Paraxial approximation of the polarization vectors in the isotropic ray theory

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Summary

The equations for the linear paraxial approximation of the polarization vectors and for the variation of the polarization vectors with a velocity perturbation are presented.

Introduction

The equations for the linear paraxial approximation of the polarization vectors and for the variation of the polarization vectors with a velocity perturbation were derived by Coates & Chapman (1990). Here the equations are derived in more detail, and the final equations presented make it more obvious which numerical quadratures along the central ray are required.

Notation and coordinates

In the case of component notation, the capital–letter indices take values $K, L, ... = 1, 2$; the lower–case indices take values $k, l, ... = 1, 2, 3$. The Einstein summation convention is used with respect to repeated subscripts. Let us introduce three coordinate systems: (a) Cartesian coordinates x_i. (b) Ray coordinates γ_i, where γ_1, γ_2 are the take-off ray parameters and γ_3 is an independent variable along rays (e.g., travel time τ, arclength s, $\sigma = \int v\mathrm{d}s$, or another parameter). (c) Ray–centred coordinates q_i connected with the central ray, where q_I are Cartesian coordinates in plane $q_3 = constant$ perpendicular to the central ray, and q_3 is the arclength s along the central ray.

Polarization vectors

We denote by H_{iK} two mutually perpendicular unit vectors, perpendicular to the ray (polarization vectors in the case of an S wave). We require them not to rotate with respect to the ray. Similarly, we denote by H_{i3},

$$H_{i3} = p_i v \quad , \tag{1}$$

the unit vector tangent to the ray (polarization vector in the case of a P wave). At the central ray,

$$H_{ik} = \frac{\partial x_i}{\partial q_k} \quad . \tag{2}$$

Geometrical spreading matrix and the travel–time derivatives

We denote by Q_{im} the matrix of geometrical spreading,

$$Q_{im} = \frac{\partial q_i}{\partial \gamma_m} \quad , \tag{3}$$

and by P_{im} the matrix

$$P_{im} = \frac{\partial^2 \tau}{\partial \gamma_m \partial q_i} = \frac{\partial}{\partial \gamma_m}\left(p_j \frac{\partial x_j}{\partial q_i}\right) = \frac{\partial p_j}{\partial \gamma_m}\frac{\partial x_j}{\partial q_i} + p_j \frac{\partial q_l}{\partial \gamma_m}\frac{\partial^2 x_j}{\partial q_l \partial q_i} = \frac{\partial p_j}{\partial \gamma_m}\frac{\partial x_j}{\partial q_i} + p_j \frac{\partial q_3}{\partial \gamma_m}\frac{\partial^2 x_j}{\partial q_3 \partial q_i} + \delta_{i3} p_j \frac{\partial q_l}{\partial \gamma_m}\frac{\partial^2 x_j}{\partial q_3 \partial q_l} \ , \tag{4}$$

where δ_{ij} is the Kronecker delta (components of the identity matrix). At the central ray, relation (4) yields

$$P_{im} = \frac{\partial p_j}{\partial \gamma_m} H_{ji} + p_j \frac{\partial H_{ji}}{\partial s} Q_{3m} + \delta_{i3} p_j \frac{\partial H_{jl}}{\partial s} Q_{lm} \quad . \tag{5}$$

Matrices Q_{im} and P_{im} are related as

$$P_{im} = M_{ij} Q_{jm} \quad , \tag{6}$$

where

$$M_{ij} = \frac{\partial^2 \tau}{\partial q_i \partial q_j} \tag{7}$$

is the matrix of the second travel–time derivatives in the ray–centred coordinates.

Equation for the polarization vectors

The very simple form of the equation to trace the polarization vectors along rays reads (Coates & Chapman 1990, equation C3)

$$\frac{\partial H_{ik}}{\partial \gamma_3} = W_{ij} H_{jk} \quad , \tag{8}$$

where

$$W_{ij}=\frac{\partial p_i v}{\partial\gamma_3}p_j v-\frac{\partial p_j v}{\partial\gamma_3}p_i v=\frac{\partial p_i}{\partial\gamma_3}p_j v^2-\frac{\partial p_j}{\partial\gamma_3}p_i v^2 \quad . \tag{9}$$

For the sake of conciseness, we shall express similar equations in a form analogous to

$$W_{ij}=\left[\frac{\partial p_i v}{\partial\gamma_3}p_j v\right]-\left[\dots_{i\leftrightarrow j}\right] \quad . \tag{10}$$

Note that the derivative with respect to γ_3 in (8) is really a partial derivative, because it is applied for γ_1, γ_2 constant.

Paraxial polarization vectors

We are interested in the derivatives

$$\frac{\partial H_{ik}}{\partial q_n}=\frac{\partial H_{ik}}{\partial\gamma_m}Q^{-1}_{mn} \tag{11}$$

of the polarization vectors in ray–centred coordinates. Equation (8) together with the ray tracing equations yields

$$\boxed{\frac{\partial H_{ik}}{\partial q_3}=\left(\frac{\partial p_i}{\partial s}p_j v^2-\frac{\partial p_j}{\partial s}p_i v^2\right)H_{jk}=\left(-\frac{\partial v}{\partial x_i}p_j+\frac{\partial v}{\partial x_j}p_i\right)H_{jk} \quad ,} \tag{12}$$

where s is the arclength along the ray. Since H_{i3} is a unit vector perpendicular to the wavefront,

$$\boxed{\frac{\partial H_{i3}}{\partial q_N}=H_{iL}M_{LN}v \quad ,} \tag{13}$$

where M_{LN} are the second travel–time derivatives in the ray–centred coordinates, see (7). Unit vectors H_{i1}, H_{i2} are mutually perpendicular and both of them are perpendicular to H_{i3}. Their derivatives can thus be expressed as

$$\boxed{\frac{\partial H_{iK}}{\partial q_N}=\epsilon_{KL}H_{iL}\Omega_N-H_{i3}M_{KN}v \quad ,} \tag{14}$$

where $\epsilon_{11}=\epsilon_{22}=0$, $\epsilon_{12}=-\epsilon_{21}=1$ and

$$\Omega_N=\frac{\partial H_{i1}}{\partial q_N}H_{i2}=-\frac{\partial H_{i2}}{\partial q_N}H_{i1} \quad . \tag{15}$$

Note that equation (14) is equivalent to equation C9 of Coates & Chapman (1990). We now have to find the equations for Ω_1 and Ω_2.

Equations for the paraxial polarization vectors

The differentiation of (8) with respect to ray coordinates γ_m yields

$$\frac{\partial}{\partial\gamma_3}\frac{\partial H_{ik}}{\partial\gamma_m}=W_{ij}\frac{\partial H_{jk}}{\partial\gamma_m}+\frac{\partial W_{ij}}{\partial\gamma_m}H_{jk} \quad . \tag{16}$$

Assume that H_{ik} is a unitary 3×3 matrix satisfying equation (8). The solution of (16) may then be expressed in the form

$$\begin{aligned}\frac{\partial H_{nl}}{\partial\gamma_m}(\gamma_3)&=H_{nk}(\gamma_3)\left[H^{-1}_{ki}(\gamma_3^0)\frac{\partial H_{il}}{\partial\gamma_m}(\gamma_3^0)+\int_{\gamma_3^0}^{\gamma_3}\mathrm{d}\gamma_3\,H^{-1}_{ki}\frac{\partial W_{ij}}{\partial\gamma_m}H_{jl}\right]\\&=H_{nk}(\gamma_3)\left[H_{ik}(\gamma_3^0)\frac{\partial H_{il}}{\partial\gamma_m}(\gamma_3^0)+\int_{\gamma_3^0}^{\gamma_3}\mathrm{d}\gamma_3\,H_{ik}\frac{\partial W_{ij}}{\partial\gamma_m}H_{jl}\right] \quad .\end{aligned} \tag{17}$$

Note that equations (16) and (17) are identical to equations C5 and C6 of Coates & Chapman (1990), who introduce 3×3 propagator matrix $E_{ni}(\gamma_3,\gamma_3^0)=H_{nk}(\gamma_3)H^{-1}_{ki}(\gamma_3^0)$ by equation C4.

We shall now consider the integral on the right–hand side of this relation,

$$\begin{aligned}\int\mathrm{d}\gamma_3\,H_{ik}\frac{\partial W_{ij}}{\partial\gamma_m}H_{jl}&=\left[\int\mathrm{d}\gamma_3\,H_{ik}H_{jl}\frac{\partial}{\partial\gamma_m}\left(\frac{\partial p_i v}{\partial\gamma_3}p_j v\right)\right]-\left[\dots_{k\leftrightarrow l}\right]\\&=\left[\int\mathrm{d}\gamma_3\,H_{ik}H_{jl}\left(p_j v\frac{\partial}{\partial\gamma_3}\frac{\partial p_i v}{\partial\gamma_m}+\frac{\partial p_i v}{\partial\gamma_3}\frac{\partial p_j v}{\partial\gamma_m}\right)\right]-\left[\dots_{k\leftrightarrow l}\right],\end{aligned} \tag{18}$$

see (10). Since $p_i v = H_{i3}$, see (1), and $H_{i3}\frac{\partial p_i v}{\partial \gamma_m} = 0$,

$$\int \mathrm{d}\gamma_3 H_{ik}\frac{\partial W_{ij}}{\partial \gamma_m}H_{jl} = \left[\int \mathrm{d}\gamma_3 \left(H_{ik}\delta_{3l}\frac{\partial}{\partial \gamma_3}\frac{\partial p_i v}{\partial \gamma_m} + H_{iK}\delta_{Kk}\frac{\partial p_i v}{\partial \gamma_3}\frac{\partial p_j v}{\partial \gamma_m}H_{jL}\delta_{Ll}\right)\right] - \left[\ldots_{k\leftrightarrow l}\right] \quad , \tag{19}$$

Since (19) is skew (antisymmetric) in indices k and l,

$$\int \mathrm{d}\gamma_3 H_{ik}\frac{\partial W_{ij}}{\partial \gamma_m}H_{jl} = \left[\int \mathrm{d}\gamma_3 \left(H_{iK}\delta_{Kk}\delta_{3l}\frac{\partial}{\partial \gamma_3}\frac{\partial p_i v}{\partial \gamma_m} + H_{iK}\delta_{Kk}\frac{\partial p_i v}{\partial \gamma_3}\frac{\partial p_j v}{\partial \gamma_m}H_{jL}\delta_{Ll}\right)\right] - \left[\ldots_{k\leftrightarrow l}\right] \quad . \tag{20}$$

Since $\frac{\partial H_{iK}}{\partial \gamma_n}\frac{\partial p_i v}{\partial \gamma_m} = 0$ and $H_{iK}\frac{\partial p_i v}{\partial \gamma_3} = -v^{-1}\frac{\partial s}{\partial \gamma_3}\frac{\partial v}{\partial q_K}$,

$$\int \mathrm{d}\gamma_3 H_{ik}\frac{\partial W_{ij}}{\partial \gamma_m}H_{jl} = \left[\int \mathrm{d}\gamma_3 \left(\delta_{Kk}\delta_{3l}\frac{\partial}{\partial \gamma_3}\left(\frac{\partial p_i v}{\partial \gamma_m}H_{iK}\right) - \delta_{Kk}v^{-1}\frac{\partial s}{\partial \gamma_3}V_K\frac{\partial p_j v}{\partial \gamma_m}H_{jL}\delta_{Ll}\right)\right] - \left[\ldots_{k\leftrightarrow l}\right] \quad , \tag{21}$$

which reads

$$\int \mathrm{d}\gamma_3 H_{ik}\frac{\partial W_{ij}}{\partial \gamma_m}H_{jl} = \left[\delta_{Kk}\delta_{3l}v\frac{\partial p_i}{\partial \gamma_m}H_{iK} - \delta_{Kk}\delta_{Ll}\int \mathrm{d}s V_K\frac{\partial p_i}{\partial \gamma_m}H_{iL}\right] - \left[\ldots_{k\leftrightarrow l}\right] \quad . \tag{22}$$

Here

$$V_k = \frac{\partial v}{\partial x_i}H_{ik} \tag{23}$$

is the velocity gradient in the ray–centred coordinate system at the central ray. Insertion of (12) into (5) yields

$$P_{km} = \frac{\partial p_j}{\partial \gamma_m}H_{jk} - 2v^{-2}\delta_{k3}V_3Q_{3m} + v^{-2}V_kQ_{3m} + v^{-2}\delta_{k3}V_lQ_{lm} \quad . \tag{24}$$

Considering (24), relation (22) takes the form

$$\begin{aligned}\int \mathrm{d}\gamma_3 H_{ik}\frac{\partial W_{ij}}{\partial \gamma_m}H_{jl} &= \left[\delta_{Kk}\delta_{3l}v\left(P_{Km} - v^{-2}V_KQ_{3m}\right) - \delta_{Kk}\delta_{Ll}\int \mathrm{d}s V_K P_{Lm}\right] - \left[\ldots_{k\leftrightarrow l}\right] \\ &= \left(\delta_{Kk}\delta_{3l} - \delta_{3k}\delta_{Kl}\right)\left(vP_{Lm} - v^{-1}V_LQ_{3m}\right) - \delta_{Kk}\delta_{Ll}\epsilon_{KL}\int \mathrm{d}s V_I\epsilon_{IJ}P_{Jm} \quad .\end{aligned} \tag{25}$$

Equations (11) and (17) yield

$$H_{iK}(\gamma_3)\frac{\partial H_{iL}}{\partial q_n}(\gamma_3) = \left[H_{iK}(\gamma_3^0)\frac{\partial H_{iL}}{\partial \gamma_m}(\gamma_3^0) + \int_{\gamma_3^0}^{\gamma_3}\mathrm{d}\gamma_3 H_{iK}\frac{\partial W_{ij}}{\partial \gamma_m}H_{jL}\right]Q^{-1}_{mn}(\gamma_3) \, . \tag{26}$$

Inserting (25) into (26), we arrive at

$$H_{iK}(\gamma_3)\frac{\partial H_{iL}}{\partial q_n}(\gamma_3) = \left[H_{iK}(\gamma_3^0)\frac{\partial H_{iL}}{\partial \gamma_m}(\gamma_3^0) - \epsilon_{KL}\int_{s(\gamma_3^0)}^{s(\gamma_3)}\mathrm{d}s V_I\epsilon_{IJ}P_{Jm}\right]Q^{-1}_{mn}(\gamma_3) \, . \tag{27}$$

Inserting this into (15) and applying (11) for $\gamma_3 = \gamma_3^0$, we obtain this expression for Ω_N:

$$\Omega_N(\gamma_3) = \left[H_{i2}(\gamma_3^0)\frac{\partial H_{i1}}{\partial q_n}(\gamma_3^0)Q_{nm}(\gamma_3^0) + \int_{s(\gamma_3^0)}^{s(\gamma_3)}\mathrm{d}s V_I\epsilon_{IJ}P_{Jm}\right]Q^{-1}_{mN}(\gamma_3) \quad . \tag{28}$$

Since

$$Q_{J3} = 0, \quad P_{J3} = 0, \quad H_{iK}\frac{\partial H_{iL}}{\partial q_3} = 0, \tag{29}$$

which follows from (3), (4) and (8), equation (28) becomes

$$\boxed{\Omega_N(\gamma_3) = \left[H_{i2}(\gamma_3^0)\frac{\partial H_{i1}}{\partial q_J}(\gamma_3^0)Q_{JM}(\gamma_3^0) + \int_{s(\gamma_3^0)}^{s(\gamma_3)}\mathrm{d}s V_I\epsilon_{IJ}P_{JM}\right]Q^{-1}_{MN}(\gamma_3) \quad ,} \tag{30}$$

which is equivalent to equation C10 of Coates & Chapman (1990). Here Q_{MN} and P_{MN} are the standard 2×2 paraxial ray tracing matrices in the ray–centred coordinate system. In order to evaluate the paraxial changes of the S–wave polarization vectors, we need to compute two integrals

$$\int_{s(\gamma_3^0)}^{s(\gamma_3)}\mathrm{d}s\left(V_1P_{2M} - V_2P_{1M}\right) \tag{31}$$

along the ray.

Paraxial polarization vectors

Variation of the polarization vectors with a velocity perturbation

We denote by δA the variation of quantity A with respect to any of the model parameters. During the velocity perturbations, we keep the coordinate systems fixed. The perturbations (i.e. the variations with respect to the model parameters) then have the properties of partial derivatives, and commute with the partial derivatives with respect to the coordinates. Equations (16)–(22) thus remain valid if we replace partial derivative $\frac{\partial}{\partial\gamma_m}$ by variation δ with respect to a model parameter. With this substitution, equations (17) and (22) yield

$$\delta H_{nl}(\gamma_3) = H_{nk}(\gamma_3)\Bigg[H_{ik}(\gamma_3^0)\delta H_{il}(\gamma_3^0) + (\delta_{Kk}\delta_{3l}-\delta_{3k}\delta_{Kl})\, v(\gamma_3)\big[H_{iK}(\gamma_3)\delta p_i(\gamma_3) - H_{iK}(\gamma_3^0)\delta p_i(\gamma_3^0)\big] - (\delta_{Kk}\delta_{Ll}-\delta_{Lk}\delta_{Kl})\int_{s(\gamma_3^0)}^{s(\gamma_3)} \mathrm{d}s V_K H_{iL}\delta p_i\Bigg]. \qquad (32)$$

This equation may be rearranged to read

$$\boxed{\delta H_{nl}(\gamma_3) = H_{nk}(\gamma_3)\Bigg[\delta_{Kk}\delta_{Ll}H_{iK}(\gamma_3^0)\delta H_{iL}(\gamma_3^0) + (\delta_{Kk}\delta_{3l}-\delta_{3k}\delta_{Kl})\times v(\gamma_3)H_{iK}(\gamma_3)\delta p_i(\gamma_3) - \epsilon_{KL}\delta_{Kk}\delta_{Ll}\int_{s(\gamma_3^0)}^{s(\gamma_3)}\mathrm{d}s V_I\epsilon_{IJ}H_{iJ}\delta p_i\Bigg]. \qquad (33)}$$

In order to evaluate the perturbation of the S–wave polarization vectors, we need to compute the integral

$$\int_{s(\gamma_3^0)}^{s(\gamma_3)} \mathrm{d}s\,(V_1\delta P_2 - V_2\delta P_1) \qquad (34)$$

along the ray. Here

$$\delta P_k = H_{ik}\delta p_i \qquad (35)$$

is the slowness vector perturbation in ray–centred coordinates.

Conclusions

Equations (12), (13) and (14) with (30) may be used to calculate the first derivatives of the polarization vectors. Two numerical quadratures (31) are required to calculate the derivatives of the S–wave polarization vectors perpendicularly to the ray.

Equation (33) may be used to calculate the variations of the polarization vectors with velocity perturbations. Equation (33) takes a very simple form for the variation of the P–wave polarization vector or for the ray–tangent component of the variation. In addition to the quadratures for the variation of the central ray (Farra & Madariaga 1987), one numerical quadrature (34) is required for the ray–normal components of the variation of the S–wave polarization vectors per each perturbation.

The numerical quadratures can be calculated after ray tracing, along rays stored together with the polarization vectors and the paraxial ray propagator matrix in disk files (Červený, Klimeš & Pšenčík 1988, section 7.26).

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