INHOMOGENEOUS HARMONIC PLANE WAVES IN VISCOELASTIC ANISOTROPIC MEDIA

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ABSTRACT

In viscoelastic media, the slowness vector $\mathbf{p}$ *of plane waves is complex-valued,* $\mathbf{p} = \mathbf{P} + \mathrm{i}\mathbf{A}$. *The real-valued vectors* $\mathbf{P}$ *and* $\mathbf{A}$ *are usually called the propagation and the attenuation vector, respectively. For* $\mathbf{P}$ *and* $\mathbf{A}$ *nonparallel, the plane wave is called inhomogeneous.*

Three basic approaches to the determination of the slowness vector of an inhomogeneous plane wave propagating in a homogeneous viscoelastic anisotropic medium are discussed. They differ in the specification of the mathematical form of the slowness vector $\mathbf{p}$. *We speak of directional specification, componental specification and mixed specification of the slowness vector. Individual specifications lead to the eigenvalue problems for* 3×3 *or* 6×6 *complex-valued matrices.*

In the directional specification of the slowness vector, the real-valued unit vectors $\mathbf{N}$ *and* $\mathbf{M}$ *in the direction of* $\mathbf{P}$ *and* $\mathbf{A}$ *are assumed to be known. This has been the most common specification of the slowness vector used in the seismological literature. In the componental specification, the real-valued unit vectors* $\mathbf{N}$ *and* $\mathbf{M}$ *are not known in advance. Instead, the complex-valued vectorial component* $\mathbf{p}^{\Sigma}$ *of slowness vector* $\mathbf{p}$ *into an arbitrary plane* Σ *with unit normal* $\mathbf{n}$ *is assumed to be known. Finally, the mixed specification is a special case of the componental specification, with* $\mathbf{p}^{\Sigma}$ *purely imaginary. In the mixed specification, plane* Σ *represents the plane of constant phase, so that* $\mathbf{N} = \pm\mathbf{n}$. *Consequently, unit vector* $\mathbf{N}$ *is known, similarly as in the directional specification. Instead of unit vector* $\mathbf{M}$, *however, the vectorial component* $\mathbf{d}$ *of the attenuation vector in the plane of constant phase is known.*

The simplest, most straightforward and transparent algorithms to determine the phase velocities and slowness vectors of inhomogeneous plane waves propagating in viscoelastic anisotropic media are obtained, if the mixed specification of the slowness vector is used. These algorithms are based on the solution of an algebraic equation of the sixth degree. Alternatively, they are based on the solution of a conventional eigenvalue problem for 6×6 *complex-valued matrices. The derived equations are quite general and universal. They can be used both for homogeneous and inhomogeneous plane waves, propagating in elastic or viscoelastic, isotropic or anisotropic media. Contrary to the mixed specification, the directional specification can hardly be used* *to determine the slowness vector of inhomogeneous plane waves propagating in viscoelastic anisotropic media. Although the procedure is based on 3×3 complex-valued matrices, it yields a cumbersome system of two coupled equations.*

Keywords: viscoelastic anisotropic media, inhomogeneous plane waves, phase velocity, slowness vector

1. INTRODUCTION

In viscoelastic anisotropic media, the determination of the slowness vector of plane waves is more complex than in perfectly elastic anisotropic media. For this reason, we shall first briefly recapitulate the relevant procedures for anisotropic elastic media. Only then shall we discuss the anisotropic viscoelastic media.

Two basic procedures to determine the real-valued slowness vector $\mathbf{p}$ of a plane wave propagating in a homogeneous perfectly elastic anisotropic medium (with real-valued elastic moduli c_{ijkl}) can be used. They differ in the specification of the slowness vector.

1) Assume that the real-valued unit vector $\mathbf{N}$ in the direction of slowness vector $\mathbf{p}$ is specified. Slowness vector $\mathbf{p}$ can then be expressed by the relation $\mathbf{p} = \mathbf{N}/\mathcal{C}$, where $\mathcal{C}$ is the phase velocity. The square of the phase velocity $\mathcal{C}^2$ is an eigenvalue of the 3×3 real-valued Christoffel matrix $\Gamma_{ik}(N_n) = (c_{ijkl}/\rho)N_j N_l$. Thus, if $\mathbf{N}$ is specified, the determination of $\mathbf{p}$ reduces to the solution of an eigenvalue problem for the 3×3 real-valued Christoffel matrix $\mathbf{\Gamma}$. The three eigenvalues of the Christoffel matrix correspond to the three plane waves (qP, qS1 and qS2), propagating in the anisotropic medium in the direction of $\mathbf{N}$. See *Fedorov (1968)*, *Musgrave (1970)*, *Helbig (1994,* p. 132–133*)*, *Červený (2001,* p. 22–25*)*, and many other references given in these books. Note that the Christoffel matrix is also called the Kelvin-Christoffel matrix.

2) Assume that the vectorial component $\mathbf{p}^{\Sigma}$ of slowness vector $\mathbf{p}$ into any plane Σ is specified, and denote the Cartesian components of this projection by p_i^{Σ}. Denote further the unit vector perpendicular to Σ by $\mathbf{n}$. Slowness vector $\mathbf{p}$ can then be expressed by the relation $\mathbf{p} = \mathbf{p}^{\Sigma} + \sigma\mathbf{n}$, where $\mathbf{p}^{\Sigma} \cdot \mathbf{n} = 0$, and where σ is a solution of an algebraic equation of the sixth degree. The six solutions of the algebraic equation correspond to the six plane waves, three of them (qP, qS1, qS2) approaching the plane Σ, and the three other (qP, qS1, qS2) leaving the plane Σ. See *Fedorov (1968)*, *Šílený (1981)*, *Gajewski and Pšenčík (1987)*, *Červený (2001,* p. 527*)*. Alternatively, σ may be determined as an eigenvalue of a 6×6 real-valued matrix $\mathbf{A}$. Thus, if p_i^{Σ} $(i = 1, 2, 3)$ are specified, the determination of $\mathbf{p}$ reduces to the solution of an eigenvalue problem for a 6×6 real-valued matrix $\mathbf{A}$. The six eigenvalues of $\mathbf{A}$ again correspond to the six plane waves, three of them approaching and three of them leaving the plane Σ, as above. This method has been traditionally used in seismology in the solution of the reflection/transmission problems, and in the displacement-stress propagator matrix computations. See more details in *Woodhouse (1974)*,

Frazer and Fryer (1989), Chapman (1994), Thomson (1996a,b), and *Červený (2001, Section 5.4.7)*, where many other references can be found. The method based on the eigenvalues of the 6×6 matrix $\mathbf{A}$ can be also formulated using the so-called Stroh formalism, see *Stroh (1962), Ting (1996)*. The last reference offers a detailed tutorial treatment of the Stroh formalism, with many references.

Let us now return to inhomogeneous plane waves (with a complex-valued slowness vector) propagating in an anisotropic viscoelastic medium (with complex-valued viscoelastic moduli c_{ijkl}). The first procedure, based on the solution of the 3×3 eigenvalue problem, is considerably more complicated in this case. The characteristic equation is then represented by a system of two coupled equations for the square $\mathcal{C}^2$ of the real-valued phase velocity $\mathcal{C}$ and for an additional real-valued parameter δ, see Section 4.

On the contrary, the second procedure, based on the solution of the eigenvalue problem for a 6×6 complex-valued matrix $\mathbf{A}$, can be used quite routinely, even for inhomogeneous plane waves propagating in viscoelastic anisotropic media. See Sections 5 and 6.

We now introduce certain notations. We use Cartesian coordinates x_i, denote the time t, the fixed circular frequency ω, and assume $\omega > 0$ throughout the paper. We denote the Cartesian components of the displacement vector $u_i(x_k, t)$, of particle velocity $v_i(x_k, t) = \dot{u}_i(x_k, t)$, and of stress tensor $\tau_{ij}(x_k, t)$. The stress tensor is presumably symmetric, $\tau_{ij} = \tau_{ji}$. The source-free elastodynamic equation reads

$$\tau_{ij,j} = \rho \dot{v}_i \ , \tag{1}$$

where ρ is the density. The dot above the letter denotes the partial derivative with respect to time t ($\dot{v}_i = \partial v_i / \partial t = \partial^2 u_i / \partial t^2$), the index following the comma in the subscript indicates the partial derivative with respect to the relevant Cartesian coordinate ($\tau_{ij,j} = \partial \tau_{ij} / \partial x_j$). The upper-case indices (I,J,K,...) take values of 1,2, the lower-case indices (i,j,k,...) the values of 1,2,3, and the Greek indices $(\alpha, \beta, \gamma, ...)$ the values of 1,2,...,6. The Einstein summation convention over repeating indices is used. In complex-valued quantities, $z = x + \mathrm{i}y$, the asterisk is used as a superscript to denote the complex-conjugate quantity, $z^* = x - \mathrm{i}y$. We shall consider the time-harmonic waves, with the exponential factor $\exp[-\mathrm{i}\omega t]$.

In the frequency domain, the generalized Hooke's law for linearized viscoelastic anisotropic solids can be expressed in the following form:

$$\tau_{ij} = c_{ijkl} u_{k,l} \ , \tag{2}$$

where $c_{ijkl} = c_{ijkl}(\omega)$ are complex-valued, frequency-dependent, viscoelastic moduli. We shall perform all computations for a fixed (but arbitrary) positive frequency $\omega > 0$. For this reason, we shall not use frequency ω as the argument of c_{ijkl}. The viscoelastic moduli satisfy the symmetry relations $c_{ijkl} = c_{jikl} = c_{ijlk} = c_{klij}$. In addition to c_{ijkl}, we shall also use the density-normalized viscoelastic moduli a_{ijkl}:

$$a_{ijkl} = c_{ijkl} / \rho \ . \tag{3}$$

Similarly as c_{ijkl}, also a_{ijkl} are complex-valued and frequency-dependent, and satisfy the above shown symmetry relations.

2. PLANE WAVES IN HOMOGENEOUS VISCOELASTIC ANISOTROPIC MEDIA

Elastodynamic equation (1) for a homogeneous viscoelastic anisotropic unbounded medium, in which Hooke's law (2) is valid, can be expressed in the following form:

$$a_{ijkl} u_{k,lj} = \dot{v}_i \ . \tag{4}$$

Now consider an ansatz plane-wave time-harmonic solution of (4),

$$u_j(x_k, t) = U_j \exp[-\mathrm{i}\omega(t - p_n x_n)] \ , \tag{5}$$

where p_n and U_n are some complex-valued constants. These constants, however, cannot be taken arbitrarily. They must satisfy the constraint relations which follow from elastodynamic equation (4). Inserting the ansatz plane wave solution (5) into elastodynamic equation (4), three constraint relations are obtained:

$$a_{ijkl} p_j p_l U_k = U_i \ , \qquad i = 1, 2, 3 \ . \tag{6}$$

Thus, Equation (5) represents a nontrivial plane wave solution of elastodynamic equation (4) if and only if complex-valued constants p_n and $U_n (n = 1, 2, 3)$ satisfy constraint relations (6).

Constraint relations (6) may be interpreted as a system of three complex-valued homogeneous linear algebraic equations for U_1, U_2, U_3. The condition of solvability of this system reads

$$\det[a_{ijkl} p_j p_l - \delta_{ik}] = 0 \ . \tag{7}$$

This is a complex-valued equation, which contains only constants p_1, p_2, p_3, not U_1, U_2, U_3. It is very close to the so-called dispersion equation, in which the wave vector $\mathbf{k} = \omega\mathbf{p}$ is used instead of the vector $\mathbf{p}$.

The complex-valued vector $\mathbf{p}$, with Cartesian components p_i satisfying (7), represents the slowness vector. It is common to express slowness vector $\mathbf{p}$ in the following form

$$\mathbf{p} = \mathbf{P} + \mathrm{i}\mathbf{A} \ . \tag{8}$$

Real-valued vector $\mathbf{P}$ is usually called the *propagation vector*, and real-valued vector $\mathbf{A}$ the *attenuation vector*. Propagation vector $\mathbf{P}$ is perpendicular to the plane of constant phase (wavefront), and is oriented in the direction of the propagation of the wavefront. The attenuation vector $\mathbf{A}$ is perpendicular to the plane of constant amplitudes, and is oriented in the direction of the maximum exponential decay of amplitudes. We introduce the real-valued unit vectors $\mathbf{N}$ and $\mathbf{M}$ in the direction of $\mathbf{P}$ and $\mathbf{A}$, respectively, and *attenuation angle* γ, $0^0 \leq \gamma < 180^0$,

$$\mathbf{N} = \mathbf{P}/|\mathbf{P}| \ , \quad \mathbf{M} = \mathbf{A}/|\mathbf{A}| \ , \quad \cos\gamma = \mathbf{N} \cdot \mathbf{M} \ . \tag{9}$$

Unit vector $\mathbf{N}$ specifies the direction of propagation of the plane of constant phase of the plane wave under consideration (briefly, the direction of propagation), and unit vector $\mathbf{M}$ the direction of the maximum exponential decay of amplitudes of the plane wave. As $0^0 \leq \gamma < 180^0$, Equation (9) can be used to determine attenuation angle γ uniquely. The plane wave is called homogeneous for $\gamma = 0$, i. e. for $\mathbf{N} \equiv \mathbf{M}$, and inhomogeneous for $\gamma \neq 0$, i. e. for $\mathbf{N} \neq \mathbf{M}$. We further introduce the *propagation-attenuation plane* $\Sigma^{\parallel}$ specified by unit vectors $\mathbf{N}$ and $\mathbf{M}$. We denote $\mathbf{d}$ the vectorial component of attenuation vector $\mathbf{A}$ into the wavefront and call it the *wavefront attenuation vector*. The wavefront attenuation vector $\mathbf{d}$ is situated in the wavefront, along its intersection with propagation-attenuation plane $\Sigma^{\parallel}$. It is perpendicular to $\mathbf{N}$, so that $\mathbf{N} \cdot \mathbf{d} = 0$. It can be determined from attenuation vector $\mathbf{A}$ using the relation $\mathbf{d} = \mathbf{N} \times (\mathbf{A} \times \mathbf{N})$.

Finally, we introduce two mutually perpendicular unit vectors $\mathbf{e}^{\parallel}$ and $\mathbf{e}^{\perp}$, perpendicular to $\mathbf{N}$, by the relations,

$$\mathbf{d} = d\,\mathbf{e}^{\parallel} \;, \qquad \mathbf{e}^{\perp} = \mathbf{N} \times \mathbf{e}^{\parallel} \;. \tag{10}$$

Here $\mathbf{e}^{\parallel}$ is a real-valued unit vector in the direction of the wavefront attenuation vector $\mathbf{d}$, and $\mathbf{e}^{\perp}$ is a real-valued unit vector perpendicular to the propagation-attenuation plane $\Sigma^{\parallel}$. The real-valued unit vector $\mathbf{e}^{\parallel}$ is situated in propagation-attenuation plane $\Sigma^{\parallel}$, but the real-valued unit vector $\mathbf{e}^{\perp}$ is perpendicular to $\Sigma^{\parallel}$. The real-valued unit vectors $\mathbf{e}^{\parallel}, \mathbf{e}^{\perp}, \mathbf{N}$ form a right-handed orthogonal triplet of unit vectors. The real-valued scalar d, $0 \leq d < \infty$, represents a measure of the "strength" of the inhomogeneity of the plane wave under consideration, and will be referred to here as the *inhomogeneity strength*. For $d = 0$, the plane wave is homogeneous.

For various approaches to studying inhomogeneous plane waves propagating in elastic and viscoelastic, isotropic and anisotropic media, see *Buchen (1971)*, *Auld (1973)*, *Borcherdt (1973)*, *Aki and Richards (1980)*, *Hayes (1984)*, *Hosten, Deschamps and Tittmann (1987)*, *Caviglia and Morro (1992, 1999)*, *Romeo (1994)*, *Carcione and Cavallini (1995)*, *Krebes and Le (1994)*, *Brokešová and Červený (1998)*, *Carcione (2001)*, *Shuvalov (2001)*, and many other references given in these books and papers. Often, the wave vector $\mathbf{k} = \omega\mathbf{p}$ has been used instead of the slowness vector $\mathbf{p}$ in Equation (8). For example, see *Aki and Richards (1980)*.

3. THREE SPECIFICATIONS OF THE SLOWNESS VECTOR OF INHOMOGENEOUS PLANE WAVES

We shall now present three suitable specifications of slowness vector (8). The first specification contains two free real-valued quantities, and will be referred to here as the directional specification of the slowness vector. The second specification contains one free complex-valued quantity, and will be referred to here as the componental specification of the slowness vector. Finally, the third specification also contains one free complex-valued quantity, and will be referred to here as the mixed specification of the slowness vector. Inserting these expressions for slowness vector $\mathbf{p}$ into the characteristic equation (7), we can determine the free quantities, and, consequently, also the slowness vector.

In the **directional specification** of the slowness vector, we express slowness vector $\mathbf{p}$ in terms of two known real-valued unit vectors $\mathbf{N}$ and $\mathbf{M}$ as follows:

$$\mathbf{p} = \mathcal{C}^{-1}(\mathbf{N} + \mathrm{i}\delta\mathbf{M}) \;. \tag{11}$$

We remind the reader that unit vector $\mathbf{N}$ is oriented in the direction of propagation, and $\mathbf{M}$ in the direction of the maximum exponential decay of amplitudes. $\mathcal{C}$ and δ are real-valued quantities, not known at this stage, which must be determined by inserting (11) into (7). $\mathcal{C}$ represents the velocity of the wavefront (plane of constant phase), and is called the *phase velocity*. Quantity δ represents the ratio of the lengths of the attenuation and propagation vectors, $\delta = |\mathbf{A}|/|\mathbf{P}|$. We refer to δ as the *attenuation amplitude ratio*. Phase velocity $\mathcal{C}$ and attenuation amplitude ratio δ should always be real-valued and nonnegative. If, for given $\mathbf{N}$ and $\mathbf{M}$, Equation (7) with (11) is not satisfied for any $\mathcal{C} \geq 0$ and $\delta \geq 0$, we speak about forbidden combinations of $\mathbf{N}$ and $\mathbf{M}$ or forbidden directions $\mathbf{N}$, $\mathbf{M}$. See *Carcione and Cavallini (1995)*.

In the **componental specification** of the slowness vector, we consider an arbitrarily oriented real-valued unit vector $\mathbf{n}$, and a plane Σ perpendicular to it. We denote the vectorial complex-valued component of slowness vector $\mathbf{p}$ into Σ by $\mathbf{p}^{\Sigma}$, and assume that $\mathbf{p}^{\Sigma}$ is known. For known real-valued unit vector $\mathbf{n}$, perpendicular to Σ, and complex-valued vector $\mathbf{p}^{\Sigma}$, situated in plane Σ, we can reconstruct slowness vector $\mathbf{p}$ from $\mathbf{p}^{\Sigma}$ as follows:

$$\mathbf{p} = \sigma\mathbf{n} + \mathbf{p}^{\Sigma} \ , \qquad \text{with} \quad \mathbf{p}^{\Sigma} \cdot \mathbf{n} = 0 \ . \tag{12}$$

Here σ is a complex quantity, not known at this stage, which must be determined by inserting (12) into (7). Note that $\mathbf{p}^{\Sigma}$ may be an arbitrary complex-valued vector, satisfying the orthogonality condition $\mathbf{p}^{\Sigma} \cdot \mathbf{n} = 0$. Unit vectors $\mathbf{N}$ and $\mathbf{M}$ are not known in advance in this specification; they can be calculated from (12) only after σ has been determined. Plane Σ is fully arbitrary, and does not have anything in common with the plane of constant phases or the plane of constant amplitudes. Only if $\mathrm{Re}(\mathbf{p}^{\Sigma}) = \mathbf{0}$, does plane Σ represent the plane of constant phases, and $\mathbf{N} = \pm\mathbf{n}$. If also $\mathrm{Im}(\mathbf{p}^{\Sigma}) = \mathbf{0}$, the plane wave is homogeneous.

The **mixed specification** of the slowness vector is a special case of componental specification (12), with $\mathbf{p}^{\Sigma}$ purely imaginary. It is particularly suitable for inhomogeneous plane waves propagating in the direction of an á priori known real-valued unit vector $\mathbf{N}$. We express slowness vector $\mathbf{p}$ in terms of two known real-valued mutually perpendicular vectors $\mathbf{n}$ and $\mathbf{d}$ as follows:

$$\mathbf{p} = \sigma\mathbf{n} + \mathrm{i}\mathbf{d} \ , \qquad \text{with} \quad \mathbf{d} \cdot \mathbf{n} = 0 \ . \tag{13}$$

Here $\mathbf{n}$ is an arbitrarily chosen real-valued unit vector, and $\mathbf{d}$ is an arbitrary real-valued vector perpendicular to $\mathbf{n}$, so that $\mathbf{n} \cdot \mathbf{d} = 0$. σ is a complex-valued quantity, which must be determined by inserting (13) into (7). It follows from (13) that the plane Σ always represents the plane of constant phase (wavefront). Consequently, $\mathbf{N} = \pm\mathbf{n}$, and (13) includes qP, qS1 and qS2 inhomogeneous plane waves propagating to both directions away from Σ (the directions of $\mathbf{N} = +\mathbf{n}$ and $\mathbf{N} = -\mathbf{n}$). This actually indicates that we can choose an arbitrarily oriented $\mathbf{N}$ by a proper specification of $\mathbf{n}$. As $\mathbf{N} = \pm\mathbf{n}$, real-valued vector $\mathbf{d}$ represents the wavefront attenuation vector, $\mathbf{d} = d\mathbf{e}^{\parallel}$. Unit vectors $\mathbf{n}$ and $\mathbf{e}^{\parallel}$ specify propagation-attenuation plane $\Sigma^{\parallel}$, and d the inhomogeneity strength. The wavefront attenuation vector $\mathbf{d}$ is simply related to attenuation vector $\mathbf{A}$, $\mathbf{d} = \mathbf{n} \times (\mathbf{A} \times \mathbf{n})$.

Let us emphasize that the assumed perpendicularity of the real-valued vectors $\mathbf{n}$ and $\mathbf{d}$ in (13) does not restrict the generality and universality of (13). There is no constraint in it. The specification (13) can be used to specify any plane wave propagating in any medium under consideration (isotropic or anisotropic, perfectly elastic or viscoelastic media, homogeneous, weakly inhomogeneous or strongly inhomogeneous plane waves). The inhomogeneity strength $d = |\mathbf{d}|$ represents a parameter of the inhomogeneous wave under consideration, and may be chosen quite freely. For $d = 0$, the plane wave is homogeneous, for d small it is weakly inhomogeneous, and for d large it is strongly inhomogeneous.

The mixed specification (13) of the slowness vector combines both previous specifications (11) and (12). Similarly as in the directional specification, it considers a known unit vector $\mathbf{N} = \mathbf{n}$, perpendicular to the plane of constant phase, and oriented in the direction of propagation. In addition to the directional specification, however, it also includes the plane waves propagating in the opposite direction, $\mathbf{N} = -\mathbf{n}$. Instead of unit vector $\mathbf{M}$ in the direction of attenuation vector $\mathbf{A}$, it considers the wavefront attenuation vector $\mathbf{d}$, representing the vectorial component of $\mathbf{A}$ into the wavefront. This is similar to the componental specification of slowness vector (12).

Let us briefly compare the three specifications (11)–(13) of the slowness vector. The most important advantage of the componental and mixed specifications (12) and (13) is that slowness vector $\mathbf{p}$ is expressed in terms of one free complex-valued quantity σ, which can be found using conventional numerical methods (solution of a algebraic equation of the sixth degree, or solution of an eigenvalue problem for a 6×6 matrix). Contrary to this, the directional specification (11) would require the solution of a complicated coupled system of two equations.

The next advantage of specifications (12) and (13) over specification (11) is as follows: Specifications (12) and (13) consider the slowness vectors not only for direction $\mathbf{n}$, but also for direction $-\mathbf{n}$. For fixed $\mathbf{d}$, there is no symmetry between the slowness vector computed for $\mathbf{n}$ and $-\mathbf{n}$ in viscoelastic anisotropic media. Among others, the phase velocities might be different in both cases.

The final advantage of specifications (12) and (13) is that they yield physically acceptable solutions for any choice of parameters. Contrary to it, the directional specification (11) exhibits forbidden regions of $\mathbf{N}$ and $\mathbf{M}$, corresponding to negative squared phase velocities ($\mathcal{C}^2 < 0$), see *Krebes and Le (1994)*, *Carcione and Cavallini (1995)*. In the componental and mixed specifications (12) and (13), such forbidden regions are not encountered; nonnegative real-valued phase velocities ($\mathcal{C} \geq 0$) are obtained for arbitrarily chosen $\mathbf{p}^{\Sigma}$ and $\mathbf{d}$. The forbidden combinations $\mathbf{N}$ and $\mathbf{M}$ still exist, but they do not arise if the computations of the complex-valued slowness vector $\mathbf{p}$ are done using the mixed specification of the slowness vector $\mathbf{p}$, with the same $\mathbf{N}$ and arbitrary $\mathbf{d}$ (perpendicular to $\mathbf{N}$). A similar fact was pointed out already by *Krebes and Le (1994)*, pages 23907 and 23915–23916. For allowed $\mathbf{N}$ and $\mathbf{M}$, both the directional and mixed specifications should give the same results.

4. DIRECTIONAL SPECIFICATION OF THE SLOWNESS VECTOR

In the whole of Section 4, we shall consider the directional specification (11) of the slowness vector, with real-valued unit vectors $\mathbf{N}$ and $\mathbf{M}$ à priori known. The unknown phase velocity $\mathcal{C}$ and amplitude attenuation ratio δ should be determined from relation (7). The approach yields simple algorithms for inhomogeneous plane waves propagating in viscoelastic isotropic media, and for homogeneous plane waves propagating in viscoelastic anisotropic media, see *Červený (2003a,b)*. For this reason, it has been very popular in the wave-propagation literature. For inhomogeneous plane waves propagating in viscoelastic anisotropic media, however, it yields considerably more complicated algorithms.

We shall now discuss the algorithms to determine phase velocity $\mathcal{C}$ and the attenuation amplitude ratio δ of inhomogeneous plane waves in greater detail. First, we shall use the standard equations (7) and (11), and derive two nonlinear coupled equations for $\mathcal{C}^2$ and δ, see Section 4.1. The numerical solution of these coupled equations, however, may be rather complicated. For this reason, an alternative procedure is proposed in Section 4.2, based on the "semi-analytical" approach. The semi-analytical approach yields one decoupled equation for δ. The decoupling simplifies the treatment, even though the equation is still very complicated. Once δ is determined, $\mathcal{C}^2$ may be computed analytically using an explicit formula.

4.1. Coupled equations for $\mathcal{C}^2$ and δ

If we insert (11) into (7), and take into account that $a_{ijkl}p_jp_l$ are homogeneous functions of the second degree in p_n, we obtain two coupled real-valued equations for two real-valued unknowns $\mathcal{C}^2 = \mathcal{C}^2(N_n, M_n)$ and $\delta = \delta(N_n, M_n)$:

$$\begin{aligned} &\mathrm{Re}\,\det[\Gamma_{ik}(N_n) + \mathrm{i}\delta E_{ik}(N_n, M_n) - \delta^2\Gamma_{ik}(M_n) - \mathcal{C}^2\delta_{ik}] = 0\ , \\ &\mathrm{Im}\,\det[\Gamma_{ik}(N_n) + \mathrm{i}\delta E_{ik}(N_n, M_n) - \delta^2\Gamma_{ik}(M_n) - \mathcal{C}^2\delta_{ik}] = 0\ . \end{aligned} \tag{14}$$

Here $\Gamma_{ik}(N_n)$ and $\Gamma_{ik}(M_n)$ are complex-valued Christoffel matrices,

$$\Gamma_{ik}(N_n) = a_{ijkl}N_jN_l\ , \quad \Gamma_{ik}(M_n) = a_{ijkl}M_jM_l\ , \tag{15}$$

corresponding to real-valued unit vectors $\mathbf{N}$ and $\mathbf{M}$. The 3×3 complex-valued matrix $E_{ik}(N_n, M_n)$ is given by the relation

$$E_{ik}(N_n, M_n) = a_{ijkl}(N_jM_l + N_lM_j)\ . \tag{16}$$

The L.H.S.'s of (14) are coupled polynomials in $\mathcal{C}^2$ and δ. These polynomials are of the third degree in $\mathcal{C}^2$, and of the sixth degree in δ. The solution of the coupled system (14) for $\mathcal{C}^2$ and δ is not simple. See the detailed treatment in *Romeo (1994)*.

Only for homogeneous plane waves, can Equations (14) be solved simply. Then $\mathbf{N} \equiv \mathbf{M}$, we obtain $\Gamma_{ik}(N_n) = \Gamma_{ik}(M_n) = \frac{1}{2}E_{ik}(N_n, M_n)$, and Equations (14) simplify:

$$\det[\Gamma_{ik}(N_n) - \mathcal{C}^2(1 + \mathrm{i}\delta)^{-2}\delta_{ik}] = 0\ .$$

This equation shows that $\mathcal{C}^2(1+\mathrm{i}\delta)^{-2}$ represents an eigenvalue $G^{(m)}(N_n)$, $m = 1,2,3$, of the complex-valued Christoffel matrix $\Gamma_{ik}(N_n)$, see (15). From the three eigenvalues $G^{(m)}(N_n)$, $m = 1,2,3$, the parameters $\mathcal{C}^{(m)}(N_n)$ and $\delta^{(m)}(N_n)$ of the three homogeneous plane waves (qP, qS1 and qS2), propagating in the direction of $\mathbf{N}$ in viscoelastic anisotropic media, can be obtained. The determination of $\mathcal{C}^{(m)}$ and $\delta^{(m)}$ from a known eigenvalue $G^{(m)}(N_n)$ is simple. For details, see *Červený (2003a)*.

4.2. Semianalytical approach

We introduce the 3×3 complex-valued matrix $\Gamma_{ik}(p_n)$ by the relation

$$\Gamma_{ik}(p_n) = a_{ijkl}p_j p_l \ . \tag{17}$$

Here p_n are Cartesian components of the complex-valued slowness vector $\mathbf{p}$. Formally, matrix $\Gamma_{ik}(p_n)$ is analogous to Christoffel matrix $\Gamma_{ik}(N_n)$, given by (15). There are, however, two basic differences between $\Gamma_{ik}(N_n)$, given by (15), and $\Gamma_{ik}(p_n)$, given by (17). First, N_n in (15) are real-valued, but p_n in (17) complex-valued. Second, matrix $\Gamma_{ik}(N_n)$ is a known matrix, as the real-valued unit vector $\mathbf{N}$ is assumed to be known. On the contrary, the components of slowness vector $\mathbf{p}$ are not known at this stage, so that $\Gamma_{ik}(p_n)$ cannot be computed. Nevertheless, it will be useful to use analytical expressions for $\Gamma_{ik}(p_n)$, given by (17), in our derivations. This is the reason why we speak of the semi-analytical approach.

To distinguish $\Gamma_{ik}(p_n)$ from $\Gamma_{ik}(N_n)$, we shall call $\Gamma_{ik}(p_n)$ the Christoffel matrix in p_n, when necessary. Note that the Christoffel matrix in p_n has been broadly used in the seismic ray method, where it is mostly called simply the Christoffel matrix. See *Červený (2001,* p. 22*)*.

We can also introduce the complex-valued eigenvalues $G^{(m)}(p_n)$ and eigenvectors $\mathbf{U}^{(m)}(p_n)$ of the matrix $\Gamma_{ik}(p_n)$,

$$[\Gamma_{ik}(p_n) - G^{(m)}(p_n)\delta_{ik}]U_k^{(m)}(p_n) = 0 \ , \quad i = 1,2,3 \ . \tag{18}$$

This is a system of three linear homogeneous algebraic equations for $U_1^{(m)}(p_n)$, $U_2^{(m)}(p_n)$ and $U_3^{(m)}(p_n)$. The condition of solvability of (18) is represented by the characteristic equation

$$\det[\Gamma_{ik}(p_n) - G^{(m)}(p_n)\delta_{ik}] = 0 \ . \tag{19}$$

Comparing (18) with the constraint relation (6), we can see that the constraint relation (6) is satisfied if any of the three eigenvalues $G^{(m)}(p_n)$, $m = 1,2,3$, equals unity,

$$G^{(m)}(p_n) = 1 \ . \tag{20}$$

Although the 3×3 matrix $\Gamma_{ik}(p_n)$ and its eigenvalues $G^{(m)}(p_n)$ cannot be numerically computed at this stage, as p_n are not yet known, we can express them analytically. ($G^{(m)}(p_n)$ are solutions of the cubic equation (19).) We can also suitably exploit certain well-known properties of $\Gamma_{ik}(p_n)$. One immediately sees from (17) that $\Gamma_{ik}(p_n)$ are homogeneous functions of the second degree in p_n. Consequently, also $G^{(m)}(p_n)$ are homogeneous functions of the second degree in p_n. Using

Euler's theorem for a homogeneous function, and the directional specification (11) of the slowness vector $\mathbf{p}$, we obtain from (20):

$$G^{(m)}(N_n + \mathrm{i}\delta M_n) = \mathcal{C}^2 \; . \tag{21}$$

This is a complex-valued equation. We can express it in terms of two real-valued equations:

$$\mathrm{Re}[G^{(m)}(N_n + \mathrm{i}\delta M_n)] = \mathcal{C}^2 \; , \tag{22}$$

$$\mathrm{Im}[G^{(m)}(N_n + \mathrm{i}\delta M_n)] = 0 \; . \tag{23}$$

For given unit vectors $\mathbf{N}$ and $\mathbf{M}$, Equations (22) and (23) represent the final two equations for δ and $\mathcal{C}^2$. Equation (23) does not depend on $\mathcal{C}^2$ and can be directly used to determine δ. Once δ is determined from (23), we can insert it into (22) and compute $\mathcal{C}^2$.

Thus, the procedure of determining δ and $\mathcal{C}^2$ for given $\mathbf{N}$ and $\mathbf{M}$, in the case of general viscoelasticity and general anisotropy, is as follows:

First, we express the eigenvalues $G^{(m)}(p_n)$ of the matrix $\Gamma_{ik}(p_n)$ analytically. The relevant analytical expressions for $G^{(m)}(p_n)$ may be cumbersome, but can be always found. Actually, they represent roots of a cubic algebraic equation. See *Červený (2001,* Eq. (2.2.28)*)*. Second, we substitute p_n by $N_n + \mathrm{i}\delta M_n$ in these analytical expressions for $G^{(m)}(p_n)$, and obtain analytical expressions for $G^{(m)}(N_n + \mathrm{i}\delta M_n)$. Third, we take the imaginary part of $G^{(m)}(N_n + \mathrm{i}\delta M_n)$ and obtain equation (23) for δ analytically. Fourth, we solve Equation (23) for δ numerically. Fifth, we insert the found numerical value of δ into (22) and obtain the relevant $\mathcal{C}^2$.

The solution is simplified if the cubic characteristic equation for $G^{(m)}(p_n)$ is factorized into one linear and one quadratic equations. This applies, for example, to the symmetry plane of a monoclinic (or orthorhombic) viscoelastic medium. See *Krebes and Le (1994)*, *Carcione and Cavallini (1995)*, and *Červený (2003b)*.

The solutions δ and $\mathcal{C}^2$ of (22) and (23) should be real-valued and nonnegative. Unfortunately, for certain combinations of unit vectors $\mathbf{N}$ and $\mathbf{M}$, such δ and $\mathcal{C}$ do not exist. Consequently, the inhomogeneous plane waves do not exist for such $\mathbf{N}$ and $\mathbf{M}$. In this case we speak of forbidden directions $\mathbf{N}$ and $\mathbf{M}$. See *Krebes and Le (1994)*, *Carcione and Cavallini (1995)*, and *Carcione (2001,* p. 146–147*)*.

Once $\delta^{(m)} \geq 0$ and $\mathcal{C}^{(m)} \geq 0$ have been determined from (23) and (22) for particular $\mathbf{N}, \mathbf{M}$ and m, the relevant complex-valued slowness vector $\mathbf{p}^{(m)}$ can be determined,

$$p_n^{(m)} = \frac{1}{\mathcal{C}^{(m)}}(N_n + \mathrm{i}\delta^{(m)} M_n) \; , \tag{24}$$

see (11).

5. COMPONENTAL SPECIFICATION OF THE SLOWNESS VECTOR

There are two main disadvantages of the directional specification of the slowness vector, in which the real-valued unit vectors $\mathbf{N}$ and $\mathbf{M}$ are à priori known. The first disadvantage is that the algorithm to determine $\mathcal{C}^2$ and δ is rather complicated. The second disadvantage is that the algorithm may yield nonphysical solutions for $\mathcal{C}^2$ and δ, for example $\mathcal{C}^2 < 0$. Such solutions, of course, must be excluded. In this section, we shall discuss the componental specification (12) of the slowness vector, which overcomes both these difficulties. We shall assume that the complex-valued vector $\mathbf{p}^\Sigma$ and the real-valued unit vector $\mathbf{n}$ are known, and try to determine the complex-valued quantity σ.

Inserting (12) into constraint relations (6) yields

$$[a_{ijkl}(p_j^\Sigma + \sigma n_j)(p_l^\Sigma + \sigma n_l) - \delta_{ik}]U_k = 0 \ , \quad i = 1, 2, 3 \ . \tag{25}$$

This represents a system of three complex-valued linear homogeneous equations for U_1, U_2, U_3. The condition of solvability of (25) reads

$$\det[a_{ijkl}(p_j^\Sigma + \sigma n_j)(p_l^\Sigma + \sigma n_l) - \delta_{ik}] = 0 \ . \tag{26}$$

Equation (26) is a algebraic equation of the sixth degree in σ. It can be solved for σ by various methods. Six solutions are obtained, corresponding to qS1, qS2 and qP plane waves, propagating away from plane Σ, to both sides of it.

For each σ, satisfying (26), we can determine the relevant slowness vector $\mathbf{p}$ from (12). The procedure can be applied both to homogeneous and inhomogeneous plane waves, to isotropic and anisotropic media, to viscoelastic and perfectly elastic media.

5.1. 6×6 matrix $\mathbf{A}$, its eigenvalues and eigenvectors

Equations (25) and (26) can be solved directly by conventional numerical methods. They can be, however, also expressed and solved in several alternative forms. We shall describe one such possibility, and transform (25) into an eigenvalue problem for a 6×6 complex-valued matrix $\mathbf{A}$. For elastic media (with real-valued elastic moduli c_{ijkl}), the derivation is given in *Červený (2001,* Section 5.4.7*)*. The derivation remains valid even for viscoelastic anisotropic media (with complex-valued c_{ijkl}). For this reason, we shall not repeat the derivation here, but merely present the final results. For other references, see Introduction. The relevant 6×6 eigenvalue problem is represented by a system of six equations:

$$A_{\alpha\beta} W_\beta = W_\alpha \sigma \ , \quad \alpha = 1, 2, ..., 6 \ . \tag{27}$$

The summation runs over $\beta = 1, 2, ...6$. Here $\mathbf{A}$ is a 6×6 complex-valued non-symmetric non-Hermitian matrix, given by relation

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{pmatrix} \ . \tag{28}$$

The 3×3 complex-valued partition matrices $\mathbf{A}_{IJ}$ are given by relations

$$\begin{aligned} &\mathbf{A}_{11} = -\mathbf{C}^{(1)-1}\mathbf{C}^{(2)} = \mathbf{A}_{22}^T \ , && \mathbf{A}_{12} = -\mathbf{C}^{(1)-1} \ , \\ &\mathbf{A}_{21} = -\rho\mathbf{I} + \mathbf{C}^{(4)} - \mathbf{C}^{(3)}\mathbf{C}^{(1)-1}\mathbf{C}^{(2)} \ , && \mathbf{A}_{22} = -\mathbf{C}^{(3)}\mathbf{C}^{(1)-1} \ . \end{aligned} \tag{29}$$

As usual, $\mathbf{I}$ is a 3×3 identity matrix. The 3×3 complex-valued matrices $\mathbf{C}^{(1)}$, $\mathbf{C}^{(2)}$, $\mathbf{C}^{(3)}$ and $\mathbf{C}^{(4)}$ are given by relations

$$\begin{aligned} &C_{ik}^{(1)} = c_{ijkl}n_j n_l \ , && C_{ik}^{(2)} = c_{ijkl}n_j p_l^\Sigma \ , \\ &C_{ik}^{(3)} = c_{ijkl}p_j^\Sigma n_l = C_{ki}^{(2)} \ , && C_{ik}^{(4)} = c_{ijkl}p_j^\Sigma p_l^\Sigma \ . \end{aligned} \tag{30}$$

Note that the three components p_i^Σ, $i = 1, 2, 3$, cannot be chosen arbitrarily; they must satisfy the orthogonality relation $p_i^\Sigma n_i = 0$, see (12). Orthogonality relation $\mathbf{p}^\Sigma \cdot \mathbf{n} = 0$ can be used to express one of the three components $p_1^\Sigma, p_2^\Sigma, p_3^\Sigma$ in terms of the remaining two; for example p_3^Σ in terms of p_1^Σ and p_2^Σ.

The eigenvalues σ of (28) are roots of the characteristic equation

$$\det[\mathbf{A} - \sigma\mathbf{I}] = 0 \ . \tag{31}$$

Here $\mathbf{I}$ is the 6×6 identity matrix. As $\mathbf{A}$ is a 6×6 complex-valued matrix, the characteristic equation (31) represents an algebraic equation of the sixth degree, and always has six roots. Some of these roots may, of course, coincide (degenerate cases). The six eigenvalues σ correspond to the qS1, qS2 and qP waves, approaching and leaving the plane Σ.

For a known σ, we can also determine the real-valued propagation vector $\mathbf{P}$ and attenuation vector $\mathbf{A}$, unit vectors $\mathbf{N}$ and $\mathbf{M}$, attenuation angle γ, phase velocity $\mathcal{C}$ and the attenuation amplitude ratio δ, see (9), (11) and (12):

$$\begin{aligned} &\mathbf{P} = \mathbf{n}\,\mathrm{Re}\sigma + \mathrm{Re}\mathbf{p}^\Sigma \ , && \mathbf{A} = \mathbf{n}\,\mathrm{Im}\sigma + \mathrm{Im}\mathbf{p}^\Sigma \ , \\ &|\mathbf{P}| = [(\mathrm{Re}\sigma)^2 + (\mathrm{Re}\mathbf{p}^\Sigma \cdot \mathrm{Re}\mathbf{p}^\Sigma)]^{1/2} \ , && |\mathbf{A}| = [(\mathrm{Im}\sigma)^2 + (\mathrm{Im}\mathbf{p}^\Sigma \cdot \mathrm{Im}\mathbf{p}^\Sigma)]^{1/2} \ , \\ &\mathbf{N} = \mathbf{P}/|\mathbf{P}| \ , && \mathbf{M} = \mathbf{A}/|\mathbf{A}| \ , \\ &\mathcal{C} = 1/|\mathbf{P}| \ , && \delta = |\mathbf{A}|/|\mathbf{P}| = \mathcal{C}|\mathbf{A}| \ , \\ &\cos\gamma = [(\mathrm{Re}\sigma)(\mathrm{Im}\sigma) + (\mathrm{Re}\mathbf{p}^\Sigma \cdot \mathrm{Im}\mathbf{p}^\Sigma)]/|\mathbf{P}||\mathbf{A}| \ . \end{aligned} \tag{32}$$

All these quantities are real-valued.

The problem of forbidden combinations of $\mathbf{N}$ and $\mathbf{M}$ does not appear in the computation of the slowness vector $\mathbf{p}$ using the eigenvalue problem for the 6×6 matrix $\mathbf{A}$. Actually, the six eigenvalues $\sigma^{(\mu)}$ are obtained for an arbitrary viscoelastic anisotropic medium, for an arbitrarily chosen unit normal $\mathbf{n}$ and for arbitrarily chosen complex-valued constants p_i^Σ, satisfying orthogonality relation $\mathbf{p}^\Sigma \cdot \mathbf{n} = 0$. The relevant unit vectors $\mathbf{N}$ and $\mathbf{M}$ are then always allowed and are given by (32). This does not mean that the forbidden combinations of $\mathbf{N}$ and $\mathbf{M}$ do not exist. We only say that they do not exist for arbitrarily chosen $\mathbf{n}$ and $\mathbf{p}^\Sigma$, satisfying orthogonality relation $\mathbf{p}^\Sigma \cdot \mathbf{n} = 0$. The values of N_i and M_i, given by (32), never represent the forbidden combinations.

5.2. Matrix **A** in a suitably chosen Cartesian coordinate system

All the equations of Section 5.1 are applicable to arbitrary viscoelastic moduli c_{ijkl}, specified in an arbitrary Cartesian coordinate system, to an arbitrary plane Σ (with unit normal $\mathbf{n}$), and to arbitrarily given complex-valued $\mathbf{p}^{\Sigma}$, satisfying orthogonality relation $\mathbf{p}^{\Sigma} \cdot \mathbf{n} = 0$. Without loss of generality, we can choose the Cartesian coordinate system in some special way. Of course, we have to use proper transformation equations for viscoelastic moduli c_{ijkl} when we pass from one Cartesian coordinate system to another. For example, we can choose the Cartesian coordinate system in such a way that plane Σ corresponds to a coordinate plane. Certain quantities involved in the equations then vanish, and the physical meaning of the eigenvalues σ becomes more transparent.

In the seismological literature, it is usual to choose the Cartesian coordinate system in such a way that the x_3-axis coincides with the normal $\mathbf{n}$ to plane Σ (*Frazer and Fryer, 1989*; *Chapman, 1994*; *Thomson, 1996a,b*). The unit normal $\mathbf{n}$ then has only one non-vanishing Cartesian component,

$$\mathbf{n} \equiv (0, 0, 1)^T \ . \tag{33}$$

Both the x_1- and x_2-axes are situated in plane Σ in this case, and the complex-valued Cartesian component p_3^{Σ} vanishes. Consequently,

$$p_1^{\Sigma} = p_1 = P_1 + \mathrm{i}A_1 \ , \quad p_2^{\Sigma} = p_2 = P_2 + \mathrm{i}A_2 \ , \quad p_3^{\Sigma} = 0 \ . \tag{34}$$

Here P_1, A_1, P_2 and A_2 are assumed to be known. Equations (30) for $\mathbf{C}^{(1)}, \mathbf{C}^{(2)}, \mathbf{C}^{(3)}$ and $\mathbf{C}^{(4)}$ now read

$$\begin{aligned} C_{ik}^{(1)} &= c_{i3k3} \ , & C_{ik}^{(2)} &= c_{i3kL}p_L \ , \\ C_{ik}^{(3)} &= c_{iJk3}p_J = C_{ki}^{(2)} \ , & C_{ik}^{(4)} &= c_{iJkL}p_J p_L \ . \end{aligned} \tag{35}$$

The summation runs over $J = 1, 2$ and $L = 1, 2$. Otherwise, Equations (27)–(30) are not changed. We can further see from (12) that eigenvalue σ represents actually the complex-valued x_3-component of the slowness vector $\mathbf{p}$,

$$\sigma = p_3 = P_3 + \mathrm{i}A_3 \ . \tag{36}$$

The four real-valued Cartesian components P_1, A_1, P_2 and A_2 may be arbitrary and are assumed to be à priori known. P_3 and A_3 can then be determined by the solution of the eigenvalue problem (31) for the 6×6 complex-valued matrix $\mathbf{A}$. Quantities $N_i, M_i, \cos\gamma, \mathcal{C}$ and δ can be determined from (32).

In addition to the x_3-axis perpendicular to Σ, we can also choose the x_1-axis in a suitable way. We can take it along the intersection of plane Σ with a plane normal to Σ, containing vector $\mathrm{Re}(\mathbf{p}^{\Sigma})$ or $\mathrm{Im}(\mathbf{p}^{\Sigma})$. In the first case, we obtain

$$P_2 = 0 \ , \tag{37}$$

and in the second case,

$$A_2 = 0 \ . \tag{38}$$

Vector $\mathbf{p}^{\Sigma}$ is then fully specified by the three real-valued quantities P_1, A_1, A_2 or P_1, P_2, A_1.

6. MIXED SPECIFICATION OF THE SLOWNESS VECTOR

The componental specification of the slowness vector, described in Section 5, yields a conventional eigenvalue algorithm to determine the slowness vector of plane inhomogeneous waves, if p_i^Σ $(i = 1, 2, 3)$ are given, satisfying orthogonality relation $p_i^\Sigma n_i = 0$. Indeed, it is sufficient to specify only p_I^Σ $(I = 1, 2)$, as p_3^Σ can be excluded using relation $p_i^\Sigma n_i = 0$. However, the direction $\mathbf{N}$ of propagation vector $\mathbf{P}$ is not known in advance; it is obtained as a result of the computations. However, our primary interest is to determine the slowness vector and other relevant quantities of the inhomogeneous plane wave for a given direction $\mathbf{N}$ of the propagation vector (perpendicular to the plane of constant phase). It is then suitable to use the mixed specification (13) of the slowness vector, $\mathbf{p} = \sigma\mathbf{n} + \mathrm{i}\mathbf{d}$, with the real-valued vector $\mathbf{d}$ perpendicular to $\mathbf{n}$ (so that $\mathbf{n} \cdot \mathbf{d} = 0$). The mixed specification (13) of the slowness vector may, in fact, be treated as a special case of componental specification (12), if we put

$$\mathbf{p}^\Sigma = \mathrm{i}\mathbf{d} , \qquad \text{with} \quad \mathbf{d} \cdot \mathbf{n} = 0 . \tag{39}$$

Consequently, (12) can then be expressed in the form of (13), $\mathbf{p} = \sigma\mathbf{n}+\mathrm{i}\mathbf{d}$. As the real part of $\mathbf{p}^\Sigma$ vanishes, plane Σ represents the plane of constant phase, and $\mathbf{N} = \pm\mathbf{n}$. Thus, $\mathbf{N}$ may be chosen in advance, similarly as in the directional specification. All equations of Section 5 remain valid also for $\mathbf{p}^\Sigma$ given by (39). The complex-valued quantity σ can then be determined as an eigenvalue of the 6×6 matrix $\mathbf{A}$, given by (28)–(31), where (39) is used. Once we have determined σ, we can also find other useful quantities, related to the inhomogeneous plane wave under consideration, see (32),

$$\begin{array}{ll}
\mathbf{P} = \mathbf{n}\,\mathrm{Re}\sigma \ , & \mathbf{A} = \mathbf{n}\,\mathrm{Im}\sigma + \mathbf{d} \ , \\
|\mathbf{P}| = |\mathrm{Re}\sigma| \ , & |\mathbf{A}| = [(\mathrm{Im}\sigma)^2 + \mathbf{d}\cdot\mathbf{d}]^{1/2} \ , \\
\mathbf{N} = \mathbf{P}/|\mathbf{P}| = \epsilon\,\mathbf{n} \ , & \mathbf{M} = \mathbf{A}/|\mathbf{A}| \ , \\
\mathcal{C} = 1/|\mathrm{Re}\sigma| \ , & \delta = |\mathbf{A}|/|\mathbf{P}| = \mathcal{C}|\mathbf{A}| \ , \\
\cos\gamma = \epsilon\,\mathrm{Im}(\sigma)/[(\mathrm{Im}\sigma)^2 + \mathbf{d}\cdot\mathbf{d}]^{1/2} \ . &
\end{array} \tag{40}$$

Here

$$\epsilon = \mathrm{Re}\sigma/|\mathrm{Re}\sigma| = \pm 1 \ . \tag{41}$$

We can also simply show that the projection of attenuation vector $\mathbf{A}$ into vector $\mathbf{d}$ is always positive, $\mathbf{A} \cdot \mathbf{d} = \mathbf{d} \cdot \mathbf{d} > 0$.

Without any loss of generality, we can choose the Cartesian coordinate system x_i in a suitable way. Similarly as in Section 5.2, we choose the x_3-axis to coincide with the unit normal $\mathbf{n}$ to plane Σ. Consequently, plane Σ coincides with coordinate plane x_1, x_2. Further, we choose the x_1-axis along the wavefront attenuation vector $\mathbf{d}$. The transformation matrix then reads

$$\begin{pmatrix} e_1^{\parallel} & e_2^{\parallel} & e_3^{\parallel} \\ e_1^{\perp} & e_2^{\perp} & e_3^{\perp} \\ n_1 & n_2 & n_3 \end{pmatrix} \ . \tag{42}$$

Here $e_i^{\parallel}$, $e_i^{\perp}$ and n_i are Cartesian components of unit vectors $\mathbf{e}^{\parallel}$, $\mathbf{e}^{\perp}$ and $\mathbf{n}$, see (10). Unit vectors $\mathbf{e}^{\parallel}$, $\mathbf{e}^{\perp}$ and $\mathbf{n}$ are orthonormal and form a right-handed system.

Applying transformation matrix (42) to $\mathbf{n}$ and $\mathbf{d}$, we obtain $\mathbf{n}$ and $\mathbf{d}$ in the new coordinate system

$$\mathbf{n} \equiv (0,0,1)^T , \qquad \mathbf{d} \equiv (d,0,0)^T , \tag{43}$$

and the relevant p_i^{Σ}:

$$p_1^{\Sigma} = \mathrm{i}d , \qquad p_2^{\Sigma} = p_3^{\Sigma} = 0 . \tag{44}$$

Of course, the viscoelastic moduli c_{ijkl} must also be transformed to the new coordinate system using (42). The determination of σ in the new coordinate system is simpler than in the general coordinate system. In the transformed Cartesian coordinate system, Equations (35) yield

$$\begin{aligned} C_{ik}^{(1)} &= \tilde{C}_{ik}^{(1)} , & C_{ik}^{(2)} &= \mathrm{i}d\tilde{C}_{ik}^{(2)} , \\ C_{ik}^{(3)} &= \mathrm{i}d\tilde{C}_{ik}^{(3)} , & C_{ik}^{(4)} &= -d^2\tilde{C}_{ik}^{(4)} , \end{aligned} \tag{45}$$

where

$$\begin{aligned} \tilde{C}_{ik}^{(1)} &= c_{i3k3} , & \tilde{C}_{ik}^{(2)} &= c_{i3k1} , \\ \tilde{C}_{ik}^{(3)} &= c_{i1k3} , & \tilde{C}_{ik}^{(4)} &= c_{i1k1} . \end{aligned} \tag{46}$$

Equations (29) then yield,

$$\begin{aligned} \mathbf{A}_{11} &= -\mathrm{i}d\mathbf{D}_{11} , & \mathbf{A}_{12} &= -\mathbf{D}_{12} , \\ \mathbf{A}_{21} &= -\rho\mathbf{I} - d^2\mathbf{D}_{21} , & \mathbf{A}_{22} &= -\mathrm{i}d\mathbf{D}_{22} . \end{aligned} \tag{47}$$

Here $\mathbf{D}_{IJ}$ are 3×3 matrices, given by relations

$$\begin{aligned} \mathbf{D}_{11} &= \tilde{\mathbf{C}}^{(1)-1}\tilde{\mathbf{C}}^{(2)} , & \mathbf{D}_{21} &= \tilde{\mathbf{C}}^{(1)-1} , \\ \mathbf{D}_{21} &= \tilde{\mathbf{C}}^{(4)} - \tilde{\mathbf{C}}^{(3)}\tilde{\mathbf{C}}^{(1)-1}\tilde{\mathbf{C}}^{(2)} , & \mathbf{D}_{22} &= \tilde{\mathbf{C}}^{(3)}\tilde{\mathbf{C}}^{(1)-1} . \end{aligned} \tag{48}$$

Note that the 3×3 matrices $\mathbf{D}_{IJ}$ do not depend on the inhomogeneity strength d. Eigenvalues σ are then solutions of the characteristic equation

$$\det\begin{pmatrix} \mathrm{i}d\mathbf{D}_{11} + \sigma\mathbf{I} & \mathbf{D}_{12} \\ \rho\mathbf{I} + d^2\mathbf{D}_{21} & \mathrm{i}d\mathbf{D}_{22} + \sigma\mathbf{I} \end{pmatrix} = 0 , \tag{49}$$

where $\mathbf{I}$ is a 3×3 identity matrix.

The eigenvalues σ in the transformed coordinate system are the same as in the original coordinate system, assuming that the viscoelastic moduli are transformed appropriately. Consequently, we can again use (40) to determine other quantities related to the inhomogeneous plane wave under consideration.

The characteristic equation (49) can be modified in several ways. For example, if $d \neq 0$, we can introduce new eigenvalues w related to σ as follows:

$$\sigma = dw . \tag{50}$$

It is not difficult to see from (49) that the characteristic equation for w reads

$$\det\begin{pmatrix} \mathrm{i}\mathbf{D}_{11} + w\mathbf{I} & \mathbf{D}_{12} \\ \mathbf{D}_{21} + \rho\mathbf{I}/d^2 & \mathrm{i}\mathbf{D}_{22} + w\mathbf{I} \end{pmatrix} = 0 . \tag{51}$$

As $\mathbf{D}_{IJ}$ are independent of the inhomogeneity strength d, characteristic equation (51) depends on d in a particularly simple way. Solving (51) for w, we obtain six roots, corresponding to qS1, qS2 and qP inhomogeneous plane waves, propagating in directions $\mathbf{N}=\mathbf{n}$ and $\mathbf{N}=-\mathbf{n}$. For each of these waves, we can use (40) to obtain simple expressions for the quantities related to these inhomogeneous plane waves:

$$\begin{aligned} &\mathbf{P}=d\,\mathbf{n}\,\mathrm{Re}w\ , && \mathbf{A}=(\mathbf{n}\,\mathrm{Im}w+\mathbf{e}^{\|})d\ ,\\ &|\mathbf{P}|=d|\mathrm{Re}w|\ , && |\mathbf{A}|=d[(\mathrm{Im}w)^2+1]^{1/2}\ ,\\ &\mathbf{N}=\mathbf{P}/|\mathbf{P}|=\epsilon\,\mathbf{n}\ , && \mathbf{M}=\mathbf{A}/|\mathbf{A}|\ ,\\ &\mathcal{C}=1/d|\mathrm{Re}w|\ , && \delta=[(\mathrm{Im}w)^2+1]^{1/2}/|\mathrm{Re}w|\ ,\\ &\cos\gamma=\epsilon\,\mathrm{Im}w/[(\mathrm{Im}w)^2+1]^{1/2}\ . && \end{aligned} \tag{52}$$

Here $\epsilon=\mathrm{Re}w/|\mathrm{Re}w|=\pm1$.

Let us now return to Equations (40) and (52) for $\cos\gamma$. As we can see, the mixed specification of the slowness vector offers explicit expressions for $\cos\gamma$, assuming that eigenvalues σ (or w) are known. This is a great advantage with respect to the directional specification, in which $\cos\gamma$ is assumed to be known à priori (as $\mathbf{N}$ and $\mathbf{M}$ are assumed to be known, and $\cos\gamma=\mathbf{N}\cdot\mathbf{M}$). Equations (40) and (52) give a very simple relation between cotanγ and eigenvalues σ or w:

$$\mathrm{cotan}\gamma=\epsilon\,\mathrm{Im}w=\epsilon\,d^{-1}\mathrm{Im}\sigma\ . \tag{53}$$

Here $\epsilon=\mathrm{Re}\sigma/|\mathrm{Re}\sigma|=\mathrm{Re}w/|\mathrm{Re}w|=\pm1$. We can also say that $\epsilon=1$ for $\mathbf{N}=\mathbf{n}$, and $\epsilon=-1$ for $\mathbf{N}=-\mathbf{n}$. Equation (53) is very interesting. It offers a very simple explanation of the physical meaning of the imaginary part of eigenvalue w: Imw equals cotanγ for $\mathbf{N}=\mathbf{n}$, and $-$cotanγ for $\mathbf{N}=-\mathbf{n}$.

The phase velocity $\mathcal{C}$ is always nonnegative, and is given by the relation

$$\mathcal{C}=1/|\mathrm{Re}\sigma|=1/d|\mathrm{Re}w|\ , \tag{54}$$

see (40) and (52). In the limiting case of $d\to\infty$, the zero phase velocity $\mathcal{C}=0$ is obtained. Thus, the eigenvalue w corresponding to the zero phase velocity is obtained from characteristic equation (51), if we insert $1/d^2=0$ into it:

$$\det\begin{pmatrix}\mathrm{i}\mathbf{D}_{11}+w\mathbf{I} & \mathbf{D}_{12}\\ \mathbf{D}_{21} & \mathrm{i}\mathbf{D}_{22}+w\mathbf{I}\end{pmatrix}=0\ . \tag{55}$$

From the eigenvalues w of (55), we can compute the quantities δ and γ corresponding to zero phase velocities, using (52). It is well known that $\gamma=\frac{1}{2}\pi$ corresponds to the zero phase velocity in an isotropic viscoelastic medium. In an anisotropic viscoelastic medium, however, the result is different: the value of γ corresponding to the zero phase velocity is, in general, different from $\frac{1}{2}\pi$.

In the directional specification of the slowness vector, the phase velocities $\mathcal{C}$ are zero along the boundaries of forbidden and allowed regions of $\mathbf{N}$ and $\mathbf{M}$. Thus, Equation (55) offers a simple tool to calculate these boundaries, and to calculate corresponding boundary attenuation angles γ. In the mixed specification of the slowness vector (13), the boundary corresponding to the zero phase velocity cannot be crossed by any choice of real-valued inhomogeneity strength d, as the zero phase velocity corresponds to $d=\infty$. This is the reason why the mixed specification never gives forbidden combinations of $\mathbf{N}$ and $\mathbf{M}$. The boundary values of $\mathbf{N}$ and $\mathbf{M}$, corresponding to the limiting value of $d=\infty$, and yielding the zero phase velocities $\mathcal{C}$, can be, however, simply calculated from (55).

7. CONCLUDING REMARKS

The mixed specification (13) of slowness vector $\mathbf{p}$ yields the most straightforward algorithm for the computation of phase velocities $\mathcal{C}$ and the slowness vectors $\mathbf{p}$ of inhomogeneous qS1, qS2 and qP plane waves propagating in a general viscoelastic anisotropic medium in the direction of an arbitrarily chosen unit vector $\mathbf{n}$. This requires the solving of an algebraic equation of the sixth degree. Alternatively, it requires the solving of a conventional eigenvalue problem for a 6×6 complex-valued matrix $\mathbf{A}$. A uniform algorithm can be used both for homogeneous and inhomogeneous plane waves, propagating in anisotropic or isotropic, viscoelastic or perfectly elastic models. Once the eigenvalue problem for the 6×6 complex-valued matrix $\mathbf{A}$ has been solved, simple explicit expressions can be used to determine phase velocities $\mathcal{C}$, attenuation angles γ, and the amplitude attenuation factors δ of the qS1, qS2 and qP plane waves propagating along and against $\mathbf{n}$.

In certain special cases, the algorithm simplifies. Let us describe several such cases.

1. For **homogeneous plane waves**, propagating in viscoelastic anisotropic media, both approaches based on the directional and mixed specifications of the slowness vector may be used. They yield the same results. Whereas the mixed specification requires the solving of the eigenvalue problem for a 6×6 complex-valued matrix $\mathbf{A}$, the directional specification, in this case, reduces to the solution of an eigenvalue problem for the 3×3 complex-valued Christoffel matrix $\mathbf{\Gamma}$. In fact, in viscoelastic anisotropic media, the case of homogeneous plane waves is the only one in which the directional specification of the slowness vector yields more efficient algorithms than the mixed specification. For a more detailed discussion of homogeneous plane waves and for the derivation of the relevant expressions for $\mathcal{C}$, δ and for the slowness vector $\mathbf{p}$, see *Červený (2003a)*.

2. Inhomogeneous plane waves can also propagate in **perfectly elastic anisotropic media** (with real-valued elastic moduli c_{ijkl}). Their properties, however, are different from those propagating in viscoelastic anisotropic media (with complex-valued viscoelastic moduli c_{ijkl}). Let us consider a selected qS1, qS2 or qP inhomogeneous plane wave, propagating in the propagation-attenuation plane $\Sigma^{\parallel}$, in the direction of $\mathbf{N}$. (The propagation-attenuation plane $\Sigma^{\parallel}$ is specified by real-valued unit vectors $\mathbf{N}$ and $\mathbf{e}^{\parallel}$.) In a viscoelastic anisotropic medium, for $\mathbf{N}$ and $\mathbf{e}^{\parallel}$ given, a one-parametric system of inhomogeneous plane waves, with different attenuation angles γ, can propagate. Each inhomogeneous plane wave of this system is parametrised by the inhomogeneity strength d, $0 \leq d < \infty$. In a perfectly elastic anisotropic medium, for $\mathbf{N}$ and $\mathbf{e}^{\parallel}$ given, only one inhomogeneous plane wave, with the attenuation angle γ^{el}, is allowed. This is common knowledge for isotropic elastic media, where the attenuation angle γ^{el} does not depend on $\mathbf{N}$ and $\mathbf{e}^{\parallel}$, and always equals $\frac{1}{2}\pi$. In anisotropic elastic media, the attenuation angle γ^{el} is, in general, different from $\frac{1}{2}\pi$, although it may be close to it in weakly anisotropic media. The attenuation angle γ^{el} in anisotropic viscoelastic media always depends on $\mathbf{N}$ and $\mathbf{e}^{\parallel}$, but not on d. The mixed specification offers explicit equations for the computation of the

attenuation angle γ^{el} for arbitrarily chosen unit vectors $\mathbf{N}$ and $\mathbf{e}^{\|}$ (with $\mathbf{N} \cdot \mathbf{e}^{\|} = 0$). For this reason, the mixed specification is preferable in studying the inhomogeneous plane waves propagating in elastic anisotropic media. For a very detailed treatment of inhomogeneous plane waves propagating in elastic anisotropic media see *Shuvalov (2001)*.

3. For inhomogeneous plane waves, propagating in **viscoelastic isotropic media**, well-known analytical solutions are obtained. Any of the three specifications of the slowness vector can be used to derive these analytical solutions. Even in this case, however, the solutions obtained using the mixed specifications are preferable. See *Červený (2003b)*.

4. For inhomogeneous plane waves, propagating in a **plane of symmetry** of certain viscoelastic anisotropic media, the characteristic equations in all the three specifications of the slowness vector can be factorized. The example is the propagation of inhomogeneous plane waves in the plane of symmetry of a monoclinic anisotropic viscoelastic medium. This, of course, also applies to planes of symmetry in simpler anisotropy systems (orthorhombic, hexagonal). A preferable option is again to use the mixed specification of the slowness vector. The characteristic equation for the 6×6 complex-valued matrix $\mathbf{A}$ is factorised into two characteristic equations, for the 2×2 and 4×4 complex-valued matrices. The first, for the qSH inhomogeneous plane waves, can be completely solved analytically. The second, for the qSV and qP inhomogeneous plane waves, requires a quartic complex-valued characteristic equation to be solved. For more details on such analytical solutions, see *Červený (2003b)*, and on numerical computations of qSV and qP inhomogeneous plane waves see *Červený and Pšenčík (2003)*.

A remark to the **forbidden combinations** in the $(\mathbf{N}, \mathbf{M})$ domain of the directional specification (11). Let us first emphasize that any "real world" inhomogeneous plane wave propagating in an anisotropic viscoelastic medium can be fully specified both by (11) and (13). The specification (13) can be simply transformed to (11), and vice versa. In addition to the physically acceptable inhomogeneous plane waves, obtained by the mixed specification (13), the directional specification (11) also includes nonphysical $(\mathbf{N}, \mathbf{M})$, corresponding to forbidden combinations. Along the boundaries between the forbidden and allowed regions of $(\mathbf{N}, \mathbf{M})$, the phase velocities $\mathcal{C}$ are zero. These boundaries cannot be simply calculated using the directional specification (11), but can be explicitly obtained if the mixed specification is used, see (55). Actually, they correspond to the infinite inhomogeneity strength ($d = \infty$). As $0 \leq d < \infty$, the boundaries between the allowed and forbidden regions cannot be crossed in the mixed specification. Consequently, the mixed parametrisation never gives forbidden combinations of $\mathbf{N}$ and $\mathbf{M}$. We repeat that the forbidden combinations physically exist in the $(\mathbf{N}, \mathbf{M})$ domain, but are automatically excluded in the $(\mathbf{N}, \mathbf{d})$ domain. In other words, the mixed specification never yields $(\mathbf{N}, \mathbf{M})$ corresponding to forbidden regions.

The directional and mixed specifications of the slowness vector are convenient in studying the properties of inhomogeneous plane waves propagating in the direction of a known unit vector $\mathbf{N}$, i. e. in the direction of propagation vector $\mathbf{P}$. The

componental specification is not so suitable in this case, as the unit vectors $\mathbf{N}$ of plane waves propagating away from Σ are not known in advance. The componental specification, however, offers some more general important applications. Let us name two such applications: a) Propagation of inhomogeneous plane waves from an initial surface Σ, along which complex-valued p_i^Σ are given. b) The reflection/transmission problem at a plane interface Σ. The complex-valued quantities p_i^Σ can be determined from the incident wave in this case.

The plane wave algorithms play an important role in the **seismic ray method**. Consequently, the proposed algorithms may find applications in the propagation of high-frequency seismic body waves in inhomogeneous anisotropic viscoelastic media.

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