Rotationally invariant bianisotropic electromagnetic medium

Luděk Klimeš

Department of Geophysics, Faculty of Mathematics and Physics, Charles University, Ke Karlovu 3, 121 16 Praha 2, Czech Republic, http://sw3d.cz/staff/klimes.htm

Summary

We determine the general form of the rotationally invariant constitutive tensor of a bianisotropic medium. In the coordinate system attached to the symmetry axis, the rotationally invariant constitutive tensor is described by twelve parameters. It is thus described by four additional parameters in comparison with an uniaxial constitutive tensor.

Keywords

Electromagnetic waves, bianisotropic media, constitutive tensor.

1. Introduction

In this paper, we determine the general form of the rotationally invariant constitutive tensor of a bianisotropic medium. We suppose that a bianisotropic medium is invariant with respect to the rotation about a given symmetry axis. We calculate the derivative of the constitutive tensor with respect to the angle of rotation in Section 3. We put the derivative equal to zero, and obtain the system of equations for the elements of the constitutive tensor.

We express and solve these equations in the coordinate system attached to the symmetry axis in Section 4. We then determine the general form of a rotationally invariant constitutive tensor in general coordinates in Section 5.

We assume Cartesian coordinates with the unit metric tensor. The lower–case Roman indices take values 1, 2 and 3. The lower–case Greek indices take values 1, 2, 3 and 4. The Einstein summation over repetitive indices is used throughout the paper.

2. Frequency–domain constitutive relations

We assume the constitutive relations in the *Boys–Post representation* which express the dependence of the *electric displacement* D^j and *magnetic field strength* H_j on *electric field strength* E_j and *magnetic induction* B^j. In this paper, we consider just the *linear* constitutive relations in the Boys–Post representation.

The linear point constitutive relations without spatial dispersion but with possible time dispersion can be expressed in the frequency domain as (Weiglhofer, 2000, eq. 1.12; 2003, eq. 57)

$$D^i = \varepsilon^{ij} E_j + \alpha^i{}_j B^j \quad , \tag{1}$$

and (Weiglhofer, 2000, eq. 1.13, 2003, eq. 58)

$$H_i = \beta_i{}^j E_j + \mu^{-1}_{ij} B^j \quad . \tag{2}$$

Electric field strength E_j, magnetic induction B^j, electric displacement D^j, magnetic field strength H_j, *permittivity tensor* ε^{ij}, *inverse permeability tensor* μ^{-1}_{ij}, and *magnetoelectric tensors* $\alpha^i{}_j$ and $\beta_i{}^j$ may depend on spatial coordinates x^m and circular frequency ω.

We define *constitutive tensor* $\chi^{\alpha\beta\gamma\delta}$ (Post, 1962, eq. 6.12; 2003, eq. 27; Hehl & Obukhov, 2003, eq. D.1.9) by relations

$$\chi^{4i4j} = -\chi^{i44j} = -\chi^{4ij4} = \chi^{i4j4} = -\varepsilon^{ij} \quad , \tag{3}$$

$$\chi^{ij4k} = -\chi^{ijk4} = \varepsilon^{ijr}\beta_r{}^k \quad , \tag{4}$$

$$\chi^{4ikl} = -\chi^{i4kl} = -\alpha^i{}_s\varepsilon^{skl} \tag{5}$$

and

$$\chi^{ijkl} = \varepsilon^{ijr}\mu^{-1}_{rs}\varepsilon^{skl} \quad . \tag{6}$$

The constitutive tensor is skew with respect to its first pair of superscripts,

$$\chi^{\alpha\beta\gamma\delta} = -\chi^{\beta\alpha\gamma\delta} \quad , \tag{7}$$

and its last pair of superscripts,

$$\chi^{\alpha\beta\gamma\delta} = -\chi^{\alpha\beta\delta\gamma} \quad , \tag{8}$$

and thus has 36 independent components. Analogously to Voigt notation in elasticity, the constitutive tensor can be expressed as the 6×6 constitutive matrix. The 36 distinct components of the constitutive tensor read

$$\chi^{\alpha\beta\gamma\delta} = \begin{array}{c} \\ 41 \\ 42 \\ 43 \\ 23 \\ 31 \\ 12 \end{array} \begin{array}{c} \begin{array}{cccccc} 41 & 42 & 43 & 23 & 31 & 12 \end{array} \\ \begin{pmatrix} -\varepsilon^{11} & -\varepsilon^{12} & -\varepsilon^{13} & -\alpha^1{}_1 & -\alpha^1{}_2 & -\alpha^1{}_3 \\ -\varepsilon^{21} & -\varepsilon^{22} & -\varepsilon^{23} & -\alpha^2{}_1 & -\alpha^2{}_2 & -\alpha^2{}_3 \\ -\varepsilon^{31} & -\varepsilon^{32} & -\varepsilon^{33} & -\alpha^3{}_1 & -\alpha^3{}_2 & -\alpha^3{}_3 \\ \beta_1{}^1 & \beta_1{}^2 & \beta_1{}^3 & \mu^{-1}_{11} & \mu^{-1}_{12} & \mu^{-1}_{13} \\ \beta_2{}^1 & \beta_2{}^2 & \beta_2{}^3 & \mu^{-1}_{21} & \mu^{-1}_{22} & \mu^{-1}_{23} \\ \beta_3{}^1 & \beta_3{}^2 & \beta_3{}^3 & \mu^{-1}_{31} & \mu^{-1}_{32} & \mu^{-1}_{33} \end{pmatrix} \end{array} \quad , \tag{9}$$

see Post (1962, eq. 6.21). Constitutive relations (1) and (2) then can be expressed as

$$\begin{pmatrix} -D^1 \\ -D^2 \\ -D^3 \\ H_1 \\ H_2 \\ H_3 \end{pmatrix} = \begin{array}{c} \\ 41 \\ 42 \\ 43 \\ 23 \\ 31 \\ 12 \end{array} \begin{array}{c} \begin{array}{cccccc} 41 & 42 & 43 & 23 & 31 & 12 \end{array} \\ \begin{pmatrix} \chi^{4141} & \chi^{4142} & \chi^{4143} & \chi^{4123} & \chi^{4131} & \chi^{4112} \\ \chi^{4241} & \chi^{4242} & \chi^{4243} & \chi^{4223} & \chi^{4231} & \chi^{4212} \\ \chi^{4341} & \chi^{4342} & \chi^{4343} & \chi^{4323} & \chi^{4331} & \chi^{4312} \\ \chi^{2341} & \chi^{2342} & \chi^{2343} & \chi^{2323} & \chi^{2331} & \chi^{2312} \\ \chi^{3141} & \chi^{3142} & \chi^{3143} & \chi^{3123} & \chi^{3131} & \chi^{3112} \\ \chi^{1241} & \chi^{1242} & \chi^{1243} & \chi^{1223} & \chi^{1231} & \chi^{1212} \end{pmatrix} \end{array} \begin{pmatrix} E_1 \\ E_2 \\ E_3 \\ B^1 \\ B^2 \\ B^3 \end{pmatrix} \quad , \tag{10}$$

see Post (1962, eq. 6.21).

3. Derivative of the constitutive tensor with respect to the angle of rotation

Transformation matrix $R_{in}(\varphi,t_a)$ corresponding to the rotation of vectors about a given unit vector t_a by angle φ is an orthogonal matrix, with $R_{in}(0,t_a)=\delta_{in}$, where Kronecker delta δ_{in} represents the elements of the identity matrix, see Klimeš (2016, eq. 4). The derivative of the transformation matrix at $\varphi=0$ reads

$$\frac{\mathrm{d}R_{in}}{\mathrm{d}\varphi}(0,t_a) = -S_{in} \quad , \tag{11}$$

where

$$S_{in} = \varepsilon_{inr} t_r \tag{12}$$

(Klimeš, 2016, eq. 3). Here ε_{ijk} is the Levi–Civita symbol.

The rotated parts of the constitutive tensor read

$$\chi^{4j4l}(\varphi,t_a) = R_{jq}(\varphi,t_b)\,R_{ls}(\varphi,t_d)\,\chi^{4q4s} \quad , \tag{13}$$

$$\chi^{ij4l}(\varphi,t_a) = R_{ip}(\varphi,t_a)\,R_{jq}(\varphi,t_b)\,R_{ls}(\varphi,t_d)\,\chi^{pq4s} \quad , \tag{14}$$

$$\chi^{4jkl}(\varphi,t_a) = R_{jq}(\varphi,t_b)\,R_{kr}(\varphi,t_c)\,R_{ls}(\varphi,t_d)\,\chi^{4qrs} \quad , \tag{15}$$

$$\chi^{ijkl}(\varphi,t_a) = R_{ip}(\varphi,t_a)\,R_{jq}(\varphi,t_b)\,R_{kr}(\varphi,t_c)\,R_{ls}(\varphi,t_d)\,\chi^{pqrs} \quad , \tag{16}$$

where $\chi^{\alpha\beta\gamma\delta}$ without arguments is the non–rotated constitutive tensor. Rotated constitutive tensor $\chi^{\alpha\beta\gamma\delta}(\varphi,t_a)$ obviously satisfies relations (7)–(8) and can be expressed in the form of the 6×6 constitutive matrix.

The derivative $\frac{\mathrm{d}\chi^{\alpha\beta\gamma\delta}}{\mathrm{d}\varphi}(0,t_a)$ of constitutive tensor $\chi^{\alpha\beta\gamma\delta}(\varphi,t_a)$ with respect to the angle φ of rotation at $\varphi=0$ follows directly from transformation (13)–(16) with derivative (11),

$$\frac{\mathrm{d}\chi^{4j4l}}{\mathrm{d}\varphi}(0,t_a) = -S_{jn}\chi^{4n4l} - S_{ln}\chi^{4j4n} \quad , \tag{17}$$

$$\frac{\mathrm{d}\chi^{ij4l}}{\mathrm{d}\varphi}(0,t_a) = -S_{in}\chi^{nj4l} - S_{jn}\chi^{in4l} - S_{ln}\chi^{ij4n} \quad , \tag{18}$$

$$\frac{\mathrm{d}\chi^{4jkl}}{\mathrm{d}\varphi}(0,t_a) = -S_{jn}\chi^{4nkl} - S_{kn}\chi^{4jnl} - S_{ln}\chi^{4jkn} \quad , \tag{19}$$

$$\frac{\mathrm{d}\chi^{ijkl}}{\mathrm{d}\varphi}(0,t_a) = -S_{in}\chi^{njkl} - S_{jn}\chi^{inkl} - S_{kn}\chi^{ijnl} - S_{ln}\chi^{ijkn} \quad . \tag{20}$$

The derivative $\frac{\mathrm{d}\chi^{\alpha\beta\gamma\delta}}{\mathrm{d}\varphi}(0,t_a)$ of constitutive tensor $\chi^{\alpha\beta\gamma\delta}(\varphi,t_a)$ obviously satisfies relations (7)–(8) and can be expressed in the form of the 6×6 matrix, analogously to the constitutive matrix.

We put $\frac{\mathrm{d}\chi^{\alpha\beta\gamma\delta}}{\mathrm{d}\varphi}(0,t_a)=0$, and obtain the system of equations

$$S_{jn}\chi^{4n4l} + S_{ln}\chi^{4j4n} = 0 \tag{21}$$

$$S_{in}\chi^{nj4l} + S_{jn}\chi^{in4l} + S_{ln}\chi^{ij4n} = 0 \quad , \tag{22}$$

$$S_{jn}\chi^{4nkl} + S_{kn}\chi^{4jnl} + S_{ln}\chi^{4jkn} = 0 \quad , \tag{23}$$

$$S_{in}\chi^{njkl} + S_{jn}\chi^{inkl} + S_{kn}\chi^{ijnl} + S_{ln}\chi^{ijkn} = 0 \quad , \tag{24}$$

for the rotationally invariant non–symmetric constitutive tensor.

4. Coordinate system attached to the symmetry axis

We choose the coordinate system which third coordinate axis coincides with the symmetry axis. In this coordinate system, the symmetry vector reads

$$t_a = (0, 0, 1) \quad . \tag{25}$$

Matrix (12) then takes form

$$S_{ia} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad . \tag{26}$$

The individual addends on the right–hand side of equation (21) then read

$$S_{in}\chi^{4n4k} = \begin{array}{c} \\ 41 \\ 42 \\ 43 \end{array} \begin{array}{c} \begin{array}{ccc} 41 & 42 & 43 \end{array} \\ \begin{pmatrix} \chi^{4241} & \chi^{4242} & \chi^{4243} \\ -\chi^{4141} & -\chi^{4142} & -\chi^{4143} \\ 0 & 0 & 0 \end{pmatrix} \end{array} \quad , \tag{27}$$

$$S_{kn}\chi^{4i4n} = \begin{array}{c} \\ 41 \\ 42 \\ 43 \end{array} \begin{array}{c} \begin{array}{ccc} 41 & 42 & 43 \end{array} \\ \begin{pmatrix} \chi^{4142} & -\chi^{4141} & 0 \\ \chi^{4242} & -\chi^{4241} & 0 \\ \chi^{4342} & -\chi^{4341} & 0 \end{pmatrix} \end{array} \quad , \tag{28}$$

and equation (21) reads

$$\begin{pmatrix} \chi^{4142} + \chi^{4241} & \chi^{4242} - \chi^{4141} & \chi^{4243} \\ \chi^{4242} - \chi^{4141} & -\chi^{4142} - \chi^{4241} & -\chi^{4143} \\ \chi^{4342} & -\chi^{4341} & 0 \end{pmatrix} = \mathbf{0} \quad . \tag{29}$$

We see that

$$\begin{gathered} \chi^{4143} = 0 \quad , \qquad \chi^{4243} = 0 \quad , \qquad \chi^{4341} = 0 \quad , \qquad \chi^{4342} = 0 \quad , \\ \chi^{4242} = \chi^{4141} \quad , \qquad \chi^{4241} = -\chi^{4142} \quad . \end{gathered} \tag{30}$$

The individual addends on the right–hand side of equation (22) read

$$S_{in}\chi^{nj4l} = \begin{array}{c} \\ 23 \\ 31 \\ 12 \end{array} \begin{array}{c} \begin{array}{ccc} 41 & 42 & 43 \end{array} \\ \begin{pmatrix} \chi^{3141} & \chi^{3142} & \chi^{3143} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{array} \quad , \tag{31}$$

$$S_{jn}\chi^{in4l} = \begin{array}{c} \\ 23 \\ 31 \\ 12 \end{array} \begin{array}{c} \begin{array}{ccc} 41 & 42 & 43 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 \\ -\chi^{2341} & -\chi^{2342} & -\chi^{2343} \\ 0 & 0 & 0 \end{pmatrix} \end{array} \quad , \tag{32}$$

$$S_{ln}\chi^{ij4n} = \begin{array}{c} \\ 23 \\ 31 \\ 12 \end{array} \begin{array}{c} \begin{array}{ccc} 41 & 42 & 43 \end{array} \\ \begin{pmatrix} \chi^{2342} & -\chi^{2341} & 0 \\ \chi^{3142} & -\chi^{3141} & 0 \\ \chi^{1242} & -\chi^{1241} & 0 \end{pmatrix} \end{array} \quad , \tag{33}$$

and equation (22) reads

$$\begin{pmatrix} \chi^{2342} + \chi^{3141} & \chi^{3142} - \chi^{2341} & \chi^{3143} \\ \chi^{3142} - \chi^{2341} & -\chi^{2342} - \chi^{3141} & -\chi^{2343} \\ \chi^{1242} & -\chi^{1241} & 0 \end{pmatrix} = \mathbf{0} \quad . \tag{34}$$

We see that

$$\chi^{2343} = 0 \quad , \quad \chi^{3143} = 0 \quad , \quad \chi^{1241} = 0 \quad , \quad \chi^{1242} = 0 \quad ,$$
$$\chi^{3142} = \chi^{2341} \quad , \quad \chi^{3141} = -\chi^{2342} \quad . \tag{35}$$

The individual addends on the right–hand side of equation (23) read

$$S_{jn}\chi^{4nkl} = \begin{array}{c} \\ 41 \\ 42 \\ 43 \end{array} \begin{array}{c} \begin{array}{ccc} 23 & 31 & 12 \end{array} \\ \begin{pmatrix} \chi^{4223} & \chi^{4231} & \chi^{4212} \\ -\chi^{4123} & -\chi^{4131} & -\chi^{4112} \\ 0 & 0 & 0 \end{pmatrix} \end{array} \quad , \tag{36}$$

$$S_{kn}\chi^{4jnl} = \begin{array}{c} \\ 41 \\ 42 \\ 43 \end{array} \begin{array}{c} \begin{array}{ccc} 23 & 31 & 12 \end{array} \\ \begin{pmatrix} \chi^{4131} & 0 & 0 \\ \chi^{4231} & 0 & 0 \\ \chi^{4331} & 0 & 0 \end{pmatrix} \end{array} \quad , \tag{37}$$

$$S_{ln}\chi^{4jkn} = \begin{array}{c} \\ 41 \\ 42 \\ 43 \end{array} \begin{array}{c} \begin{array}{ccc} 23 & 31 & 12 \end{array} \\ \begin{pmatrix} 0 & -\chi^{4123} & 0 \\ 0 & -\chi^{4223} & 0 \\ 0 & -\chi^{4323} & 0 \end{pmatrix} \end{array} \quad , \tag{38}$$

and equation (23) reads

$$\begin{pmatrix} \chi^{4223} + \chi^{4131} & \chi^{4231} - \chi^{4123} & \chi^{4212} \\ \chi^{4231} - \chi^{4123} & -\chi^{4131} - \chi^{4223} & -\chi^{4112} \\ \chi^{4331} & -\chi^{4323} & 0 \end{pmatrix} = \mathbf{0} \quad . \tag{39}$$

We see that

$$\chi^{4112} = 0 \quad , \quad \chi^{4212} = 0 \quad , \quad \chi^{4323} = 0 \quad , \quad \chi^{4331} = 0 \quad ,$$
$$\chi^{4231} = \chi^{4123} \quad , \quad \chi^{4223} = -\chi^{4131} \quad . \tag{40}$$

The individual addends on the right–hand side of equation (24) read

$$S_{in}\chi^{njkl} = \begin{array}{c} \\ 23 \\ 31 \\ 12 \end{array} \begin{array}{c} \begin{array}{ccc} 23 & 31 & 12 \end{array} \\ \begin{pmatrix} \chi^{3123} & \chi^{3131} & \chi^{3112} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{array} \quad , \tag{41}$$

$$S_{jn}\chi^{inkl} = \begin{array}{c} \\ 23 \\ 31 \\ 12 \end{array} \begin{array}{c} \begin{array}{ccc} 23 & 31 & 12 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 \\ -\chi^{2323} & -\chi^{2331} & -\chi^{2312} \\ 0 & 0 & 0 \end{pmatrix} \end{array} \quad , \tag{42}$$

$$S_{kn}\chi^{ijnl} = \begin{array}{c} \\ 23 \\ 31 \\ 12 \end{array} \begin{array}{c} \begin{array}{ccc} 23 & 31 & 12 \end{array} \\ \begin{pmatrix} \chi^{2331} & 0 & 0 \\ \chi^{3131} & 0 & 0 \\ \chi^{1231} & 0 & 0 \end{pmatrix} \end{array} \quad , \tag{43}$$

$$S_{ln}\chi^{ijkn} = \begin{array}{c} \\ 23 \\ 31 \\ 12 \end{array} \begin{array}{c} \begin{array}{ccc} 23 & 31 & 12 \end{array} \\ \begin{pmatrix} 0 & -\chi^{2323} & 0 \\ 0 & -\chi^{3123} & 0 \\ 0 & -\chi^{1223} & 0 \end{pmatrix} \end{array} \quad , \tag{44}$$

and equation (24) reads

$$\begin{pmatrix} \chi^{3123}+\chi^{2331} & \chi^{3131}-\chi^{2323} & \chi^{3112} \\ \chi^{3131}-\chi^{2323} & -\chi^{2331}-\chi^{3123} & -\chi^{2312} \\ \chi^{1231} & -\chi^{1223} & 0 \end{pmatrix} = \mathbf{0} \quad . \tag{45}$$

We see that

$$\chi^{2312}=0 \quad , \qquad \chi^{3112}=0 \quad , \qquad \chi^{1223}=0 \quad , \qquad \chi^{1231}=0 \quad ,$$
$$\chi^{3131}=\chi^{2323} \quad , \qquad \chi^{3123}=-\chi^{2331} \quad . \tag{46}$$

Considering conditions (30), (35), (40) and (46), we observe that constitutive matrix (9) which is rotationally invariant about symmetry vector $t_a=(0,0,1)$ takes form

$$\chi^{\alpha\beta\gamma\delta} = \begin{array}{c} \\ 41 \\ 42 \\ 43 \\ 23 \\ 31 \\ 12 \end{array} \begin{array}{c} \begin{array}{cccccc} 41 & 42 & 43 & 23 & 31 & 12 \end{array} \\ \begin{pmatrix} -\varepsilon & -\breve{\varepsilon} & 0 & -\alpha & -\breve{\alpha} & 0 \\ \breve{\varepsilon} & -\varepsilon & 0 & \breve{\alpha} & -\alpha & 0 \\ 0 & 0 & -(\varepsilon+\tilde{\varepsilon}) & 0 & 0 & -(\alpha+\tilde{\alpha}) \\ \beta & \breve{\beta} & 0 & \mu^{-1} & \breve{\mu}^{-1} & 0 \\ -\breve{\beta} & \beta & 0 & -\breve{\mu}^{-1} & \mu^{-1} & 0 \\ 0 & 0 & \beta+\tilde{\beta} & 0 & 0 & \mu^{-1}+\tilde{\mu}^{-1} \end{pmatrix} \end{array} \tag{47}$$

specified by 12 parameters.

Constitutive matrix (47) may be non–symmetric. For example, for natural optical activity $\beta \neq -\alpha$ (Post, 2003, table 3), and for Faraday effect $\breve{\varepsilon} \neq 0$ (Post, 2003, table 4).

A rotationally invariant medium is referred to as *uniaxial* if $\breve{\varepsilon}=0$, $\breve{\alpha}=0$, $\breve{\beta}=0$ and $\breve{\mu}^{-1}=0$. The constitutive matrix (47) then reads

$$\chi^{\alpha\beta\gamma\delta} = \begin{array}{c} \\ 41 \\ 42 \\ 43 \\ 23 \\ 31 \\ 12 \end{array} \begin{array}{c} \begin{array}{cccccc} 41 & 42 & 43 & 23 & 31 & 12 \end{array} \\ \begin{pmatrix} -\varepsilon & 0 & 0 & -\alpha & 0 & 0 \\ 0 & -\varepsilon & 0 & 0 & -\alpha & 0 \\ 0 & 0 & -(\varepsilon+\tilde{\varepsilon}) & 0 & 0 & -(\alpha+\tilde{\alpha}) \\ \beta & 0 & 0 & \mu^{-1} & 0 & 0 \\ 0 & \beta & 0 & 0 & \mu^{-1} & 0 \\ 0 & 0 & \beta+\tilde{\beta} & 0 & 0 & \mu^{-1}+\tilde{\mu}^{-1} \end{pmatrix} \end{array} \quad . \tag{48}$$

The constitutive matrix of an uniaxial bianisotropic electromagnetic medium may be non–symmetric, whereas the stiffness matrix of an uniaxial (transversely isotropic) viscoelastic medium is symmetric.

A *biisotropic medium* is invariant with respect to rotations about all three coordinate axes. If we imagine the constitutive matrices analogous to matrix (47) but corresponding to symmetry vectors $t_a=(1,0,0)$ and $t_a=(0,1,0)$, we immediately see that the constitutive matrix of a biisotropic medium reads

$$\chi^{\alpha\beta\gamma\delta} = \begin{array}{c} \\ 41 \\ 42 \\ 43 \\ 23 \\ 31 \\ 12 \end{array} \begin{array}{c} \begin{array}{cccccc} 41 & 42 & 43 & 23 & 31 & 12 \end{array} \\ \begin{pmatrix} -\varepsilon & 0 & 0 & -\alpha & 0 & 0 \\ 0 & -\varepsilon & 0 & 0 & -\alpha & 0 \\ 0 & 0 & -\varepsilon & 0 & 0 & -\alpha \\ \beta & 0 & 0 & \mu^{-1} & 0 & 0 \\ 0 & \beta & 0 & 0 & \mu^{-1} & 0 \\ 0 & 0 & \beta & 0 & 0 & \mu^{-1} \end{pmatrix} \end{array} \quad . \tag{49}$$

The constitutive matrix of a biisotropic electromagnetic medium may be non–symmetric, whereas the stiffness matrix of an isotropic viscoelastic medium is symmetric. For example, a biisotropic electromagnetic medium exhibiting natural optical activity has a non–symmetric constitutive matrix (Post, 2003, table 3).

5. General form of a rotationally invariant constitutive tensor

We transform the 3×3 submatrices of rotationally invariant constitutive matrix (47) from the coordinate system attached to the symmetry axis to a general coordinate system, and obtain the general forms of rotationally invariant 3×3 submatrices of constitutive matrix (9),

$$\varepsilon^{ij} = \varepsilon\,\delta_{ij} + \tilde{\varepsilon}\,t_i t_j + \breve{\varepsilon}\,\varepsilon_{ijr} t_r \quad , \tag{50}$$

$$\beta_i{}^j = \beta\,\delta_{ij} + \tilde{\beta}\,t_i t_j + \breve{\beta}\,\varepsilon_{ijr} t_r \quad , \tag{51}$$

$$\alpha^i{}_j = \alpha\,\delta_{ij} + \tilde{\alpha}\,t_i t_j + \breve{\alpha}\,\varepsilon_{ijr} t_r \quad , \tag{52}$$

$$\mu^{-1}_{ij} = \mu^{-1}\,\delta_{ij} + \tilde{\mu}^{-1}\,t_i t_j + \breve{\mu}^{-1}\,\varepsilon_{ijr} t_r \quad , \tag{53}$$

where Kronecker delta δ_{ij} represents the elements of the identity matrix and ε_{ijk} represents the Levi–Civita symbol.

We insert relations (50)–(53) into definitions (3)–(6) and arrive at

$$\chi^{4j4l} = -\varepsilon\,\delta_{jl} - \tilde{\varepsilon}\,t_j t_l - \breve{\varepsilon}\,\varepsilon_{jlr} t_r \quad , \tag{54}$$

$$\chi^{ij4l} = \beta\,\varepsilon_{ijl} + \tilde{\beta}\,\varepsilon_{ijr} t_r\, t_l + \breve{\beta}\,(\delta_{il} t_j - t_i \delta_{jl}) \quad , \tag{55}$$

$$\chi^{4jkl} = -\alpha\,\varepsilon_{jkl} - \tilde{\alpha}\,t_j\;\varepsilon_{klr} t_r - \breve{\alpha}\,(\delta_{jl} t_k - t_l \delta_{jk}) \quad , \tag{56}$$

$$\chi^{ijkl} = \mu^{-1}\,(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + \tilde{\mu}^{-1}\,\varepsilon_{ijr} t_r \varepsilon_{kls} t_s + \breve{\mu}^{-1}\,\varepsilon_{ijr}\varepsilon_{rsn}\varepsilon_{skl} t_n \quad . \tag{57}$$

We may also express the last term of relation (57) in alternative forms,

$$\chi^{ijkl} = \mu^{-1}\,(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + \tilde{\mu}^{-1}\,\varepsilon_{ijr} t_r \varepsilon_{kls} t_s + \breve{\mu}^{-1}\,(\varepsilon_{ikl} t_j - \varepsilon_{jkl} t_i) \quad , \tag{58}$$

or

$$\chi^{ijkl} = \mu^{-1}\,(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + \tilde{\mu}^{-1}\,\varepsilon_{ijr} t_r \varepsilon_{kls} t_s + \breve{\mu}^{-1}\,(\varepsilon_{ijl} t_k - \varepsilon_{ijk} t_l) \quad . \tag{59}$$

6. Conclusions

In the coordinate system which third coordinate axis coincides with the symmetry axis, the constitutive tensor of a rotationally invariant bianisotropic electromagnetic medium has form (47). It is described by four additional parameters in comparison with the constitutive tensor of an uniaxial electromagnetic medium.

In a general coordinate system, the constitutive tensor of a rotationally invariant bianisotropic electromagnetic medium has form (54)–(57).

Whereas the stiffness tensor of an isotropic viscoelastic medium is symmetric, the constitutive tensor of a biisotropic electromagnetic medium may be non–symmetric and may exhibit an optical activity.

Acknowledgements

The research has been supported by the Grant Agency of the Czech Republic under contract 16-01312S, and by the members of the consortium "Seismic Waves in Complex 3–D Structures" (see "http://sw3d.cz").

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