FERMAT'S VARIATIONAL PRINCIPLE
FOR ANISOTROPIC INHOMOGENEOUS MEDIA

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ABSTRACT

Fermat's variational principle states that the signal propagates from point S to R along a curve which renders Fermat's functional $\mathcal{I}(l)$ stationary. Fermat's functional $\mathcal{I}(l)$ depends on curves l which connect points S and R, and represents the travel times from S to R along l. In seismology, it is mostly expressed by the integral $\mathcal{I}(l) = \int_S^R \mathcal{L}(x^k, x^{k\prime})\mathrm{d}u$, taken along curve l, where $\mathcal{L}(x^k, x^{k\prime})$ is the relevant Lagrangian, x^k are coordinates, u is a parameter used to specify the position of points along l, and $x^{k\prime} = \mathrm{d}x^k/\mathrm{d}u$. If Lagrangian $\mathcal{L}(x^k, x^{k\prime})$ is a homogeneous function of the first degree in $x^{k\prime}$, Fermat's principle is valid for arbitrary monotonic parameter u. We then speak of the first–degree Lagrangian $\mathcal{L}^{(1)}(x^k, x^{k\prime})$. It is shown that the conventional Legendre transform cannot be applied to the first–degree Lagrangian $\mathcal{L}^{(1)}(x^k, x^{k\prime})$ to derive the relevant Hamiltonian $\mathcal{H}^{(1)}(x^k, p_k)$, and Hamiltonian ray equations. The reason is that the Hessian determinant of the transform vanishes identically for first–degree Lagrangians $\mathcal{L}^{(1)}(x^k, x^{k\prime})$. The Lagrangians must be modified so that the Hessian determinant is different from zero. A modification to overcome this difficulty is proposed in this article, and is based on second–degree Lagrangians $\mathcal{L}^{(2)}$. Parameter u along the curves is taken to correspond to travel time τ, and the second–degree Lagrangian $\mathcal{L}^{(2)}(x^k, \dot{x}^k)$ is then introduced by the relation $\mathcal{L}^{(2)}(x^k, \dot{x}^k) = \frac{1}{2}[\mathcal{L}^{(1)}(x^k, \dot{x}^k)]^2$, with $\dot{x}^k = \mathrm{d}x^k/\mathrm{d}\tau$. The second–degree Lagrangian $\mathcal{L}^{(2)}(x^k, \dot{x}^k)$ yields the same Euler–Lagrange equations for rays as the first–degree Lagrangian $\mathcal{L}^{(1)}(x^k, \dot{x}^k)$. The relevant Hessian determinant, however, does not vanish identically. Consequently, the Legendre transform can then be used to compute Hamiltonian $\mathcal{H}^{(2)}(x^k, p_k)$ from Lagrangian $\mathcal{L}^{(2)}(x^k, \dot{x}^k)$, and vice versa, and the Hamiltonian canonical equations can be derived from the Euler–Lagrange equations. Both $\mathcal{L}^{(2)}(x^k, \dot{x}^k)$ and $\mathcal{H}^{(2)}(x^k, p_k)$ can be expressed in terms of the wave propagation metric tensor $g_{ij}(x^k, \dot{x}^k)$, which depends not only on position x^k, but also on the direction of vector $\dot{x}^k$. It is defined in a Finsler space, in which the distance is measured by the travel time. It is shown that the standard form of the Hamiltonian, derived from the elastodynamic equation and representing the eikonal equation, which has been broadly used in the seismic ray method, corresponds to the second–degree Lagrangian $\mathcal{L}^{(2)}(x^k, \dot{x}^k)$, not to the first–degree Lagrangian $\mathcal{L}^{(1)}(x^k, \dot{x}^k)$. It is also shown that relations $\mathcal{L}^{(2)}(x^k, \dot{x}^k) = \frac{1}{2}$ and $\mathcal{H}^{(2)}(x^k, p_k) = \frac{1}{2}$ are valid at any

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point of the ray and that they represent the group velocity surface and the slowness surface, respectively. All procedures and derived equations are valid for general anisotropic inhomogeneous media, and for general curvilinear coordinates x^i. To make certain procedures and equations more transparent and objective, the simpler cases of isotropic and ellipsoidally anisotropic media are briefly discussed as special cases.

Keywords: Fermat's principle, anisotropic media, Lagrangian, Hamiltonian, Finsler space, wave propagation metric tensor.

1. INTRODUCTION

The problem of Fermat's variational principle in an anisotropic inhomogeneous medium is mostly of academic interest only. Practically everything that follows from Fermat's variational principle can also be derived by applying high–frequency asymptotic methods to the elastodynamic equation, and from the consequent eikonal equation. The eikonal equation, following from the anisotropic elastodynamic equation using asymptotic methods, reads

$$G(x^k, p_k) = 1 \ , \tag{1}$$

where $G(x^k, p_k)$ is the eigenvalue of Christoffel matrix $\Gamma_{ik} = a_{ik}^{jl} p_j p_l$. Here p_i are covariant components of the slowness vector, x^i the general curvilinear coordinates, and a_{ik}^{jl} the density normalized elastic moduli. The eikonal equation (1) corresponds to any of the three elementary waves (qP, qS1 or qS2) which may propagate in anisotropic inhomogeneous medium. Eikonal equation (1) may be used to write the Hamiltonian ray tracing equations:

$$\frac{\mathrm{d}x^i}{\mathrm{d}\tau} = \frac{1}{2}\,\frac{\partial G}{\partial p_i} \ , \quad \frac{\mathrm{d}p_i}{\mathrm{d}\tau} = -\frac{1}{2}\,\frac{\partial G}{\partial x^i} \ , \tag{2}$$

see *Červený* (*2001,* p. 149). Here τ denotes the parameter along the ray, corresponding to the travel time. It is not difficult to prove that $G(x^i, p_i)$ is a homogeneous function of the second degree in p_i. By homogeneous function $f(p_i)$ of the k-th degree in p_i, we understand function $f(p_i)$ which satisfies the relation

$$f(a p_i) = a^k f(p_i) \ , \tag{3}$$

for any nonvanishing constant a. The homogeneous function $f(p_i)$ of the k-th degree satisfies Euler's theorem

$$p_j\,\frac{\partial f(p_i)}{\partial p_j} = k\,f(p_i) \ . \tag{4}$$

The ray tracing equations (2) take the standard form of Hamiltonian canonical equations

$$\frac{\mathrm{d}x^i}{\mathrm{d}\tau} = \frac{\partial \mathcal{H}^{(2)}}{\partial p_i} \ , \quad \frac{\mathrm{d}p_i}{\mathrm{d}\tau} = -\frac{\partial \mathcal{H}^{(2)}}{\partial x^i} \ , \tag{5}$$

if we introduce Hamiltonian $\mathcal{H}^{(2)}(x^k, p_k)$ by the relation

$$\mathcal{H}^{(2)}(x^k, p_k) = \tfrac{1}{2}G(x^k, p_k) \; . \tag{6}$$

Hamiltonian $\mathcal{H}^{(2)}(x^k, p_k)$ is again a homogeneous function of the second degree in p_i, similarly as $G(x^k, p_k)$. To emphasize this fact, we use notation $\mathcal{H}^{(2)}(x^k, p_k)$ for the Hamiltonian. The relation

$$\mathcal{H}^{(2)}(x^k, p_k) = \tfrac{1}{2} \tag{7}$$

is satisfied along the whole ray. It is well known that relations (1) and (7) represent the slowness surface in a phase space specified by coordinates p_i.

Note that the same Hamiltonian ray equations (5) are also obtained if Hamiltonian $\mathcal{H}^{(2)}(x^k, p_k) = \frac{1}{2}(G(x^k, p_k) - 1)$ is used. Then $\mathcal{H}^{(2)}(x^k, p_k)$ equals zero along the whole ray. This form is also common in the seismological literature, see *Červený* (*2001,* Eq. (3.6.3)). Here, however, we shall use the Hamiltonian given by (6) consistently.

In this paper, we wish to derive general equations (1)–(2), (5)–(7) using Fermat's variational principle. We would also like to find the Lagrangian corresponding to Hamiltonian (6), and to establish general rules of computing Lagrangians from Hamiltonians, and vice versa. The Lagrangians are introduced here in terms of the wave propagation metric tensor, explained in Section 2, see (9). The wave propagation metric tensor is very closely related to the phase and group velocity surfaces.

The Legendre transform will be used to compute the Lagrangians from Hamiltonians, and vice versa. It is shown that the Legendre transform cannot be used if the conventional Fermat's variational principle in parametric form is used, as the Hessian determinant of the transform vanishes in this case. The Lagrangian must be modified so that the Hessian determinant is different from zero. See Section 3.

We consider general curvilinear coordinates x^i, $i = 1, 2, 3$. The Einstein summation convention is used and applies to repeated subscripts and superscripts. It is only in the example in subsection 5.1, related to isotropic media, that we use standard Cartesian coordinates. In this case, all indices are written as subscripts, and the Einstein summation convention applies to repeated lower indices.

2. FERMAT'S VARIATIONAL PRINCIPLE

Fermat's principle, in its original version, states that the ray connecting two points, S and R, is represented by a curve along which the least time for a signal to travel from S to R is required. We shall speak of Fermat's minimum–time principle. Fermat's minimum–time principle has been used recently in network ray tracing and yields the rays corresponding to first arrival travel times. See *Moser* (*1991*), *Klimeš and Kvasnička* (*1994*), and other references given there.

In this paper, we shall consider the variational form of Fermat's principle, also sometimes called Fermat's variational principle. Fermat's variational principle also yields the rays corresponding to later arrival travel times. Fermat's variational principle for anisotropic inhomogeneous media has also been discussed in *Babich*

(*1961*), *Kline and Kay* (*1965*), *Hanyga, Lenartowicz and Pajchel* (*1984*), *Chapman and Pratt* (*1992*), *Epstein and Śniatycki* (*1992*), *Pšenčík* (*1997*), *Hanyga* (*2000*), and *Stephen* (*2000*).

We introduce Fermat's functional $\mathcal{I}(l)$ as the travel time of the signal from S to R along curve l. Fermat's functional, defined in this way, depends, of course, on the curve l, connecting S and R. Fermat's variational principle then reads: The signal propagates from point S to R along curve $\bar{l}$ which renders Fermat's functional stationary.

We now have to specify what we understand by the travel time $\mathcal{I}(l)$ of a signal along curve l. In general, $\mathcal{I}(l)$ is given by the relation

$$\mathcal{I}(l) = \int_S^R \mathrm{d}\tau \ , \tag{8}$$

where $\mathrm{d}\tau$ is the elementary time increment measured along the elementary arclength increment $\mathrm{d}l$ of curve l. To specify $\mathrm{d}\tau$, we shall introduce the metric tensor g_{ij} of a specially chosen Finsler space which depends not only on position, but also on direction (*Asanov, 1985*). The square of the distance $\mathrm{d}s^2$ between two adjacent points in a Finsler space is given by the relation $\mathrm{d}s^2 = g_{ij}\mathrm{d}x^i\mathrm{d}x^j$. The distances may have various physical meanings. We shall consider the Finsler space in which *distance s is measured by travel time τ*, so that

$$\mathrm{d}\tau^2 = g_{ij}\mathrm{d}x^i\mathrm{d}x^j \ . \tag{9}$$

We shall refer to the relevant metric tensor g_{ij} as the *Finslerian metric tensor*, or, more specifically, the *wave propagation metric tensor*. Fermat's functional (8) then reads

$$\mathcal{I}(l) = \int_S^R [g_{ij}x^{i\prime}x^{j\prime}]^{1/2}\mathrm{d}u \ , \qquad x^{i\prime} = \mathrm{d}x^i/\mathrm{d}u \ . \tag{10}$$

Here u is an arbitrary parameter which specifies the position of a point on curve l.

Actually, the introduction of the Finslerian metric tensor does not play a decisive role here. It would be possible to avoid it. However, it makes our treatment more transparent.

Several notes on the Finslerian metric tensor g_{ij}. It is a function both of position x^k and of the direction of vector $x^{k\prime}$, tangential to curve l at x^k,

$$g_{ij} = g_{ij}(x^k, x^{k\prime}) \ . \tag{11}$$

A great difference of the Finslerian metric tensor $g_{ij}(x^k, x^{k\prime})$ compared to the Riemannian metric tensor $g_{ij}(x^k)$ is that it reflects the directional dependence of velocities in anisotropic media. The Finslerian metric tensor $g_{ij}(x^k, x^{k\prime})$ reduces to the Riemannian metric tensor $g_{ij}(x^k)$, if g_{ij} is independent of direction. The so–called *Cartan torsion tensor* C_{ijl} can been introduced,

$$C_{ijl}(x^k, x^{k\prime}) = \frac{1}{2}\frac{\partial g_{ij}(x^k, x^{k\prime})}{\partial x^{l\prime}} \ . \tag{12}$$

In Riemannian geometry, the Cartan torsion tensor C_{ijl} vanishes identically, $C_{ijl}(x^k, x^{k\prime}) = 0$. In Finslerian geometry, however, it is generally different from zero. For more details, see *Asanov* (*1985*).

We shall assume that the Finslerian metric tensor $g_{ij}(x^k, x^{k\prime})$ satisfies the following properties:

1) $g_{ij}(x^k, x^{k\prime})$ is symmetric,

$$g_{ij}(x^k, x^{k\prime}) = g_{ji}(x^k, x^{k\prime}) \ . \tag{13}$$

2) $g_{ij}(x^k, x^{k\prime})$ is a homogeneous function of degree zero in $x^{k\prime}$, so that

$$g_{ij}(x^k, ax^{k\prime}) = a^0 g_{ij}(x^k, x^{k\prime}) = g_{ij}(x^k, x^{k\prime}) \ , \tag{14}$$

see (3). Thus, $g_{ij}(x^k, x^{k\prime})$ is not changed if the vector with the contravariant components $x^{k\prime}$ is replaced by a vector with a different length, but the same direction. This implies that the Finslerian metric tensor $g_{ij}(x^k, x^{k\prime})$ is a function of position and direction. Euler's theorem (4) also yields

$$x^{n\prime} \partial g_{ij}(x^k, x^{k\prime}) / \partial x^{n\prime} = 0 \ . \tag{15}$$

3) $g_{ij}(x^k, x^{k\prime})$ is nondegenerate,

$$\det(g_{ij}(x^k, x^{k\prime})) \neq 0 \ . \tag{16}$$

4) $g_{ij}(x^k, x^{k\prime})$ satisfies the following relation,

$$\frac{\partial g_{ij}(x^k, x^{k\prime})}{\partial x^{n\prime}} = \frac{\partial g_{in}(x^k, x^{k\prime})}{\partial x^{j\prime}} \ . \tag{17}$$

For a detailed discussion of (17) see *Asanov* (*1985,* p. 266).

In mathematical works devoted to the Finsler geometry, it has often been assumed that g_{ij} is also positive definite. Such an assumption would have serious consequences for our studies of anisotropic elastic media, and would exclude many important anisotropic media from our treatment. However, the Finsler geometry can be developed even for indefinite g_{ij} (indefinite Finsler spaces). See *Asanov* (*1985,* p. 21), and other references given therein (for example, *Beem, 1970*; *Beem and Kishta, 1974*).

Actually, even the Riemannian metric tensor may be used in some simple cases of anisotropic media, such as ellipsoidally anisotropic media, but not in a general case. In Section 4, we shall explain how the Finslerian metric tensor can be determined from travel–time measurements.

We shall now use the notation

$$\mathcal{L}^{(1)}(x^k, x^{k\prime}) = [g_{ij}(x^k, x^{k\prime}) x^{i\prime} x^{j\prime}]^{1/2} \ , \qquad x^{i\prime} = \mathrm{d}x^i / \mathrm{d}u \ , \tag{18}$$

and call $\mathcal{L}^{(1)}(x^k, x^{k\prime})$ the Lagrangian. Clearly, $\mathcal{L}^{(1)}(x^k, x^{k\prime})$, given by (18), is a homogeneous function of the first degree. To emphasize this fact, we use the notation $\mathcal{L}^{(1)}(x^k, x^{k\prime})$. Fermat's functional $\mathcal{I}(l)$, given by (10), then reads

$$\mathcal{I}(l) = \int_S^R \mathcal{L}^{(1)}(x^k, x^{k\prime}) \mathrm{d}u \ . \tag{19}$$

The statement that functional $\mathcal{I}(l)$ is stationary along curve $\bar{l}$ means that the first variation of the functional vanishes along $\bar{l}$:

$$\delta \mathcal{I}(l) = \delta \int_S^R \mathrm{d}\tau = \delta \int_{u(S)}^{u(R)} \mathcal{L}^{(1)}(x^k, x^{k\prime}) \mathrm{d}u = 0 \ , \quad x^{k\prime} = \mathrm{d}x^k / \mathrm{d}u \ . \tag{20}$$

Here $u(S)$ and $u(R)$ are the values of the parameter u corresponding to fixed points S and R, respectively. Curve $\bar{l}$, for which Fermat's functional is stationary, is called the extremal curve, or the *extremal of Fermat's functional*. The ray may then be identified as the *extremal of Fermat's functional*. It also represents the geodesic curve connecting points S and R in a Finsler space specified by the wave propagation metric tensor g_{ij} (11). Consequently, the ray in an anisotropic inhomogeneous medium is also represented by a *Finslerian geodesic curve*.

It is well known that the extremal of Fermat's functional (19) satisfies the Euler–Lagrange equations:

$$\frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{\partial\mathcal{L}^{(1)}}{\partial x^{i\prime}}\right) - \frac{\partial\mathcal{L}^{(1)}}{\partial x^{i}} = 0 \ , \qquad i = 1,2,3 \ . \tag{21}$$

The Euler–Lagrange equations (21) represent the ray equations in Lagrangian form, also called Lagrangian ray equations.

The Lagrangian ray equations (21) consist of three ordinary differential equations of the second order. It would be useful to transform (21) into a system of six ordinary differential equations of the first order. The well–known procedure of doing this is based on the so–called dual Legendre transform. However, some restrictions are imposed on this transform. For this reason, we shall briefly recapitulate it here. For more details, see *Babich and Buldyrev (1972)*, *Forbes (1991)*, *Goldstein (1980)*, *Kline and Kay (1965)*, etc.

The discussion here is related to any Lagrangian $\mathcal{L}(x^k, x^{k\prime})$, not only to the first–degree Lagrangian $\mathcal{L}^{(1)}(x^k, x^{k\prime})$. We introduce "momenta" p_i by the relation

$$p_i = \partial\mathcal{L}(x^k, x^{k\prime})/\partial x^{i\prime} \ , \qquad i = 1,2,3 \ . \tag{22}$$

The most important step in the dual Legendre transform consists in the determination of $x^{i\prime}$ $(i = 1,2,3)$ from known p_i $(i = 1,2,3)$, using (22). This is possible only if the relevant Hessian determinant is different from zero,

$$\det\left(\frac{\partial^2\mathcal{L}}{\partial x^{i\prime}\partial x^{j\prime}}\right) \neq 0 \ . \tag{23}$$

If (23) is satisfied, we can solve (22) for $x^{i\prime}$ and determine $x^{i\prime} = x^{i\prime}(x^k, p_k)$. We can then determine Hamiltonian $\mathcal{H}(x^k, p_k)$ from Lagrangian $\mathcal{L}(x^k, x^{k\prime})$ by the relation

$$\mathcal{H}(x^k, p_k) = [x^{i\prime}p_i - \mathcal{L}(x^k, x^{k\prime})]_{x^{i\prime}=x^{i\prime}(x^k,p_k)} \ . \tag{24}$$

Differential $\mathrm{d}\mathcal{L}$ can be expressed in two alternative ways:

$$\mathrm{d}\mathcal{L} = \frac{\partial\mathcal{L}}{\partial x^i}\,\mathrm{d}x^i + \frac{\partial\mathcal{L}}{\partial x^{i\prime}}\,\mathrm{d}x^{i\prime} = -\frac{\partial\mathcal{H}}{\partial x^i}\,\mathrm{d}x^i - \frac{\partial\mathcal{H}}{\partial p_i}\,\mathrm{d}p_i + x^{i\prime}\mathrm{d}p_i + p_i\mathrm{d}x^{i\prime} \ . \tag{25}$$

In the second relation of (25), we have used $\mathcal{L} = x^{i\prime}p_i - \mathcal{H}(x^k, p_k)$. Equation (25) then yields two important relations

$$\frac{\partial\mathcal{L}}{\partial x^i} = -\frac{\partial\mathcal{H}}{\partial x^i} \ , \quad x^{i\prime} = \frac{\partial\mathcal{H}}{\partial p_i} \ . \tag{26}$$

Equations (21), (22) and (26) yield the final Hamiltonian canonical system of six ordinary differential equations of the first order, alternative to the Euler–Lagrange equations (21),

$$\frac{\mathrm{d}x^i}{\mathrm{d}u} = \frac{\partial \mathcal{H}}{\partial p_i} , \qquad \frac{\mathrm{d}p_i}{\mathrm{d}u} = -\frac{\partial \mathcal{H}}{\partial x^i} . \tag{27}$$

The system of equations (27) represents the ray tracing system in Hamiltonian form, or the Hamiltonian ray equations.

Thus, if Hessian (23) is different from zero, we can construct Hamiltonian $\mathcal{H}(x^k, p_k)$ from Lagrangian $\mathcal{L}(x^k, x^{k\prime})$, and vice versa. Condition (23), however, is sometimes not satisfied. We shall now explain the main problem, connected with Lagrangians represented by homogeneous functions of the first degree in $x^{i\prime}$.

In seismology, Fermat's principle has mostly been used in parametric form,

$$\delta\mathcal{I}(l) = \delta \int_S^R \mathcal{L}(x^k, x^{k\prime})\mathrm{d}u , \quad x^{k\prime} = \mathrm{d}x^k/\mathrm{d}u , \tag{28}$$

see (20) with (18) as an example. It is required that the Euler–Lagrange equations (21) of (28) hold for any parameter u. For parameter σ, different from u, (28) reads

$$\delta\mathcal{I}(l) = \delta \int_S^R \mathcal{L}(x^k, x^{k\prime})\mathrm{d}\sigma = 0 , \quad x^{k\prime} = \mathrm{d}x^k/\mathrm{d}\sigma . \tag{29}$$

Substituting $\mathrm{d}u = (\mathrm{d}u/\mathrm{d}\sigma)\mathrm{d}\sigma$ into (28) yields

$$\delta\mathcal{I}(l) = \delta \int_S^R \mathcal{L}(x^k, a^{-1}x^{k\prime})a\mathrm{d}\sigma , \quad a = \mathrm{d}u/\mathrm{d}\sigma , \quad x^{k\prime} = \mathrm{d}x^k/\mathrm{d}\sigma .$$

Both expressions for $\delta\mathcal{I}(l)$ are the same if $a\mathcal{L}(x^k, a^{-1}x^{k\prime}) = \mathcal{L}(x^k, x^{k\prime})$, that is, if $\mathcal{L}(x^k, x^{k\prime})$ is a homogeneous function of the first degree in $x^{i\prime}$. Thus (28) may be used for arbitrarily chosen parameter u only if $\mathcal{L}(x^k, x^{k\prime})$ is a homogeneous function of the first degree in $x^{k\prime}$, $\mathcal{L}(x^k, x^{k\prime}) = \mathcal{L}^{(1)}(x^k, x^{k\prime})$.

It is, however, possible to prove that the Hessian determinant (23) vanishes identically if $\mathcal{L}(x^k, x^{k\prime})$ is a homogeneous function of the first degree in $x^{i\prime}$. Function $\partial\mathcal{L}/\partial x^{j\prime}$ is then a homogeneous function of degree zero in $x^{j\prime}$, and the Euler theorem yields

$$x^{i\prime}\frac{\partial^2\mathcal{L}(x^k, x^{k\prime})}{\partial x^{i\prime}\partial x^{j\prime}} = x^{i\prime}\frac{\partial}{\partial x^{i\prime}}\left(\frac{\partial\mathcal{L}(x^k, x^{k\prime})}{\partial x^{j\prime}}\right) = 0 .$$

Thus, for any fixed j, vector $\partial^2\mathcal{L}/\partial x^{i\prime}\partial x^{j\prime}$ is perpendicular to vector $x^{i\prime}$. This shows that all the three column vectors in the Hessian determinant are situated in a plane. Consequently, they are linearly dependent and the Hessian determinant vanishes.

The final result is that the dual Legendre transformation cannot be used to compute the Hamiltonian from the Lagrangian, if Lagrangian $\mathcal{L}(x^k, x^{k\prime})$ is a homogeneous function of the first degree in $x^{i\prime}$, $\mathcal{L}(x^k, x^{k\prime}) = \mathcal{L}^{(1)}(x^k, x^{k\prime})$. Consequently, the Hamiltonian ray tracing equations cannot be obtained from the Lagrangian ray tracing equations by the application of the standard dual Legendre transform.

Nevertheless, it is possible to modify the Lagrangian into a new form, for which the Euler–Lagrange equations are still satisfied, but the Hessian determinant does not identically vanish. In this contribution, we shall describe one such possibility, see the next section.

3. SECOND–DEGREE LAGRANGIANS AND HAMILTONIANS

We shall consider travel time τ to be a variable along the curves connecting S and R. The choice of the parameter $u = \tau$ plays a decisive role in the whole treatment; the procedure fails for other choices of u. Fermat's variational principle (20) then reads

$$\delta \int_S^R \mathcal{L}^{(1)}(x^k, \dot{x}^k)\mathrm{d}\tau = 0 \ , \qquad \dot{x}^k = \mathrm{d}x^k/\mathrm{d}\tau \ , \tag{30}$$

and the relevant Euler–Lagrange equations are

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial \mathcal{L}^{(1)}}{\partial \dot{x}^i}\right) - \frac{\partial \mathcal{L}^{(1)}}{\partial x^i} = 0 \ , \qquad i = 1, 2, 3 \ . \tag{31}$$

Here $\dot{x}^i$ is used to denote $\mathrm{d}x^i/\mathrm{d}\tau$ to distinguish it from $x^{i\prime} = \mathrm{d}x^i/\mathrm{d}u$, valid for any parameter u.

Lagrangian $\mathcal{L}^{(1)}(x^k, \dot{x}^k)$ is given by the relation

$$\mathcal{L}^{(1)}(x^k, \dot{x}^k) = [g_{ij}(x^k, \dot{x}^k)\dot{x}^i\dot{x}^j]^{1/2} \ . \tag{32}$$

It immediately follows from (9) that

$$\mathcal{L}^{(1)}(x^k, \dot{x}^k) = 1 \tag{33}$$

along the ray. As the Finslerian metric tensor $g_{ij}(x^k, \dot{x}^k)$ is a homogeneous function of degree zero in $\dot{x}^k$, Lagrangian $\mathcal{L}^{(1)}(x^k, \dot{x}^k)$ is a homogeneous function of the first degree in $\dot{x}^i$, so that the relevant Hessian determinant $\det(\partial^2\mathcal{L}^{(1)}/\partial\dot{x}^i\partial\dot{x}^j)$ vanishes. Consequently, the dual Legendre transform cannot be used to derive Hamiltonian canonical equations.

We now introduce a new Lagrangian $\mathcal{L}^{(2)}(x^k, \dot{x}^k)$ by the relation

$$\mathcal{L}^{(2)}(x^k, \dot{x}^k) = \tfrac{1}{2}[\mathcal{L}^{(1)}(x^k, \dot{x}^k)]^2 = \tfrac{1}{2}g_{ij}(x^k, \dot{x}^k)\dot{x}^i\dot{x}^j \ . \tag{34}$$

Consequently, Lagrangian $\mathcal{L}^{(2)}(x^k, \dot{x}^k)$ is a homogeneous function of the second degree in $\dot{x}^k$. To distinguish between $\mathcal{L}^{(1)}(x^k, \dot{x}^k)$ and $\mathcal{L}^{(2)}(x^k, \dot{x}^k)$, we shall call $\mathcal{L}^{(1)}(x^k, \dot{x}^k)$ the **first–degree Lagrangian**, $\mathcal{L}^{(2)}(x^k, \dot{x}^k)$ the **second–degree Lagrangian**. These terms are used to emphasize that $\mathcal{L}^{(1)}(x^k, \dot{x}^k)$ is a homogeneous function of the first degree in $\dot{x}^i$, and that $\mathcal{L}^{(2)}(x^k, \dot{x}^k)$ is a homogeneous function of the second degree in $\dot{x}^i$. It follows from (33) that

$$\mathcal{L}^{(2)}(x^k, \dot{x}^k) = \tfrac{1}{2} \ . \tag{35}$$

Equations (34), (15) and (17) also immediately yield an important expression for g_{ij} in terms of $\mathcal{L}^{(2)}$,

$$g_{ij}(x^k, \dot{x}^k) = \frac{\partial^2\mathcal{L}^{(2)}(x^k, \dot{x}^k)}{\partial\dot{x}^i\partial\dot{x}^j} \ , \tag{36}$$

and the relation of g_{ij} to the Hessian determinant:

$$\det\left(\frac{\partial^2\mathcal{L}^{(2)}}{\partial\dot{x}^i\partial\dot{x}^j}\right) = \det(g_{ij}) \ . \tag{37}$$

We shall now evaluate $\mathrm{d}(\partial\mathcal{L}^{(2)}/\mathrm{d}\dot{x}^i)/\mathrm{d}\tau - \partial\mathcal{L}^{(2)}/\partial x^i$. We obtain

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial\mathcal{L}^{(2)}}{\partial\dot{x}^i}\right) - \frac{\partial\mathcal{L}^{(2)}}{\partial x^i} = \left[\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\mathcal{L}^{(1)}\frac{\partial\mathcal{L}^{(1)}}{\partial\dot{x}^i}\right) - \mathcal{L}^{(1)}\frac{\partial\mathcal{L}^{(1)}}{\partial x^i}\right] = \left[\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial\mathcal{L}^{(1)}}{\partial\dot{x}^i}\right) - \frac{\partial\mathcal{L}^{(1)}}{\partial x^i}\right] = 0 \ ,$$

as $\mathcal{L}^{(1)}(x^k,\dot{x}^k) = 1$ along the ray, see (33) and (31). Thus, Lagrangian $\mathcal{L}^{(2)}(x^k,\dot{x}^k)$ satisfies the Euler–Lagrange equations,

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial\mathcal{L}^{(2)}}{\partial\dot{x}^i}\right) - \frac{\partial\mathcal{L}^{(2)}}{\partial x^i} = 0 \ , \tag{38}$$

corresponding to functional

$$\mathcal{I}(l) = 2\int_S^R \mathcal{L}^{(2)}(x^k,\dot{x}^k)\mathrm{d}\tau = \int_S^R g_{ij}(x^k,\dot{x}^k)\dot{x}^i\dot{x}^j\,\mathrm{d}\tau \ . \tag{39}$$

We have introduced the factor 2 before the integral, as we wish to keep the same variable τ in (39) as in (30), corresponding to the travel time, see (35). This factor does not influence Fermat's variational principle and the consequent Euler–Lagrange equations at all. Consequently, variational principle

$$\delta\int_S^R 2\mathcal{L}^{(2)}(x^k,\dot{x}^k)\mathrm{d}\tau = \delta\int_S^R g_{ij}(x^k,\dot{x}^k)\dot{x}^i\dot{x}^j\,\mathrm{d}\tau = 0 \tag{40}$$

yields the same extremals as Fermat's variational principle (30). As g_{ij} is presumably nondegenerate, Equation (37) shows that the Hessian determinant is different from zero. Thus, the dual Legendre transform can be applied to compute $\mathcal{H}^{(2)}(x^i,p_i)$ from $\mathcal{L}^{(2)}(x^i,\dot{x}^i)$, and vice versa.

The statement that the Hessian is not identically equal zero does not mean that the Hessian cannot vanish for some specific directions. Actually, such situations may appear when the Hamiltonian (and/or the Lagrangian) does not represent globally convex surfaces.

We introduce momenta p_i and the second–degree Hamiltonian $\mathcal{H}^{(2)}(x^k,p_k)$ using the Legendre transform:

$$p_i = \frac{\partial\mathcal{L}^{(2)}}{\partial\dot{x}^i} \ , \quad \mathcal{H}^{(2)}(x^k,p_k) = [\dot{x}^i p_i - \mathcal{L}^{(2)}(x^k,\dot{x}^k)]_{\dot{x}^i=\dot{x}^i(x^k,p_k)} \ . \tag{41}$$

If we take into account that $\mathcal{L}^{(2)}(x^k,\dot{x}^k)$ is a homogeneous function of the second order in $\dot{x}^k$, we can use Euler's theorem (4) with $k=2$, and obtain

$$\dot{x}^i p_i = \dot{x}^i\frac{\partial\mathcal{L}^{(2)}}{\partial\dot{x}^i} = 2\mathcal{L}^{(2)}(x^k,\dot{x}^k) \ .$$

The second equation of (41) then simplifies:

$$\mathcal{H}^{(2)}(x^k,p_k) = [\mathcal{L}^{(2)}(x^k,\dot{x}^k)]_{\dot{x}^i=\dot{x}^i(x^k,p_k)} \ . \tag{42}$$

Quite analogously, we obtain,

$$\mathcal{L}^{(2)}(x^k,\dot{x}^k) = [\mathcal{H}^{(2)}(x^k,p_k)]_{p_i=p_i(x^k,\dot{x}^k)} \ . \tag{43}$$

As the Hessian determinant for $\mathcal{L}^{(2)}$ does not vanish, the first equation of (41) can be inverted to yield $\dot{x}^i = \dot{x}^i(x^k, p_k)$. In the following, we shall refer to $\mathcal{H}^{(2)}(x^k, p_k)$ as the **second–degree Hamiltonian**. Using also (26),

$$\frac{\partial\mathcal{L}^{(2)}}{\partial x^i} = -\frac{\partial\mathcal{H}^{(2)}}{\partial x^i} \ , \quad \dot{x}^i = \frac{\partial\mathcal{H}^{(2)}}{\partial p_i} \ , \tag{44}$$

we arrive at the Hamiltonian ray equations in the following form,

$$\frac{\mathrm{d}x^i}{\mathrm{d}\tau} = \frac{\partial\mathcal{H}^{(2)}}{\partial p_i} \ , \quad \frac{\mathrm{d}p_i}{\mathrm{d}\tau} = -\frac{\partial\mathcal{H}^{(2)}}{\partial x^i} \ . \tag{45}$$

This is the final result. Later on, we shall show that the second–degree Hamiltonian $\mathcal{H}^{(2)}(x^k, p_k)$, obtained in this way, corresponds to the Hamiltonian defined by (6). See Section 4.

4. DISCUSSION OF THE SECOND–DEGREE LAGRANGIANS $\mathcal{L}^{(2)}$ AND RELEVANT HAMILTONIANS $\mathcal{H}^{(2)}$

We shall now discuss the second–degree Lagrangian $\mathcal{L}^{(2)}(x^k, x^{k\prime})$ and the second–degree Hamiltonian $\mathcal{H}^{(2)}(x^k, p_k)$ in some detail. We shall also add several notes to clarify the meaning and the properties of certain quantities introduced above. We shall try to use the terminology common in the seismic ray theory.

1) Momenta p_i represent the covariant components of the slowness vector. It should be noted that p_i computed from the second–degree Lagrangian $\mathcal{L}^{(2)}$ using (41) are exactly the same as the p_i computed from the first–degree Lagrangian $\mathcal{L}^{(1)}$ using (22) for $u = \tau$, $p_i = \partial\mathcal{L}^{(2)}/\partial\dot{x}^i = \mathcal{L}^{(1)}\partial\mathcal{L}^{(1)}/\partial\dot{x}^i = \partial\mathcal{L}^{(1)}/\partial x^i$. Similarly, quantities $\dot{x}^i = \mathrm{d}x^i/\mathrm{d}\tau$ represent the contravariant components of group velocity vector $\vec{\mathcal{U}}$,

$$\dot{x}^i = \mathrm{d}x^i/\mathrm{d}\tau = \mathcal{U}^i \ . \tag{46}$$

Multiplying the first equation of (41) by $\dot{x}^i$ and using Euler's theorem, we obtain

$$\dot{x}^i p_i = \dot{x}^i \frac{\partial\mathcal{L}^{(2)}}{\partial\dot{x}^i} = 2\mathcal{L}^{(2)} = 1 \ .$$

This yields the well–known relation connecting p_i and $\mathcal{U}^i$,

$$p_i\mathcal{U}^i = 1 \ . \tag{47}$$

2) The second–degree Lagrangian $\mathcal{L}^{(2)}(x^k, \dot{x}^k)$ can be expressed in terms of group velocity $\mathcal{U}$. If we use (35) and Euler's theorem (4), we obtain

$$\mathcal{L}^{(2)}(x^k,\mathcal{U}^k) = \mathcal{U}^2\mathcal{L}^{(2)}(x^k,\mathcal{U}^k/\mathcal{U}) = \tfrac{1}{2} \ .$$

We now take into account that $\mathcal{U}^k/\mathcal{U} = \mathrm{d}x^k/\mathrm{d}s$, where s is the arclength along the ray. This yields

$$\mathcal{U}^2 = \frac{1}{2\mathcal{L}^{(2)}(x^k, \mathrm{d}x^k/\mathrm{d}s)} = \frac{1}{[\mathcal{L}^{(1)}(x^k, \mathrm{d}x^k/\mathrm{d}s)]^2} , \qquad \mathcal{U} = \frac{1}{\mathcal{L}^{(1)}(x^k, \mathrm{d}x^k/\mathrm{d}s)} . \tag{48}$$

3) We shall now show that surface $\mathcal{L}^{(2)}(x^k, \mathcal{U}^k) = \frac{1}{2}$ in the $\mathcal{U}^i$–space represents the group velocity surface for a fixed x^k. We define the group velocity surface as the locus of the end points of group velocity vectors $\vec{\mathcal{U}}$, constructed in all possible directions at point x^i. Consequently, the group velocity surface represents a polar graph of group velocity $\mathcal{U}$ as a function of two take–off angles that specify the direction of unit vector $\vec{t} = \vec{\mathcal{U}}/\mathcal{U}$. We use the relation

$$\mathcal{U}^2 \mathcal{L}^{(2)}(x^k, t^k) = \tfrac{1}{2} , \tag{49}$$

where $t^k = \mathrm{d}x^k/\mathrm{d}s$ are components of the unit vector tangent to the ray. We further use the polar representation for $\mathcal{U}^k$, $\mathcal{U}^k = \rho t^k$. Then $\mathcal{L}^{(2)}(x^k, t^k) = \rho^{-2}\mathcal{L}^{(2)}(x^k, \mathcal{U}^k)$, and

$$(\mathcal{U}/\rho)^2 \mathcal{L}^{(2)}(x^k, \mathcal{U}^k) = \tfrac{1}{2} . \tag{50}$$

This shows that $\mathcal{U} = \rho$ along surface $\mathcal{L}^{(2)}(x^k, \mathcal{U}^k) = \frac{1}{2}$. Consequently, surface $\mathcal{L}^{(2)}(x^k, \mathcal{U}^k) = \frac{1}{2}$ in the $\mathcal{U}^i$–space represents the group velocity surface for a fixed x^k.

In the same way, it is simple to prove that an alternative expression for the group velocity surface is $\mathcal{L}^{(1)}(x^k, \mathcal{U}^k) = 1$. Thus, we can express the group velocity surface by any of the two equations

$$\mathcal{L}^{(2)}(x^k, \mathcal{U}^k) = \tfrac{1}{2} , \qquad \mathcal{L}^{(1)}(x^k, \mathcal{U}^k) = 1 . \tag{51}$$

4) Similar arguments may be used to prove that equation $\mathcal{H}^{(2)}(x^k, p_k) = \frac{1}{2}$ represents the slowness surface in the p_i–phase space for a fixed x^i. By the slowness surface we understand the locus of the end points of slowness vector $\vec{p}$ constructed in all directions at point x^i . Often, the eigenvalues $G(x^i, p_i)$ of the Christoffel matrix are used in the seismic ray theory to specify the slowness surface. It is related to the second–degree Hamiltonian as follows:

$$\mathcal{H}^{(2)}(x^k, p_k) = \tfrac{1}{2} G(x^k, p_k) . \tag{52}$$

Both $\mathcal{H}^{(2)}(x^k, p_k)$ and $G(x^k, p_k)$ are homogeneous functions of the second degree in p_i, and the slowness surface is given by two alternative equations:

$$\mathcal{H}^{(2)}(x^k, p_k) = \tfrac{1}{2} , \qquad G(x^k, p_k) = 1 . \tag{53}$$

Thus, function $G(x^k, p_k)$ represents the eigenvalue of the Christoffel matrix Γ_{ij}, see (1) and (6). However, to obtain the actual form of slowness surface $G(x^k, p_k) = 1$ from the known group velocity surface $\mathcal{L}^{(2)}(x^k, \dot{x}^k) = \frac{1}{2}$, it would be necessary to solve equation $p_i = \partial\mathcal{L}^{(2)}/\partial\dot{x}^i$ for $\dot{x}^i = \dot{x}^i(x^k, p_k)$, and insert it into (41). Also a graphical construction can be used, as both these surfaces are polarly reciprocal (see *Helbig, 1994*).

5) We shall now introduce the contravariant components of wave propagation metric tensor $\overline{g}^{ij}$ in the x^k, p_k domain. To distinguish it from the expressions for

the metric tensor in the $x^k, \dot{x}^k$ domain, we shall use a bar above the letter. We define $\overline{g}^{ij}$ as

$$\overline{g}^{mn}(x^k, p_k) = \frac{1}{2}\,\frac{\partial^2 G}{\partial p_m \partial p_n} = \frac{\partial^2 \mathcal{H}^{(2)}}{\partial p_m \partial p_n} \ . \tag{54}$$

Using (52) and the first equation of the Hamilton canonical equations (45), we obtain

$$\dot{x}^i = \mathcal{U}^i = \frac{1}{2}\,\frac{\partial G}{\partial p_i} = \frac{1}{2} p_m \,\frac{\partial^2 G}{\partial p_m \partial p_i} = p_m \overline{g}^{mi}(x^k, p_k) \ . \tag{55}$$

The contravariant components of metric tensor $\overline{g}^{mi}$ are related to its covariant components $\overline{g}_{ij}$ as

$$\overline{g}_{ik}\overline{g}^{ij} = \delta^j_k \ , \tag{56}$$

where δ^j_k is the mixed co- and contravariant Kronecker delta symbol. It equals 1 for $j = k$, and 0 for $j \neq k$. Equation (55) then yields

$$p_m = \overline{g}_{mk}(x^i, p_i)\dot{x}^k \ . \tag{57}$$

We shall now use the relation

$$p_i p_m \frac{\partial^2 \mathcal{H}^{(2)}}{\partial p_i \partial p_m} = p_i p_m \overline{g}^{im}(x^k, p_k) = \overline{g}_{ik}\dot{x}^k \overline{g}_{ml}\dot{x}^l \overline{g}^{im} = \dot{x}^m \dot{x}^l \overline{g}_{ml}(x^k, p_k) \ ,$$

and obtain

$$p_i p_m \overline{g}^{im}(x^k, p_k) = \dot{x}^m \dot{x}^l \overline{g}_{ml}(x^k, p_k) = 1 \ . \tag{58}$$

Relation (58) is satisfied along the ray. Thus, $\overline{g}_{ml}(x^k, p_k)$ is the p_i–domain alternative of the wave propagation metric tensor $g_{ml}(x^k, \dot{x}^k)$, given in (9).

A similar expression for the contravariant components of the wave propagation metric tensor was derived earlier by *Hanyga, Lenartowicz and Pajchel (1984)*, and by *Klimeš (1994)*. *Hanyga et al. (1984)* even refer to an older reference, *Rund (1966)*. The wave propagation metric tensor $\overline{g}^{mn}$ is also closely related to the metric tensor of the general theory of relativity, see *Synge (1960)*.

6) The first equation of (41) can also be expressed as follows, see (36),

$$p_i = \frac{\partial \mathcal{L}^{(2)}}{\partial \dot{x}^i} = \dot{x}^j \frac{\partial^2 \mathcal{L}^{(2)}}{\partial \dot{x}^i \partial \dot{x}^j} = g_{ij}(x^k, \dot{x}^k)\dot{x}^j \ , \tag{59}$$

$\mathcal{L}^{(2)}(x^k, \dot{x}^k)$ being a homogeneous function of the second degree in $\dot{x}^i$. This also yields

$$\dot{x}^i = g^{ij}(x^k, \dot{x}^k)p_j \ . \tag{60}$$

Here $g^{ij}(x^k, \dot{x}^k)$ are the contravariant components of the wave propagation metric tensor, and can be determined by inverting $g_{ij}(x^k, \dot{x}^k)$.

We now take into account the transformation relations between the covariant y_i and contravariant y^i components of vector $\vec{y}$, in coordinates specified by the Finslerian metric tensor $g_{ij}(x_k, \dot{x}_k)$,

$$y_i = g_{ij}y^j \ , \qquad y^i = g^{ij}y_j \ .$$

Equations (59) and (60) then indicate that the covariant components p_i of the slowness vector and the contravariant components $\dot{x}^i$ of the group velocity vector represent the components of the same vector in the Finsler space.

7) The comparison of Equation (55) with (60) and of (57) with (59) shows that

$$\overline{g}_{mn} = g_{mn} \ , \qquad \overline{g}^{mn} = g^{mn} \ . \tag{61}$$

The only difference is that $\overline{g}_{mn}$ and $\overline{g}^{mn}$ are expressed in terms of p_i, but g_{mn} and g^{mn} in terms of $\dot{x}^i$. Relation (56) can then be expressed in the following form: $g_{ij}\overline{g}^{in} = \delta_j^n$. This yields

$$\frac{\partial^2 \mathcal{L}^{(2)}(x^k,\dot{x}^k)}{\partial \dot{x}^i \partial \dot{x}^j} \, \frac{\partial^2 \mathcal{H}^{(2)}(x^k,p_k)}{\partial p_i \partial p_n} = \delta_i^n \ . \tag{62}$$

Using (32), (34) and (58), we can also express Lagrangians $\mathcal{L}^{(1)}(x^k,\dot{x}^k)$ and $\mathcal{L}^{(2)}(x^k,\dot{x}^k)$ in the p_i–domain as follows:

$$\mathcal{L}^{(1)}(x^k,\dot{x}^k) = [\overline{g}^{im}(x^k,p_k)p_i p_m]^{1/2} \ , \quad \mathcal{L}^{(2)}(x^k,\dot{x}^k) = \tfrac{1}{2}\overline{g}^{im}(x^k,p_k)p_i p_m \ . \tag{63}$$

As $\overline{g}^{im}$ is given by (54), we further obtain, in the p_i domain,

$$\begin{aligned} \mathcal{L}^{(1)}(x^k,\dot{x}^k) &= [2\mathcal{H}^{(2)}(x^k,p_k)]^{1/2} = [G(x^k,p_k)]^{1/2} \ , \\ \mathcal{L}^{(2)}(x^k,\dot{x}^k) &= \mathcal{H}^{(2)}(x^k,p_k) = \tfrac{1}{2}G(x^k,p_k) \ . \end{aligned} \tag{64}$$

This clearly reflects the relations between the Lagrangians in the $\dot{x}^i$–domain and the Hamiltonians in the p_i–domain.

Thus, for parameter u equal to travel time τ, we have obtained two important pairs of Lagrangians and relevant Hamiltonians. The first pair is $\mathcal{L}^{(2)}(x^k,\dot{x}^k) = \frac{1}{2}g_{ij}\,\dot{x}^i\dot{x}^j$, and $\mathcal{H}^{(2)}(x^k,p_k) = \frac{1}{2}G(x^k,p_k)$. The second pair is Lagrangian $\mathcal{L}^{(1)}(x^k,\dot{x}^k) = [g_{ij}\,\dot{x}^i\dot{x}^j]^{1/2}$ and the relevant Hamiltonian $[G(x^k,p_k)]^{1/2}$. For the first pair, the Hessian determinant does not vanish, and the dual Legendre transform can be safely used. However, for the second pair, the Hessian determinant vanishes, and the dual Legendre transform fails. Nevertheless, for $u=\tau$ and g_{ij} positive definite, the second pair may also be safely used in applications, without any problem.

8) Using (62), we obtain

$$\det(\partial^2\mathcal{L}^{(2)}/\partial\dot{x}^i\partial\dot{x}^j) = 1/\det(\partial^2\mathcal{H}^{(2)}/\partial p_i\partial p_j) \ . \tag{65}$$

Thus, the anomalous behaviour of the second–degree Hamiltonian $\mathcal{H}^{(2)}$ for some direction specified by slowness vector p_i corresponds to the anomalous behaviour of second–degree Lagrangian $\mathcal{L}^{(2)}$, for the direction of the relevant group velocity vector. Well–known examples are the parabolic lines in the slowness surface and the corresponding cuspoidal fold axes in the group velocity surface. See *Shuvalov and Every* (*1996, 1997*).

9) Second–degree Lagrangian $\mathcal{L}^{(2)}$ may be multivalued for certain directions of the group velocity vector, even if only one selected wave is considered. The exception is the qP wave. The slowness surface of the qP wave is always completely convex,

and the relevant group velocity surface is always completely single–valued and convex. The slowness surfaces of qS waves, however, may be hyperbolic or concave for certain directions of the slowness vector. The explanation of this fact is well–known. Equation $\mathcal{U}^i = \partial\mathcal{H}^{(2)}/\partial p_i$ may yield the same direction of $\vec{\mathcal{U}}$ for several different points along the slowness surface, if the slowness surface is not fully convex. This yields the multivaluedness of the modified second–degree Lagrangian, the multivaluedness of the group velocity surface, and the multivaluedness of wave propagation metric tensor $g_{ij}(x^k,\dot{x}^k)$. In this case, typical loops appear on the group velocity surface.

The multivalued group velocity surface, however, may always be decomposed into several single–valued branches. Along these branches, the wave propagation metric tensor $g_{ij}(x^k,\dot{x}^k)$ is single–valued. The proposed procedure, based on the second–degree Lagrangians and Hamiltonians, may then be applied to a selected single–valued branch. Note that the first–degree Lagrangians and Hamiltonians can be used only in regions, where the slowness surface is convex.

10) Let us return to metric tensor $g_{ij}(x^k,\dot{x}^k)$. It is given by relation (36). Thus, metric tensor g_{ij} is related to the curvature of the group velocity surface and may be calculated from it. It is well known that the group velocity surface may be determined from travel–time measurements. This suggests that even the wave propagation metric tensor $g_{ij}(x^k,\dot{x}^k)$ may be determined from the travel–time measurements. We only need to determine the second derivatives (36) of the group velocity surface. As the simplest example, let us consider an isotropic medium, and Cartesian coordinates x^i. The group velocity surface is then a sphere, given by the relation $\mathcal{L}^{(2)} = \frac{1}{2}V^{-2}(x^k)[(\dot{x}^1)^2 + (\dot{x}^2)^2 + (\dot{x}^3)^2] = \frac{1}{2}$. Equation (36) then yields $g_{ij} = V^{-2}(x^k)\delta_{ij}$. In this case, metric tensor g_{ij} does not depend on $\dot{x}^k$.

11) The determinant of the wave propagation metric tensor, $\det(g^{mn})$, is simply related to the Gaussian curvature K^S of the slowness surface,

$$\det(g^{mn}) = \det(\partial^2\mathcal{H}^{(2)}/\partial p_m \partial p_n) = \mathcal{U}^4 K^S \ , \tag{66}$$

where $\mathcal{U}$ is the group velocity. For the derivation of (66), see *Klimeš* (*2002*). Thus, the sign of $\det(g^{mn})$ is the same as the sign of K^S, and $\det(g^{mn})$ can be used to investigate the parabolic lines on the slowness surface (*Vavryčuk, 2002*).

12) We shall now write the equations for travel times $T(R,S)$ along the extremal of Fermat's functional, that is, along the ray connecting S and R. Using (19), (39) and (64), we obtain several alternative expressions in the x'_i– and p_i– domains:

$$T(R,S) = \int_S^R \mathcal{L}^{(1)}(x^k,\dot{x}^k)\mathrm{d}\tau = \int_S^R [G(x^k,p_k)]^{1/2}\mathrm{d}\tau \ . \tag{67}$$

Here $\mathcal{L}^{(1)} = [g_{mn}(x^k,\dot{x}^k)\dot{x}^m\dot{x}^n]^{1/2}$, $G = g^{mn}(x^k,p_k)p_m p_n$. This expression, however, may be used only in regions where g_{mn} is positive definite. See also *Pšenčík* (*1997*). Alternatively,

$$T(R,S) = 2\int_S^R \mathcal{L}^{(2)}(x^k,\dot{x}^k)\mathrm{d}\tau = 2\int_S^R \mathcal{H}^{(2)}(x^k,p_k)\mathrm{d}\tau = \int_S^R G(x^k,p_k)\mathrm{d}\tau \ , \tag{68}$$

where $\mathcal{L}^{(2)} = \frac{1}{2}g_{mn}(x^k,\dot{x}^k)\dot{x}^m\dot{x}^n$, $\mathcal{H}^{(2)} = \frac{1}{2}G = \frac{1}{2}g^{mn}(x^k,p_k)p_m p_n$.

The first relation of (67) is very useful, as it can be used for any parameter u along the ray, see (19),

$$T(R,S) = \int_S^R \mathcal{L}^{(1)}(x^k, \mathrm{d}x^k/\mathrm{d}u)\mathrm{d}u \ . \tag{69}$$

We remind the reader that the Hessian determinant corresponding to Lagrangian $\mathcal{L}^{(1)}$ vanishes, as $\mathcal{L}^{(1)}$ is a homogeneous function of the first degree in $\mathrm{d}x^k/\mathrm{d}u$. Consequently, the Legendre transform cannot be used to determine the associated Hamiltonian directly.

If we use the parameter u representing arclength s along the ray, (69) will yield

$$T(R,S) = \int_S^R \frac{\mathrm{d}s}{\mathcal{U}(x^k, \mathrm{d}x^k/\mathrm{d}s)} \ , \tag{70}$$

where $\mathcal{U}$ is the group velocity, see (48). This relation has often been used in the seismological literature and corresponds fully to simple physical considerations. It should be emphasized that the integrations in (67)–(70) are performed along a ray.

5. ISOTROPIC MEDIUM

The ray tracing equations derived in Sections 3 and 4, and all other relations of these sections, are valid for general anisotropic inhomogeneous media and for arbitrary curvilinear coordinates. They may be simplified in many ways. If we consider an isotropic medium, the metric tensor g_{ij} does not depend on $\dot{x}^i$. In addition, we can also use orthogonal coordinates x^i, or even Cartesian coordinates. In Sec. 5.1, we apply the general procedure to isotropic inhomogeneous media and use Cartesian coordinates in order to make certain steps in the derivation as well as some of the equations more transparent and objective. In such a case, all mathematical operations can be simply performed analytically. In Sec. 5.2, we shall return to general curvilinear coordinates.

5.1. Cartesian coordinates

We shall denote the Cartesian coordinates using standard lower–case indices and assume the summation over the same lower–case indices: $x^1 = x_1$, $x^2 = x_2$, $x^3 = x_3$. The covariant components of the wave propagation metric tensor are then given by relation

$$g_{ij} = V^{-2}(x_k)\delta_{ij} \ . \tag{71}$$

First–degree Lagrangian $\mathcal{L}^{(1)}(x_k, x'_k)$ reads,

$$\mathcal{L}^{(1)}(x_k, x'_k) = V^{-1}(x_k)(x'_i x'_i)^{1/2} \ , \qquad x'_i = \mathrm{d}x_i/\mathrm{d}u \ . \tag{72}$$

The relevant Fermat's variational principle is as follows:

$$\delta \int_S^R V^{-1}(x_k)(x'_i x'_i)^{1/2}\mathrm{d}u = 0 \ , \qquad x'_i = \mathrm{d}x_i/\mathrm{d}u \ . \tag{73}$$

and the Euler–Lagrange equations read

$$\frac{\mathrm{d}}{\mathrm{du}}(V^{-1}(x_k)D^{-1}x_i') = D\frac{\partial}{\partial x_i}\left(\frac{1}{V(x_k)}\right) , \quad D = (x_k' x_k')^{1/2} . \tag{74}$$

Note that $D = V$ for $u = \tau$, and the Euler–Lagrange equations read:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{1}{V^2(x_k)}\dot{x}_i\right) - V(x_k)\frac{\partial}{\partial x_i}\left(\frac{1}{V(x_k)}\right) = 0 , \quad \dot{x}_i = \mathrm{d}x_i/\mathrm{d}\tau . \tag{75}$$

We shall now calculate Hessian determinant (23). We obtain

$$\partial^2\mathcal{L}^{(1)}/\partial x_i'\partial x_j' = (\delta_{ij}D^2 - x_i'x_j')/VD^3 ,$$

so that

$$\det(\partial^2\mathcal{L}^{(1)}/\partial x_i'\partial x_j') = V^{-3}D^{-9}\begin{pmatrix} {x_2'}^2 + {x_3'}^2 & x_1'x_2' & x_1'x_3' \\ x_1'x_2' & {x_1'}^2 + {x_3'}^2 & x_2'x_3' \\ x_1'x_3' & x_2'x_3' & {x_1'}^2 + {x_2'}^2 \end{pmatrix} = 0 . \tag{76}$$

As the Hessian determinant vanishes, the dual Legendre transform cannot be used to determine Hamiltonian $\mathcal{H}^{(1)}(x_i, p_i)$ from Lagrangian (72) and to write the canonical Hamiltonian equations.

For this reason, we shall introduce the second–degree Lagrangian $\mathcal{L}^{(2)}$. We use $u = \tau$, where τ is the travel time along the curves, and use (34)

$$\mathcal{L}^{(2)}(x_k, \dot{x}_k) = \tfrac{1}{2}V^{-2}(x_k)(\dot{x}_i\dot{x}_i) , \quad \dot{x}_i = \mathrm{d}x_i/\mathrm{d}\tau . \tag{77}$$

Variational principle (40) then reads

$$\delta\int_S^R V^{-2}(x_k)(\dot{x}_i\dot{x}_i)\mathrm{d}\tau = 0 . \tag{78}$$

The corresponding Euler–Lagrange equations are

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{2}{V^2(x_k)}\dot{x}_i\right) - (\dot{x}_i\dot{x}_i)\frac{\partial}{\partial x_i}\left(\frac{1}{V^2(x_k)}\right) = 0 . \tag{79}$$

If we take into account $\dot{x}_i\dot{x}_i = V^2$, we can see that (79) is fully equivalent to (75). Thus, Fermat's variational principles (73) and (78) yield the same extremals, even though Fermat's variational principle in (78) is not constructed in the conventional way of (73).

It is simple to determine the Hessian determinant for the second–degree Lagrangian $\mathcal{L}^{(2)}$. From (77), we obtain

$$\frac{\partial^2\mathcal{L}^{(2)}(x_k, \dot{x}_k)}{\partial\dot{x}_i\partial\dot{x}_j} = V^{-2}(x_k)\delta_{ij} ,$$

so that

$$\det\left(\frac{\partial^2\mathcal{L}^{(2)}(x_k, \dot{x}_k)}{\partial\dot{x}_i\partial\dot{x}_j}\right) = V^{-6} \neq 0 . \tag{80}$$

Thus, the dual Legendre transform can be used to compute the relevant second–degree Hamiltonian and find the Hamiltonian canonical equations. Before we do this, we only note that the second–degree Lagrangian (77) is constant along the whole extremal and equals $\frac{1}{2}$,

$$\mathcal{L}^{(2)}(x_k, \dot{x}_k) = \tfrac{1}{2} \; , \tag{81}$$

see (35). This also immediately follows from (77), if we take into account that $\dot{x}_i \dot{x}_i = V^2$.

Using (41), we obtain

$$p_i = \frac{\partial \mathcal{L}^{(2)}}{\partial \dot{x}_i} = V^{-2} \dot{x}_i \; . \tag{82}$$

This can be simply inverted to yield

$$\dot{x}_i = V^2 p_i \; . \tag{83}$$

The second–degree Hamiltonian $\mathcal{H}^{(2)}(x_k, p_k)$ is then obtained by the Legendre transform (41)

$$\mathcal{H}^{(2)}(x_k, p_k) = [\dot{x}_i p_i - \mathcal{L}^{(2)}(x_k, \dot{x}_k)]_{\dot{x}_i = V^2 p_i} = V^2 p_i p_i - \tfrac{1}{2} V^2 p_i p_i \; .$$

Thus,

$$\mathcal{H}^{(2)}(x_k, p_k) = \tfrac{1}{2} V^2(x_k) p_i p_i \; , \tag{84}$$

see also (42), (77) and (83). The second–degree Hamiltonian is constant along the whole extremal,

$$\mathcal{H}^{(2)}(x_k, p_k) = \tfrac{1}{2} \; ,$$

see (53). As we know, $V^2(x_k) p_i p_i$ is the eigenvalue $G(x_k, p_k)$ of the Christoffel matrix for isotropic media, $G(x_k, p_k) = V^2(x_k) p_i p_i$. Consequently,

$$G(x_k, p_k) = V^2 p_i p_i = 1 \; . \tag{85}$$

along the whole extremal. This represents the eikonal equation for isotropic media.

The Hamiltonian ray tracing system (45) for isotropic media in Cartesian coordinates then reads

$$\frac{\mathrm{d}x_i}{\mathrm{d}\tau} = V^2 p_i \; , \quad \frac{\mathrm{d}p_i}{\mathrm{d}\tau} = -\frac{1}{2} p_k p_k \frac{\partial V^2}{\partial x_i} \; .$$

Using (85), we obtain $p_k p_k = 1/V^2$, so that

$$\frac{\mathrm{d}x_i}{\mathrm{d}\tau} = V^2 p_i \; , \quad \frac{\mathrm{d}p_i}{\mathrm{d}\tau} = -\frac{\partial \ln V}{\partial x_i} \; . \tag{86}$$

This corresponds fully to *Červený* (*2001,* Equation (3.1.19)).

It is possible to see from (77) and (81) that group velocity $\mathcal{U} = |\vec{\mathcal{U}}| = (\dot{x}_i \dot{x}_i)^{1/2}$ equals V and does not depend on the direction of propagation. The group velocity surface is spherical, with radius V. Similarly, (85) shows that $|\vec{p}| = (p_i p_i)^{1/2} = 1/V$ and that it does not depend on the direction. The slowness surface is spherical, with radius $1/V$ and with constant Gaussian curvature $K^S = V^2$. As $\det(g^{mn}) = V^6$, Equation (66) is satisfied. Equation (83) shows that group velocity vector $\vec{\mathcal{U}}$ and slowness vector $\vec{p}$ are parallel in isotropic media, and that $\vec{\mathcal{U}} \cdot \vec{p} = 1$.

5.2. General curvilinear coordinates

We shall now return briefly to curvilinear coordinates. In isotropic media, the relations between second–degree Lagrangians and second–degree Hamiltonians are simple, even in general curvilinear coordinates.

We shall introduce metric tensor $G_{ij}(x^k)$ in standard Riemannian space, in which the distance is measured by length. In isotropic media, the relation between metric tensor G_{ij} and wave propagation metric tensor g_{ij}, introduced by (9), is as follows:

$$g_{ij}(x^k, \dot{x}^k) = V^{-2}(x^k) G_{ij}(x^k) \ . \tag{87}$$

All relations derived in Section 3.1 simplify in this case. The second–degree Lagrangian reads, see (34):

$$\mathcal{L}^{(2)}(x^k, \dot{x}^k) = \tfrac{1}{2} V^{-2}(x^k) G_{ij}(x^k) \dot{x}^i \dot{x}^j \ . \tag{88}$$

The relevant Euler–Lagrange equations (38) are as follows:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(V^{-2} G_{ij} \dot{x}^j) - \frac{1}{2} \dot{x}^k \dot{x}^l \frac{\partial}{\partial x^i}(V^{-2} G_{kl}) = 0 \ . \tag{89}$$

For p_i, we can use (41) to obtain

$$p_i = \frac{\partial \mathcal{L}^{(2)}}{\partial \dot{x}^i} = V^{-2}(x^k) G_{ij}(x^k) \dot{x}^j \ . \tag{90}$$

This can be simply inverted for $\dot{x}^j$. We take into account that $G_{ij} G^{ik} = \delta_j^k$, where G^{ik} are the contravariant components of the metric tensor. Then

$$\dot{x}^j = V^2(x^k) G^{ji}(x^k) p_i \ . \tag{91}$$

The second–degree Lagrangian $\mathcal{L}^{(2)}$ can then be expressed in terms of p_k, see (88) and (91):

$$\mathcal{L}^{(2)} = \tfrac{1}{2} V^2 p_j p_k G^{jk} \ . \tag{92}$$

The second–degree Hamiltonian $\mathcal{H}^{(2)}(x^k, p_k)$ is then obtained from (42) and (92),

$$\mathcal{H}^{(2)}(x^k, p_k) = \tfrac{1}{2} V^2(x^k) G^{ij}(x^k) p_i p_j \ . \tag{93}$$

We remind the reader that $\mathcal{L}^{(2)}(x^k, \dot{x}^k) = \frac{1}{2}$ and $\mathcal{H}^{(2)}(x^k, p_k) = \frac{1}{2}$ along the whole ray, see (35) and (53). This yields, along the whole ray,

$$G_{ij}(x^k) \dot{x}^i \dot{x}^j = V^2(x^k) \ , \quad G^{ij}(x^k) p_i p_j = V^{-2}(x^k) \ . \tag{94}$$

Finally, the Hamiltonian ray tracing system is as follows:

$$\frac{\mathrm{d}x^i}{\mathrm{d}\tau} = \frac{\partial \mathcal{H}^{(2)}}{\partial p_i} = V^2 G^{ik} p_k \ ,$$

$$\frac{\mathrm{d}p_i}{\mathrm{d}\tau} = -\frac{\partial \mathcal{H}^{(2)}}{\partial x^i} = -\frac{1}{2} G^{jk} p_j p_k \frac{\partial V^2}{\partial x^i} - \frac{1}{2} V^2 p_j p_k \frac{\partial G^{jk}}{\partial x^i} \ .$$

As $G^{jk} p_j p_k \partial V^2 / \partial x^i = V^{-2}(x^k) \partial V^2 / \partial x^i = 2 \partial \ln V / \partial x^i$, we obtain

$$\frac{\mathrm{d}x^i}{\mathrm{d}\tau} = V^2 G^{ik} p_k \ , \quad \frac{\mathrm{d}p_i}{\mathrm{d}\tau} = -\frac{\partial \ln V}{\partial x^i} - \frac{1}{2} V^2 p_j p_k \frac{\partial G^{jk}}{\partial x^i} \ . \tag{95}$$

This corresponds fully to the Hamiltonian ray equations (3.5.54) in *Červený (2001)* for $n = 0$, if we take into account that

$$\lim_{n\to 0}\left(n^{-1}\,\frac{\partial(1/V)^n}{\partial x^i}\right) = -\frac{\partial \ln V}{\partial x^i}\ .$$

Note that the second–degree Lagrangian for isotropic media and curvilinear coordinates was also used by *Babich and Buldyrev (1972)*. The authors applied the Legendre transformation to it to derive the relevant second–degree Hamiltonian.

6. ELLIPSOIDAL ANISOTROPY. CARTESIAN COORDINATES

In the next analytical example, we shall consider ellipsoidal anisotropy. Ellipsoidal anisotropy has been traditionally introduced in the p_i–domain, see *Burridge, Chadwick and Norris (1993)*, *Mensch and Farra (1999)*, *Ettrich, Gajewski and Kashtan (2001)*. For this reason, we shall choose the reverse approach to that in Section 5: We shall start from a known second–degree Hamiltonian, and derive the relevant second–degree Lagrangian. The Hamiltonian ray tracing equations are obtained automatically in this case, as the second–degree Hamiltonian is assumed to be known. Using the dual Legendre transform, we shall derive the Lagrangian and the Lagrangian ray tracing system. In this case, all computations can be performed analytically.

We shall consider general curvilinear coordinates x^i and assume that the second–degree Hamiltonian $\mathcal{H}^{(2)}(x^k, p_k)$ is given by the relation

$$\mathcal{H}^{(2)}(x^k, p_k) = \tfrac{1}{2}G(x^k, p_k)\ , \quad G(x^k, p_k) = a^{ij}(x^k)p_i p_j\ . \tag{96}$$

Presumably, $a^{ij}(x^k)$ does not depend on $\dot{x}^k$, and is positive definite. Using (54) and (61), we immediately see that the contravariant components of the wave propagation metric tensor g^{ij} are given by the relation

$$g^{ij}(x^k, \dot{x}^k) = a^{ij}(x^k)\ . \tag{97}$$

Consequently, g^{ij} does not depend on $\dot{x}^k$, and in this case the standard Riemannian space may be considered instead of the Finslerian space, and we can use $g^{ij}(x^k)$ instead of $g^{ij}(x^k, \dot{x}^k)$. Slowness surface $\mathcal{H}^{(2)}(x^k, p_k) = \frac{1}{2}$ is described by the relation

$$a^{ij}(x^k)p_i p_j = 1\ . \tag{98}$$

As $a^{ij}(x^k)$ is positive definite, the eigenvalues of $g^{ij}(x^k)$ are always real–valued and positive. We denote them by $A^2(x^k)$, $B^2(x^k)$ and $C^2(x^k)$. In Cartesian coordinates x^i, for fixed x^i, the slowness surface (98) is an ellipsoid, with half–axes $1/A(x^k)$, $1/B(x^k)$ and $1/C(x^k)$. For this reason, we speak of ellipsoidal anisotropy.

Hamiltonian ray tracing system (45) is obtained from (96),

$$\frac{\mathrm{d}x^i}{\mathrm{d}\tau} = a^{in}(x^k)p_n\ , \quad \frac{\mathrm{d}p_i}{\mathrm{d}\tau} = -\frac{1}{2}\frac{\partial a^{mn}(x^k)}{\partial x^i}p_m p_n\ . \tag{99}$$

Thus, the contravariant components of group velocity vector $\vec{\mathcal{U}}$ are given by relations

$$\mathcal{U}^i(x^k) = a^{in}(x^k)p_n\ . \tag{100}$$

The inversion of (100) yields

$$p_j = a_{ij}(x^k)\mathcal{U}^i \ . \tag{101}$$

Now we shall use Legendre transform (41) to compute the second–degree Lagrangian, $\mathcal{L}^{(2)}(x^k, \dot{x}^k)$, from the second–degree Hamiltonian $\mathcal{H}^{(2)}(x^k, p_k)$. We can safely use the Legendre transform, as the Hessian determinant is different from zero for positive definite $g^{ij}(x^k)$. Using (43), (96) and (101), we obtain

$$\mathcal{L}^{(2)}(x^k, \dot{x}^k) = \tfrac{1}{2} a_{ij}(x^k)\dot{x}^i \dot{x}^j \ . \tag{102}$$

Thus, in Cartesian coordinates x^i, for fixed x^i, group velocity surface $\mathcal{L}^{(2)}(x^k, \dot{x}^k) = \frac{1}{2}$ is again ellipsoidal,

$$a_{ij}(x^k)\dot{x}^i \dot{x}^j = 1 \ , \tag{103}$$

with half–axes $A(x^k)$, $B(x^k)$ and $C(x^k)$, reciprocal to the half–axes of the slowness ellipsoid.

Finally, the Euler–Lagrange equations (38) corresponding to the second–degree Lagrangian $\mathcal{L}^{(2)}(x^k, \dot{x}^k)$, given by (102), read

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(a_{ij}(x^k)\dot{x}^j) - \frac{1}{2}\frac{\partial a_{nj}(x^k)}{\partial x^i}\dot{x}^n \dot{x}^j = 0 \ . \tag{104}$$

This represents the Lagrangian ray tracing system. The Lagrangian ray tracing system (104) also simply follows from the Hamiltonian ray tracing system (99).

Both the Hamiltonian ray tracing system (99) and the Lagrangian ray tracing system (104) are valid in any curvilinear coordinates, including the Cartesian coordinates.

It is not difficult to verify analytically that the general equation (62) is satisfied in our case. From (96) and (102), we obtain

$$\frac{\partial^2 \mathcal{H}^{(2)}(x^k, p_k)}{\partial p_i \partial p_j} = a^{ij}(x^k) \ , \quad \frac{\partial^2 \mathcal{L}^{(2)}(x^k, \dot{x}^k)}{\partial \dot{x}^i \partial \dot{x}^j} = a_{ij}(x^k) \ . \tag{105}$$

This shows that (62) is satisfied.

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