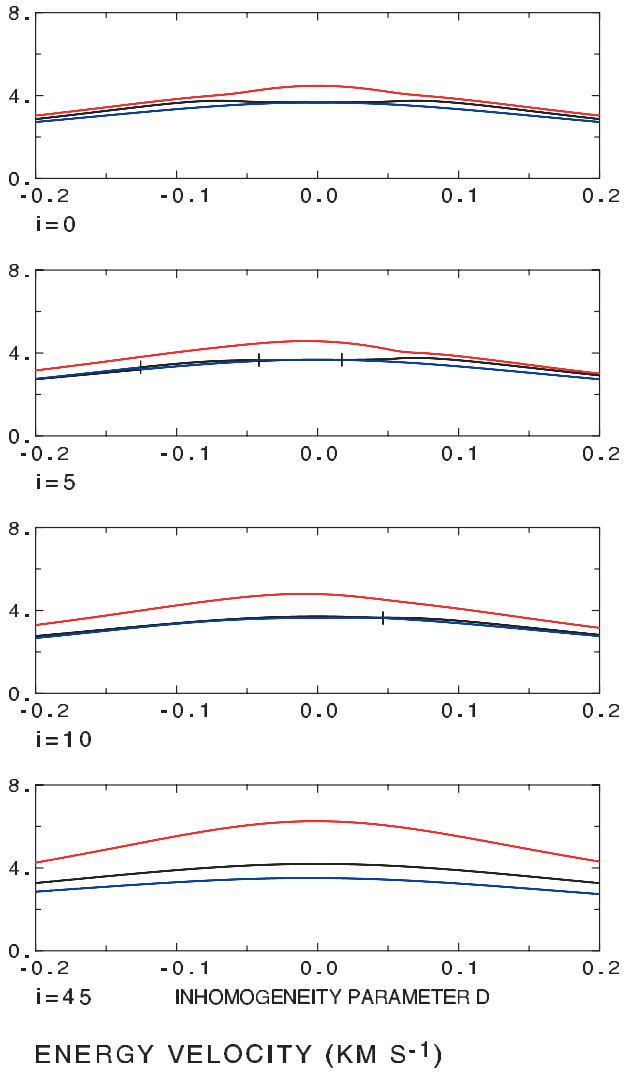


**Figure 9.** Variations with $D$ ($-0.2 \leq D \leq 0.2$) of energy velocity $\mathcal{U}$ (km s$^{-1}$), of homogeneous/inhomogeneous plane waves in a symmetry plane of the medium (eq. 73) for four values of propagation angle $i$. Fastest wave (red; $P$ wave), intermediate wave (black; mostly $SH$ wave), slowest wave (blue; mostly $SV$ wave). Ticks indicate changes of blue into black and vice versa.

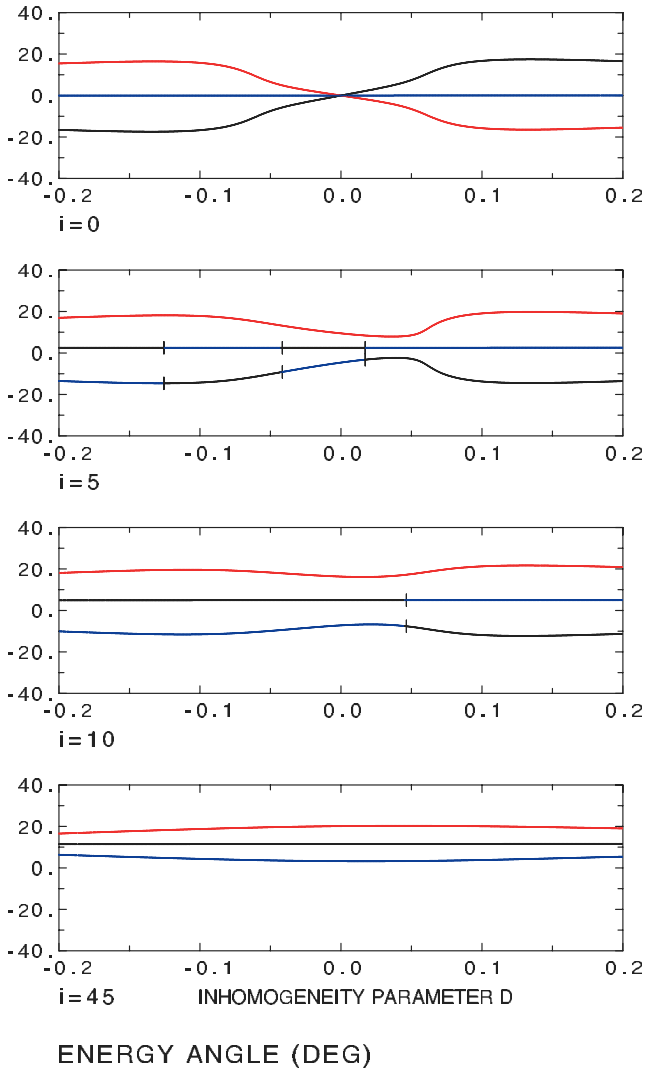


**Figure 10.** The same as in Fig. 9, but for energy angle $i_{en}$ (degrees).

10° and 45°. Lack of symmetry is most pronounced for propagation angle $i = 5°$. On the contrary for $i = 0°$, the plot is fully symmetric.

Except for $i = 0°$, we can observe a shift of the maximum of the energy velocity from $D = 0$ to a non-zero value of $D$. See a similar observation for $SH$ wave in Fig. 1.

An interesting feature is the decrease of the $SV$-wave energy velocity around $D = 0$ for $i = 0°$ and partially also for $i = 5°$. Another interesting feature is the coincidence of the shear-wave energy velocities for certain ranges of $D$. Note also places with marks indicating changes of colours specifying the $S$ waves. At these places the phase velocity of the $SV$ wave becomes greater than that of the $SH$ wave or vice versa.

In a perfectly elastic medium, the most pronounced difference with Fig. 9 is the strict symmetry with respect to $D = 0$.

Fig. 10 shows the behaviour of the **energy angle** as a function of $D$. We can again see that only the plot for $i = 45°$ resembles the corresponding plots for the $SH$ wave in Figs 1 and 3. For other propagation directions, which are close to the singular direction of the perfectly elastic case, the behaviour of the energy angles is more complicated. For $i = 0°$ and for the $S$ wave with smaller phase velocity, which is an $SH$ wave in this case, the energy angle is independent of $D$ and zero, that is, the directions of the energy–velocity vector and the propagation vector are the same. For the $P$ wave and the $SV$ wave, the energy angle is zero for $D = 0$ and increases, with opposite signs for increasing $|D|$. For $|D| = 0.2$, it reaches nearly 20°. For $i = 5°$, the plot becomes strongly non-symmetric with respect to $D = 0$. The energy angle of the $SH$ wave is again independent of $D$, but it differs slightly from zero. From the changing colour with varying $D$ (see the marks on the curves), we can see that for some $D$ the $SH$-wave phase velocity is smaller than that of the $SV$ wave, for other $D$, it is greater. The energy angles of the $P$ and $SV$ waves are symmetric with respect to the value of the energy angle of the $SH$ wave. For $i = 10°$, the variation of energy angles of all waves becomes less pronounced, for $i = 45°$, it is nearly constant.

Except for $i = 0°$, the plots for a perfectly elastic medium would be again similar to those shown in Fig. 10.

The **energy-attenuation angles** of all three waves shown in Fig. 11 behave similarly as the same angles of the $SH$ waves in Figs 1 and 3. They are always less than 90°, which distinguishes them from

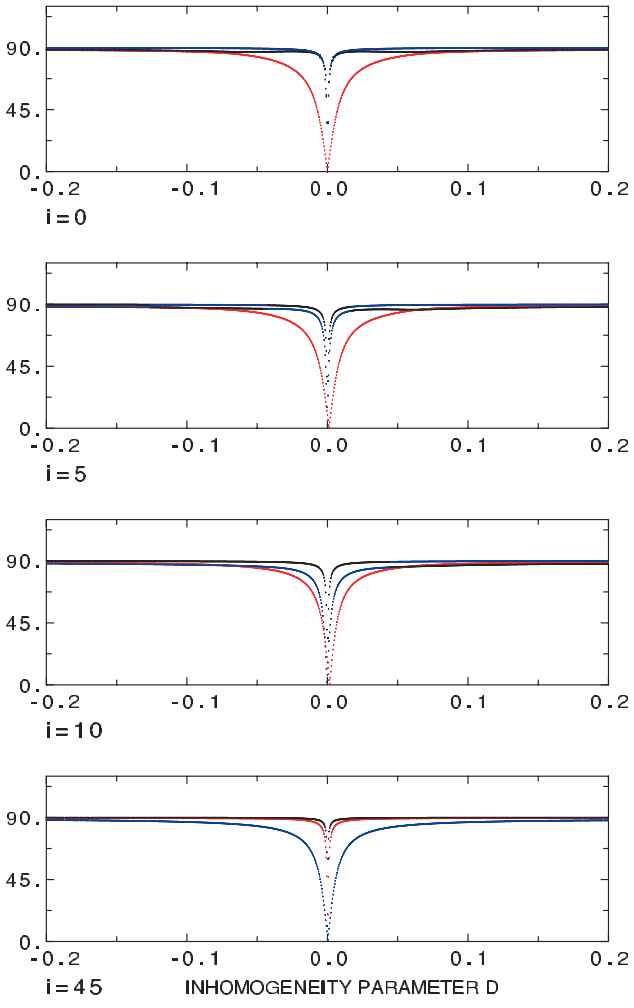


**Figure 11.** The same as in Fig. 9, but for energy-attenuation angle $\gamma_{en}$ (degrees).

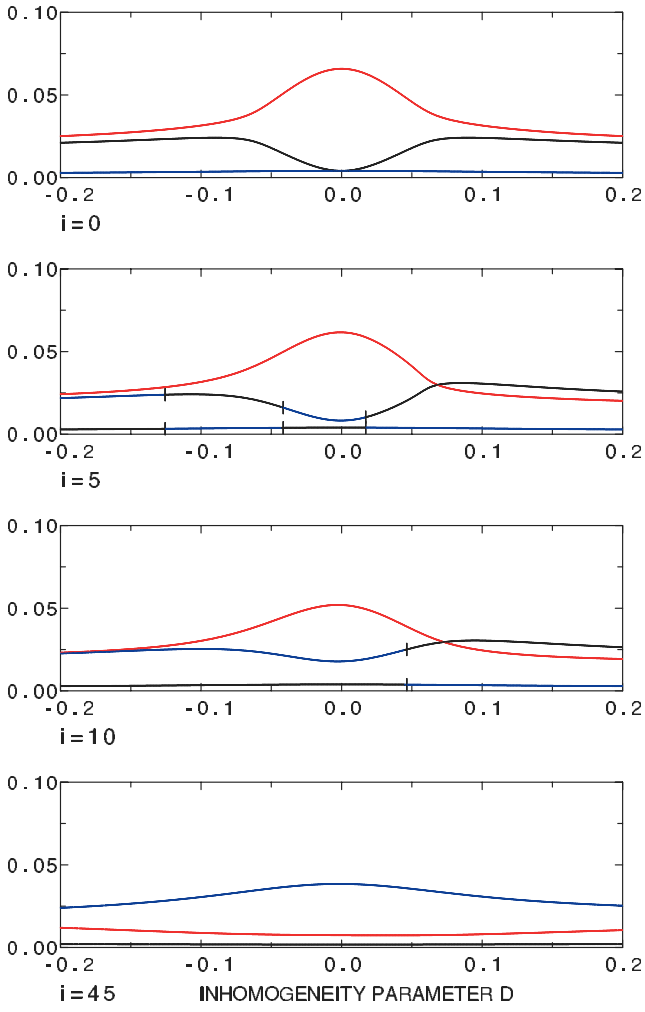


**Figure 12.** The same as in Fig. 9, but for loss factor $\mathcal{L}$.

the attenuation angles, see Červený & Pšenčík (2005b), which may be greater or less than 90°. As in Figs 1 and 3, the energy-attenuation angle becomes zero for $D$, which is, in most cases, slightly different from $D = 0$. For propagation angle $i = 0°$, the energy-attenuation angle is, however, zero for exactly $D = 0$.

The energy-attenuation angles of the $P$, $SV$ and $SH$ plane waves for a perfectly elastic medium are constant and equal to 90°.

Finally, in Fig. 12, we show the behaviour of the loss factor for the above set of propagation angles. We can see that the behaviour of the loss factor of the $P$ and $SV$ waves is quite different from the behaviour of the $SH$ waves, see the nearly constant curve in Fig. 12, or the curves in Figs 1 and 3. We can see that, except for $i = 45°$, the $P$-wave loss factor has a pronounced maximum for $D$ close to zero, whereas the $SV$-wave loss factor has pronounced minimum there. This means that in directions close to the vertical axis, homogeneous and weakly inhomogeneous $P$ waves will be substantially more dissipated than the corresponding $SV$ waves. For $i = 45°$, the situation changes. The dissipation of $SH$ waves is always small and practically independent of $D$. This is a consequence of small imaginary parts of moduli $A_{44}$ and $A_{66}$, see eq. (73).

## 8 CONCLUSIONS

Time-averaged complex-valued energy flux of homogeneous and inhomogeneous time-harmonic plane waves propagating in unbounded viscoelastic anisotropic media can be computed very effectively using the mixed specification of slowness vector, proposed by Červený & Pšenčík (2005a). In a medium with unrestricted anisotropy and viscoelasticity, the relevant numerical algorithm requires a solution of a complex-valued algebraic equation of the sixth degree. For simpler anisotropy symmetries, the algebraic equation may reduce to fourth or even second degree. The same algorithm can also be used to compute other time-averaged energy quantities like the time-averaged energy velocity and time-averaged dissipated energy.

Presented numerical examples show several interesting phenomena. When studying the behaviour of energy quantities of inhomogeneous waves as functions of direction (specified either by the propagation vector perpendicular to the wave front or by the energy flux vector), we can observe lack of symmetry, existing for homogeneous waves, with respect to the direction representing a singular direction in the corresponding perfectly elastic medium. Energy quantities behave in a peculiar way in a vicinity of these directions. This phenomenon deserves further investigation. When we

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study the energy quantities as a function of varying inhomogeneity parameter $D$, we observe loss of symmetry of the behaviour of these quantities with respect to $D = 0$. The non-symmetry is again more pronounced in directions close to directions, which are singular in the perfectly elastic case.

The algorithms proposed here open some new possibilities in the investigation of properties of time-harmonic plane waves propagating in anisotropic viscoelastic media. This applies mainly to the instantaneous energy flux and other instantaneous energy quantities. It would also be useful to investigate, in a greater detail, the imaginary part of the complex-valued energy flux, and its relation to the dissipated energy and to the difference between the kinetic and stress energies. Further studies of energy-attenuation vector and of intrinsic inhomogeneous plane waves (see Section 3.2/4) seem to be also desirable.

Similarity of presented numerical examples for weakly viscoelastic anisotropic media to examples computed for perfectly elastic media and similarity of results obtained for homogeneous and inhomogeneous waves open possibilities for use of various perturbation approaches. It should also be mentioned that the used algorithms need only a slight generalization to make them applicable to study of homogeneous and inhomogeneous plane waves propagating in more general types of media, for example, in porous viscoelastic anisotropic media. In the general case, this would require to solve a complex-valued algebraic equation of the eight degree instead of the sixth degree necessary in the present study.

## ACKNOWLEDGMENTS

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