

# Transformation rules for weak-anisotropy parameters

Ivan Pšenčík

Institute of Geophysics, The Czech Academy of Sciences, Boční II, 141 31 Praha 4, Czech Republic. E-mail: ip@ig.cas.cz

## Summary

Weak-anisotropy (WA) parameters turn out to be a useful way of parameterization of anisotropic media. They represent a generalization of the so-called Thomsen's parameters, which were designed for the parameterization of VTI anisotropy, to anisotropic media of arbitrary symmetry, strength and orientation. In this paper, we present some useful transformation rules for the WA parameters.

## Introduction

Weak-anisotropy (WA) parameters are used in different applications in this Report. Farra and Pšenčík (2017) use them for their alternative expressions for reflection moveout formulae, Jakobsen et al. (2017) use them in the ray-Born inversion, Pšenčík et al. (2017) in travelttime inversion.

WA parameters are slightly modified medium parameters introduced by Mensch and Rasolofosaon (1997), see also Pšenčík and Gajewski (1998) or Farra and Pšenčík (2003). WA parameters represent an alternative to density-normalized elastic parameters  $A_{\alpha\beta}$  in the Voigt notation or to elements of density-normalized stiffness tensor  $a_{ijkl}$ . WA parameters represent a generalization of Thomsen (1986) parameters introduced for VTI media. WA parameters can, however, be used for description of anisotropy of arbitrary symmetry, strength and orientation. They are non-dimensional. Their definition requires introduction of reference P- and S-wave velocities  $\alpha$  and  $\beta$  in a reference isotropic medium.

In this contribution, we present several useful transformation relations. First, equations describing the relation of WA parameters to elements of the  $6 \times 6$  matrix  $\mathbf{A}$  of density-normalized elastic moduli in the Voigt notation are presented. Important equations for the re-definition of WA parameters with respect to new reference velocities are also shown. Perhaps, the most important are equations describing transformation of WA parameters from one Cartesian coordinate system to another. They allow to express WA parameters in a global Cartesian coordinate system in terms of WA parameters in a local, rotated, coordinate system and corresponding Euler angles.

## Transformations of elastic moduli to WA parameters

We use 21 WA parameters for the specification of an arbitrary anisotropic medium.

The WA parameters are related to the elastic parameters in the Voigt notation,  $A_{\alpha\beta}$ , in the following way:

$$\begin{aligned}
\epsilon_x &= \frac{A_{11} - \alpha^2}{2\alpha^2}, & \epsilon_y &= \frac{A_{22} - \alpha^2}{2\alpha^2}, & \epsilon_z &= \frac{A_{33} - \alpha^2}{2\alpha^2}, \\
\delta_x &= \frac{A_{23} + 2A_{44} - \alpha^2}{\alpha^2}, & \delta_y &= \frac{A_{13} + 2A_{55} - \alpha^2}{\alpha^2}, & \delta_z &= \frac{A_{12} + 2A_{66} - \alpha^2}{\alpha^2}, \\
\epsilon_{15} &= \frac{A_{15}}{\alpha^2}, & \epsilon_{16} &= \frac{A_{16}}{\alpha^2}, & \epsilon_{24} &= \frac{A_{24}}{\alpha^2}, & \epsilon_{26} &= \frac{A_{26}}{\alpha^2}, & \epsilon_{34} &= \frac{A_{34}}{\alpha^2}, & \epsilon_{35} &= \frac{A_{35}}{\alpha^2}, \\
\chi_x &= \frac{A_{14} + 2A_{56}}{\alpha^2}, & \chi_y &= \frac{A_{25} + 2A_{46}}{\alpha^2}, & \chi_z &= \frac{A_{36} + 2A_{45}}{\alpha^2}, \\
\gamma_x &= \frac{A_{44} - \beta^2}{2\beta^2}, & \gamma_y &= \frac{A_{55} - \beta^2}{2\beta^2}, & \gamma_z &= \frac{A_{66} - \beta^2}{2\beta^2}, \\
\epsilon_{46} &= \frac{A_{46}}{\beta^2}, & \epsilon_{56} &= \frac{A_{56}}{\beta^2}, & \epsilon_{45} &= \frac{A_{45}}{\beta^2}. 
\end{aligned} \tag{1}$$

The symbols  $\alpha$  and  $\beta$  in equations (1)-(3) denote the P- and S-wave velocities of a reference isotropic medium.

### Transformations of WA parameters to elastic moduli

$$\begin{aligned}
A_{11} &= \alpha^2(1 + 2\epsilon_x), & A_{22} &= \alpha^2(1 + 2\epsilon_y), & A_{33} &= \alpha^2(1 + 2\epsilon_z), \\
A_{23} &= \alpha^2(1 + \delta_x) - 2\beta^2(1 + 2\gamma_x), & A_{13} &= \alpha^2(1 + \delta_y) - 2\beta^2(1 + 2\gamma_y), & A_{12} &= \alpha^2(1 + \delta_z) - 2\beta^2(1 + 2\gamma_z), \\
A_{15} &= \alpha^2\epsilon_{15}, & A_{16} &= \alpha^2\epsilon_{16}, & A_{24} &= \alpha^2\epsilon_{24}, & A_{26} &= \alpha^2\epsilon_{26}, & A_{34} &= \alpha^2\epsilon_{34}, & A_{35} &= \alpha^2\epsilon_{35}, \\
A_{14} &= \alpha^2\chi_x - 2\beta^2\epsilon_{56}, & A_{25} &= \alpha^2\chi_y - 2\beta^2\epsilon_{46}, & A_{36} &= \alpha^2\chi_z - 2\beta^2\epsilon_{45}, \\
A_{44} &= \beta^2(1 + 2\gamma_x), & A_{55} &= \beta^2(1 + 2\gamma_y), & A_{66} &= \beta^2(1 + 2\gamma_z), \\
A_{46} &= \beta^2\epsilon_{46}, & A_{56} &= \beta^2\epsilon_{56}, & A_{45} &= \beta^2\epsilon_{45}.
\end{aligned} \tag{2}$$

In the matrix form, the above transformation relations from WA parameters to the density-normalized parameters  $A_{\alpha\beta}$  have the form:

$$\mathbf{A} = \mathbf{A}^{iso} + \mathbf{A}^{per}, \tag{3}$$

where

$$\mathbf{A} = \begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} & A_{16} \\ & A_{22} & A_{23} & A_{24} & A_{25} & A_{26} \\ & & A_{33} & A_{34} & A_{35} & A_{36} \\ & & & A_{44} & A_{45} & A_{46} \\ & & & & A_{55} & A_{56} \\ & & & & & A_{66} \end{pmatrix}, \tag{4}$$

$$\mathbf{A}^{iso} = \begin{pmatrix} \alpha^2 & \alpha^2 - 2\beta^2 & \alpha^2 - 2\beta^2 & 0 & 0 & 0 \\ & \alpha^2 & \alpha^2 - 2\beta^2 & 0 & 0 & 0 \\ & & \alpha^2 & 0 & 0 & 0 \\ & & & \beta^2 & 0 & 0 \\ & & & & \beta^2 & 0 \\ & & & & & \beta^2 \end{pmatrix} \quad (5)$$

and

$$\mathbf{A}^{per} = \begin{pmatrix} 2\alpha^2\epsilon_x & \alpha^2\delta_z - 4\beta^2\gamma_z & \alpha^2\delta_y - 4\beta^2\gamma_y & \alpha^2\chi_x - 2\beta^2\epsilon_{56} & \alpha^2\epsilon_{15} & \alpha^2\epsilon_{16} \\ & 2\alpha^2\epsilon_y & \alpha^2\delta_x - 4\beta^2\gamma_x & \alpha^2\epsilon_{24} & \alpha^2\chi_y - 2\beta^2\epsilon_{46} & \alpha^2\epsilon_{26} \\ & & 2\alpha^2\epsilon_z & \alpha^2\epsilon_{34} & \alpha^2\epsilon_{35} & \alpha^2\chi_z - 2\beta^2\epsilon_{45} \\ & & & 2\beta^2\gamma_x & \beta^2\epsilon_{45} & \beta^2\epsilon_{46} \\ & & & & 2\beta^2\gamma_y & \beta^2\epsilon_{56} \\ & & & & & 2\beta^2\gamma_z \end{pmatrix}. \quad (6)$$

## Transformations of WA parameters for new reference velocities

Let us introduce new reference velocities  $\alpha'$  and  $\beta'$  instead of original reference velocities  $\alpha$  and  $\beta$ . WA parameters expressed with respect to primed reference velocities are distinguished by prime from WA parameters related to non-primed reference velocities. Let us introduce factors  $k_\alpha$  and  $k_\beta$  defined as ratios of original and new reference velocities,  $k_\alpha = \alpha/\alpha'$ ,  $k_\beta = \beta/\beta'$ . Then the transformation rules are:

$$\begin{aligned} \epsilon'_x &= 1/2(k_\alpha^2 - 1) + k_\alpha^2\epsilon_x, & \epsilon'_y &= 1/2(k_\alpha^2 - 1) + k_\alpha^2\epsilon_y, & \epsilon'_z &= 1/2(k_\alpha^2 - 1) + k_\alpha^2\epsilon_z, \\ \delta'_x &= k_\alpha^2(1 + \delta_x) - 1, & \delta'_y &= k_\alpha^2(1 + \delta_y) - 1, & \delta'_z &= k_\alpha^2(1 + \delta_z) - 1, \\ \epsilon'_{15} &= k_\alpha^2\epsilon_{15}, & \epsilon'_{16} &= k_\alpha^2\epsilon_{16}, & \epsilon'_{24} &= k_\alpha^2\epsilon_{24}, & \epsilon'_{26} &= k_\alpha^2\epsilon_{26}, & \epsilon'_{34} &= k_\alpha^2\epsilon_{34}, & \epsilon'_{35} &= k_\alpha^2\epsilon_{35}, \\ \chi'_x &= k_\alpha^2\chi_x, & \chi'_y &= k_\alpha^2\chi_y, & \chi'_z &= k_\alpha^2\chi_z, \\ \gamma'_x &= 1/2(k_\beta^2 - 1) + k_\beta^2\gamma_x, & \gamma'_y &= 1/2(k_\beta^2 - 1) + k_\beta^2\gamma_y, & \gamma'_z &= 1/2(k_\beta^2 - 1) + k_\beta^2\gamma_z, \\ \epsilon'_{46} &= k_\beta^2\epsilon_{46}, & \epsilon'_{56} &= k_\beta^2\epsilon_{56}, & \epsilon'_{45} &= k_\beta^2\epsilon_{45}. \end{aligned} \quad (7)$$

## Transformations of WA parameters from one Cartesian coordinate system to another

Let us consider two Cartesian coordinate systems, a global one and a local one. All quantities in the global coordinate system are primed to distinguish them from the quantities in the local coordinate system. The local coordinate system can be chosen as a coordinate system whose coordinate planes or axes coincide with the symmetry elements of the considered anisotropy symmetry. The transformation from the local (non-primed) coordinate system to the global one is controlled by the following  $3 \times 3$  rotation matrix:

$$\mathbf{R} = \begin{pmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{pmatrix}. \quad (8)$$

The elements of the transformation matrix (8) can be expressed in terms of Euler angles.

21 WA parameters specified in the local coordinate system transform into 21 WA parameters in the global coordinate system through the equations given below. The equations below represent an alternative to the transformation rules for the  $6 \times 6$  stiffness matrix in the Voigt notation proposed by Bond (1943), see also Chapman (2004). The transformation equations read:

$$\begin{aligned}\epsilon'_x &= \epsilon_x R_{11}^4 + \epsilon_y R_{12}^4 + \epsilon_z R_{13}^4 + \delta_x R_{12}^2 R_{13}^2 + \delta_y R_{11}^2 R_{13}^2 + \delta_z R_{11}^2 R_{12}^2 \\ &\quad + 2\chi_x R_{11}^2 R_{12} R_{13} + 2\chi_y R_{12}^2 R_{11} R_{13} + 2\chi_z R_{13}^2 R_{11} R_{12} \\ &\quad + 2\epsilon_{15} R_{11}^3 R_{13} + 2\epsilon_{16} R_{11}^3 R_{12} + 2\epsilon_{24} R_{12}^3 R_{13} + 2\epsilon_{26} R_{11} R_{12}^3 + 2\epsilon_{34} R_{13}^3 R_{12} + 2\epsilon_{35} R_{13}^3 R_{11} \\ \epsilon'_y &= \epsilon_x R_{21}^4 + \epsilon_y R_{22}^4 + \epsilon_z R_{23}^4 + \delta_x R_{22}^2 R_{23}^2 + \delta_y R_{21}^2 R_{23}^2 + \delta_z R_{21}^2 R_{22}^2 \\ &\quad + 2\chi_x R_{21}^2 R_{22} R_{23} + 2\chi_y R_{22}^2 R_{21} R_{23} + 2\chi_z R_{23}^2 R_{21} R_{22} \\ &\quad + 2\epsilon_{15} R_{21}^3 R_{23} + 2\epsilon_{16} R_{21}^3 R_{22} + 2\epsilon_{24} R_{22}^3 R_{23} + 2\epsilon_{26} R_{21} R_{22}^3 + 2\epsilon_{34} R_{23}^3 R_{22} + 2\epsilon_{35} R_{23}^3 R_{21} \\ \epsilon'_z &= \epsilon_x R_{31}^4 + \epsilon_y R_{32}^4 + \epsilon_z R_{33}^4 + \delta_x R_{32}^2 R_{33}^2 + \delta_y R_{31}^2 R_{33}^2 + \delta_z R_{31}^2 R_{32}^2 \\ &\quad + 2\chi_x R_{31}^2 R_{32} R_{33} + 2\chi_y R_{32}^2 R_{31} R_{33} + 2\chi_z R_{33}^2 R_{31} R_{32} \\ &\quad + 2\epsilon_{15} R_{31}^3 R_{33} + 2\epsilon_{16} R_{31}^3 R_{32} + 2\epsilon_{24} R_{32}^3 R_{33} + 2\epsilon_{26} R_{31} R_{32}^3 + 2\epsilon_{34} R_{33}^3 R_{32} + 2\epsilon_{35} R_{33}^3 R_{31}\end{aligned}$$

$$\begin{aligned}\delta'_x &= 6\epsilon_x R_{21}^2 R_{31}^2 + 6\epsilon_y R_{22}^2 R_{32}^2 + 6\epsilon_z R_{23}^2 R_{33}^2 \\ &\quad + \delta_x(D_{11}^2 + 2R_{22} R_{23} R_{32} R_{33}) + \delta_y(D_{12}^2 + 2R_{21} R_{31} R_{23} R_{33}) + \delta_z(D_{13}^2 + 2R_{21} R_{22} R_{31} R_{32}) \\ &\quad + 2\chi_x(D_{12} D_{13} + R_{21} R_{31} D_{11}) + 2\chi_y(D_{13} D_{11} + R_{22} R_{32} D_{12}) + 2\chi_z(D_{11} D_{12} + R_{23} R_{33} D_{13}) \\ &\quad + 6\epsilon_{15} R_{21} R_{31} D_{12} + 6\epsilon_{16} R_{21} R_{31} D_{13} + 6\epsilon_{24} R_{22} R_{32} D_{11} \\ &\quad + 6\epsilon_{26} R_{22} R_{32} D_{13} + 6\epsilon_{34} R_{23} R_{33} D_{11} + 6\epsilon_{35} R_{23} R_{33} D_{12}\end{aligned}$$

$$\begin{aligned}\delta'_y &= 6\epsilon_x R_{11}^2 R_{31}^2 + 6\epsilon_y R_{12}^2 R_{32}^2 + 6\epsilon_z R_{13}^2 R_{33}^2 \\ &\quad + \delta_x(D_{21}^2 + 2R_{12} R_{32} R_{13} R_{33}) + \delta_y(D_{22}^2 + 2R_{11} R_{33} R_{13} R_{31}) + \delta_z(D_{23}^2 + 2R_{11} R_{12} R_{31} R_{32}) \\ &\quad + 2\chi_x(D_{22} D_{23} + R_{11} R_{31} D_{21}) + 2\chi_y(D_{23} D_{21} + R_{12} R_{32} D_{22}) + 2\chi_z(D_{21} D_{22} + R_{13} R_{33} D_{23}) \\ &\quad + 6\epsilon_{15} R_{11} R_{31} D_{22} + 6\epsilon_{16} R_{11} R_{31} D_{23} + 6\epsilon_{24} R_{12} R_{32} D_{21} \\ &\quad + 6\epsilon_{26} R_{12} R_{32} D_{23} + 6\epsilon_{34} R_{13} R_{33} D_{21} + 6\epsilon_{35} R_{13} R_{33} D_{22}\end{aligned}$$

$$\begin{aligned}\delta'_z &= 6\epsilon_x R_{11}^2 R_{21}^2 + 6\epsilon_y R_{12}^2 R_{22}^2 + 6\epsilon_z R_{13}^2 R_{23}^2 \\ &\quad + \delta_x(D_{31}^2 + 2R_{12} R_{22} R_{13} R_{23}) + \delta_y(D_{32}^2 + 2R_{11} R_{13} R_{21} R_{23}) + \delta_z(D_{33}^2 + 2R_{11} R_{22} R_{12} R_{21}) \\ &\quad + 2\chi_x(D_{32} D_{33} + R_{11} R_{21} D_{31}) + 2\chi_y(D_{33} D_{31} + R_{12} R_{22} D_{32}) + 2\chi_z(D_{31} D_{32} + R_{13} R_{23} D_{33})\end{aligned}$$

$$\begin{aligned}
& +6\epsilon_{15}R_{11}R_{21}D_{32}+6\epsilon_{16}R_{11}R_{21}D_{33}+6\epsilon_{24}R_{12}R_{22}D_{31} \\
& +6\epsilon_{26}R_{12}R_{22}D_{33}+6\epsilon_{34}R_{13}R_{23}D_{31}+6\epsilon_{35}R_{13}R_{23}D_{32}
\end{aligned}$$

$$\begin{aligned}
\chi'_x &= 6\epsilon_x R_{11}^2 R_{21} R_{31} + 6\epsilon_y R_{12}^2 R_{22} R_{32} + 6\epsilon_z R_{13}^2 R_{23} R_{33} \\
& + \delta_x(D_{21}D_{31} + R_{12}R_{13}D_{11}) + \delta_y(D_{32}D_{22} + R_{11}R_{13}D_{12}) + \delta_z(D_{33}D_{23} + R_{11}R_{12}D_{13}) \\
& + \chi_x(R_{11}R_{13}D_{13} + R_{11}R_{12}D_{12} + D_{22}D_{33} + D_{23}D_{32}) \\
& + \chi_y(R_{12}R_{13}D_{13} + R_{12}R_{11}D_{11} + D_{23}D_{31} + D_{21}D_{33}) \\
& + \chi_z(R_{13}R_{12}D_{12} + R_{13}R_{11}D_{11} + D_{21}D_{32} + D_{22}D_{31}) \\
& + 3\epsilon_{15}R_{11}(R_{31}D_{32} + R_{21}D_{22}) + 3\epsilon_{16}R_{11}(R_{31}D_{33} + R_{21}D_{23}) \\
& + 3\epsilon_{24}R_{12}(R_{32}D_{31} + R_{22}D_{21}) + 3\epsilon_{26}R_{12}(R_{32}D_{33} + R_{22}D_{23}) \\
& + 3\epsilon_{34}R_{13}(R_{33}D_{31} + R_{23}D_{21}) + 3\epsilon_{35}R_{13}(R_{33}D_{32} + R_{23}D_{22})
\end{aligned}$$

$$\begin{aligned}
\chi'_y &= 6\epsilon_x R_{21}^2 R_{11} R_{31} + 6\epsilon_y R_{22}^2 R_{12} R_{32} + 6\epsilon_z R_{23}^2 R_{13} R_{33} \\
& + \delta_x(D_{11}D_{31} + R_{22}R_{23}D_{21}) + \delta_y(D_{12}D_{32} + R_{21}R_{23}D_{22}) + \delta_z(D_{13}D_{33} + R_{21}R_{22}D_{23}) \\
& + \chi_x(R_{21}R_{23}D_{23} + R_{21}R_{22}D_{22} + D_{12}D_{33} + D_{13}D_{32}) \\
& + \chi_y(R_{22}R_{23}D_{23} + R_{22}R_{21}D_{21} + D_{13}D_{31} + D_{11}D_{33}) \\
& + \chi_z(R_{23}R_{22}D_{22} + R_{23}R_{21}D_{21} + D_{11}D_{32} + D_{12}D_{31}) \\
& + 3\epsilon_{15}R_{21}(R_{11}D_{12} + R_{31}D_{32}) + 3\epsilon_{16}R_{21}(R_{11}D_{13} + R_{31}D_{33}) \\
& + 3\epsilon_{24}R_{22}(R_{12}D_{11} + R_{32}D_{31}) + 3\epsilon_{26}R_{22}(R_{12}D_{13} + R_{32}D_{33}) \\
& + 3\epsilon_{34}R_{23}(R_{13}D_{11} + R_{33}D_{31}) + 3\epsilon_{35}R_{23}(R_{13}D_{12} + R_{33}D_{32})
\end{aligned}$$

$$\begin{aligned}
\chi'_z &= 6\epsilon_x R_{31}^2 R_{21} R_{11} + 6\epsilon_y R_{32}^2 R_{22} R_{12} + 6\epsilon_z R_{33}^2 R_{23} R_{13} \\
& + \delta_x(D_{11}D_{21} + R_{32}R_{33}D_{31}) + \delta_y(D_{12}D_{22} + R_{31}R_{33}D_{32}) + \delta_z(D_{13}D_{23} + R_{31}R_{32}D_{33}) \\
& + \chi_x(R_{31}R_{33}D_{33} + R_{31}R_{32}D_{32} + D_{12}D_{23} + D_{13}D_{22}) \\
& + \chi_y(R_{32}R_{33}D_{33} + R_{32}R_{31}D_{31} + D_{13}D_{21} + D_{11}D_{23}) \\
& + \chi_z(R_{33}R_{32}D_{32} + R_{33}R_{31}D_{31} + D_{11}D_{22} + D_{12}D_{21}) \\
& + 3\epsilon_{15}R_{31}(R_{11}D_{12} + R_{21}D_{22}) + 3\epsilon_{16}R_{31}(R_{11}D_{13} + R_{21}D_{23}) \\
& + 3\epsilon_{24}R_{32}(R_{12}D_{11} + R_{22}D_{21}) + 3\epsilon_{26}R_{32}(R_{12}D_{13} + R_{22}D_{23}) \\
& + 3\epsilon_{34}R_{33}(R_{13}D_{11} + R_{23}D_{21}) + 3\epsilon_{35}R_{33}(R_{13}D_{12} + R_{23}D_{22})
\end{aligned}$$

$$\begin{aligned}
\epsilon'_{15} &= 2\epsilon_x R_{11}^3 R_{31} + 2\epsilon_y R_{12}^3 R_{32} + 2\epsilon_z R_{13}^3 R_{33} \\
& + \delta_x R_{12}R_{13}D_{21} + \delta_y R_{11}R_{13}D_{22} + \delta_z R_{11}R_{12}D_{23} \\
& + \chi_x R_{11}(R_{12}D_{22} + R_{13}D_{23}) + \chi_y R_{12}(R_{11}D_{21} + R_{13}D_{23}) + \chi_z R_{13}(R_{11}D_{21} + R_{12}D_{22})
\end{aligned}$$

$$\begin{aligned}
& + \epsilon_{15} R_{11}^2 (D_{22} + 2R_{13}R_{31}) + \epsilon_{16} R_{11}^2 (D_{23} + 2R_{12}R_{31}) + \epsilon_{24} R_{12}^2 (D_{21} + 2R_{13}R_{32}) \\
& + \epsilon_{26} R_{12}^2 (D_{23} + 2R_{11}R_{32}) + \epsilon_{34} R_{13}^2 (D_{21} + 2R_{12}R_{33}) + \epsilon_{35} R_{13}^2 (D_{22} + 2R_{11}R_{33})
\end{aligned}$$

$$\begin{aligned}
\epsilon'_{16} = & 2\epsilon_x R_{11}^3 R_{21} + 2\epsilon_y R_{12}^3 R_{22} + 2\epsilon_z R_{13}^3 R_{23} + \delta_x R_{12} R_{13} D_{31} + \delta_y R_{11} R_{13} D_{32} + \delta_z R_{11} R_{12} D_{33} \\
& + \chi_x R_{11} (R_{12} D_{32} + R_{13} D_{33}) + \chi_y R_{12} (R_{11} D_{31} + R_{13} D_{33}) + \chi_z R_{13} (R_{11} D_{31} + R_{12} D_{32}) \\
& + \epsilon_{15} R_{11}^2 (D_{32} + 2R_{13}R_{21}) + \epsilon_{16} R_{11}^2 (D_{33} + 2R_{12}R_{21}) + \epsilon_{24} R_{12}^2 (D_{31} + 2R_{13}R_{22}) \\
& + \epsilon_{26} R_{12}^2 (D_{33} + 2R_{11}R_{22}) + \epsilon_{34} R_{13}^2 (D_{31} + 2R_{12}R_{23}) + \epsilon_{35} R_{13}^2 (D_{32} + 2R_{11}R_{23})
\end{aligned}$$

$$\begin{aligned}
\epsilon'_{24} = & 2\epsilon_x R_{21}^3 R_{31} + 2\epsilon_y R_{22}^3 R_{32} + 2\epsilon_z R_{23}^3 R_{33} + \delta_x R_{22} R_{23} D_{11} + \delta_y R_{21} R_{23} D_{12} + \delta_z R_{21} R_{22} D_{13} \\
& + \chi_x R_{21} (R_{22} D_{12} + R_{23} D_{13}) + \chi_y R_{22} (R_{21} D_{11} + R_{23} D_{13}) + \chi_z R_{23} (R_{21} D_{11} + R_{22} D_{12}) \\
& + \epsilon_{15} R_{21}^2 (D_{12} + 2R_{23}R_{31}) + \epsilon_{16} R_{21}^2 (D_{13} + 2R_{22}R_{31}) + \epsilon_{24} R_{22}^2 (D_{11} + 2R_{23}R_{32}) \\
& + \epsilon_{26} R_{22}^2 (D_{13} + 2R_{21}R_{32}) + \epsilon_{34} R_{23}^2 (D_{11} + 2R_{22}R_{33}) + \epsilon_{35} R_{23}^2 (D_{12} + 2R_{21}R_{33})
\end{aligned}$$

$$\begin{aligned}
\epsilon'_{26} = & 2\epsilon_x R_{21}^3 R_{11} + 2\epsilon_y R_{22}^3 R_{12} + 2\epsilon_z R_{23}^3 R_{13} + \delta_x R_{22} R_{23} D_{31} + \delta_y R_{21} R_{23} D_{32} + \delta_z R_{21} R_{22} D_{33} \\
& + \chi_x R_{21} (R_{22} D_{32} + R_{23} D_{33}) + \chi_y R_{22} (R_{21} D_{31} + R_{23} D_{33}) + \chi_z R_{23} (R_{21} D_{31} + R_{22} D_{32}) \\
& + \epsilon_{15} R_{21}^2 (D_{32} + 2R_{23}R_{11}) + \epsilon_{16} R_{21}^2 (D_{33} + 2R_{22}R_{11}) + \epsilon_{24} R_{22}^2 (D_{31} + 2R_{23}R_{12}) \\
& + \epsilon_{26} R_{22}^2 (D_{33} + 2R_{21}R_{12}) + \epsilon_{34} R_{23}^2 (D_{31} + 2R_{22}R_{13}) + \epsilon_{35} R_{23}^2 (D_{32} + 2R_{21}R_{13})
\end{aligned}$$

$$\begin{aligned}
\epsilon'_{34} = & 2\epsilon_x R_{21} R_{31}^3 + 2\epsilon_y R_{22} R_{32}^3 + 2\epsilon_z R_{23} R_{33}^3 + \delta_x R_{32} R_{33} D_{11} + \delta_y R_{31} R_{33} D_{12} + \delta_z R_{31} R_{32} D_{13} \\
& + \chi_x R_{31} (R_{32} D_{12} + R_{33} D_{13}) + \chi_y R_{32} (R_{31} D_{11} + R_{33} D_{13}) + \chi_z R_{33} (R_{31} D_{11} + R_{32} D_{12}) \\
& + \epsilon_{15} R_{31}^2 (2R_{21}R_{33} + D_{12}) + \epsilon_{16} R_{31}^2 (2R_{21}R_{32} + D_{13}) + \epsilon_{24} R_{32}^2 (2R_{22}R_{33} + D_{11}) \\
& + \epsilon_{26} R_{32}^2 (2R_{22}R_{31} + D_{13}) + \epsilon_{34} R_{33}^2 (2R_{23}R_{32} + D_{11}) + \epsilon_{35} R_{33}^2 (2R_{23}R_{31} + D_{12})
\end{aligned}$$

$$\begin{aligned}
\epsilon'_{35} = & 2\epsilon_x R_{11} R_{31}^3 + 2\epsilon_y R_{12} R_{32}^3 + 2\epsilon_z R_{13} R_{33}^3 + \delta_x R_{32} R_{33} D_{21} + \delta_y R_{31} R_{33} D_{22} + \delta_z R_{31} R_{32} D_{23} \\
& + \chi_x R_{31} (R_{32} D_{22} + R_{33} D_{23}) + \chi_y R_{32} (R_{31} D_{21} + R_{33} D_{23}) + \chi_z R_{33} (R_{31} D_{21} + R_{32} D_{22}) \\
& + \epsilon_{15} R_{31}^2 (2R_{11}R_{33} + D_{22}) + \epsilon_{16} R_{31}^2 (2R_{11}R_{32} + D_{23}) + \epsilon_{24} R_{32}^2 (2R_{12}R_{33} + D_{21}) \\
& + \epsilon_{26} R_{32}^2 (2R_{12}R_{31} + D_{23}) + \epsilon_{34} R_{33}^2 (2R_{13}R_{32} + D_{21}) + \epsilon_{35} R_{33}^2 (2R_{13}R_{31} + D_{22})
\end{aligned}$$

$$\begin{aligned}
\gamma'_x &= \alpha^2/\beta^2 [\epsilon_x R_{21}^2 R_{31}^2 + \epsilon_y R_{22}^2 R_{32}^2 + \epsilon_z R_{23}^2 R_{33}^2 \\
&\quad + \delta_x R_{22} R_{32} R_{23} R_{33} + \delta_y R_{21} R_{31} R_{23} R_{33} + \delta_z R_{21} R_{31} R_{22} R_{32} \\
&\quad + \chi_x R_{21} R_{31} D_{11} + \chi_y R_{22} R_{32} D_{12} + \chi_z R_{23} R_{33} D_{13} \\
&\quad + \epsilon_{15} R_{21} R_{31} D_{12} + \epsilon_{16} R_{21} R_{31} D_{13} + \epsilon_{24} R_{22} R_{32} D_{11} \\
&\quad + \epsilon_{26} R_{22} R_{32} D_{13} + \epsilon_{34} R_{23} R_{33} D_{11} + \epsilon_{35} R_{23} R_{33} D_{12}] \\
&\quad + \gamma_x (R_{22} R_{33} - R_{23} R_{32})^2 + \gamma_y (R_{21} R_{33} - R_{23} R_{31})^2 + \gamma_z (R_{21} R_{32} - R_{22} R_{31})^2 \\
&\quad + \epsilon_{45} (R_{21} R_{33} - R_{23} R_{31})(R_{22} R_{33} - R_{23} R_{32}) + \epsilon_{46} (R_{21} R_{32} - R_{22} R_{31})(R_{22} R_{33} - R_{23} R_{32}) \\
&\quad + \epsilon_{56} (R_{21} R_{32} - R_{22} R_{31})(R_{21} R_{33} - R_{23} R_{31})
\end{aligned}$$

$$\begin{aligned}
\gamma'_y &= \alpha^2/\beta^2 [\epsilon_x R_{11}^2 R_{31}^2 + \epsilon_y R_{12}^2 R_{32}^2 + \epsilon_z R_{13}^2 R_{33}^2 \\
&\quad + \delta_x R_{12} R_{32} R_{13} R_{33} + \delta_y R_{13} R_{11} R_{31} R_{33} + \delta_z R_{11} R_{31} R_{12} R_{32} \\
&\quad + \chi_x R_{11} R_{31} D_{21} + \chi_y R_{12} R_{32} D_{22} + \chi_z R_{13} R_{33} D_{23} \\
&\quad + \epsilon_{15} R_{11} R_{31} D_{22} + \epsilon_{16} R_{11} R_{31} D_{23} + \epsilon_{24} R_{12} R_{32} D_{21} \\
&\quad + \epsilon_{26} R_{12} R_{32} D_{23} + \epsilon_{34} R_{13} R_{33} D_{21} + \epsilon_{35} R_{13} R_{33} D_{22}] \\
&\quad + \gamma_x (R_{12} R_{33} - R_{13} R_{32})^2 + \gamma_y (R_{11} R_{33} - R_{13} R_{31})^2 + \gamma_z (R_{11} R_{32} - R_{12} R_{31})^2 \\
&\quad + \epsilon_{45} (R_{11} R_{33} - R_{13} R_{31})(R_{12} R_{33} - R_{13} R_{32}) + \epsilon_{46} (R_{11} R_{32} - R_{12} R_{31})(R_{12} R_{33} - R_{13} R_{32}) \\
&\quad + \epsilon_{56} (R_{11} R_{32} - R_{12} R_{31})(R_{11} R_{33} - R_{13} R_{31})
\end{aligned}$$

$$\begin{aligned}
\gamma'_z &= \alpha^2/\beta^2 [\epsilon_x R_{11}^2 R_{21}^2 + \epsilon_y R_{12}^2 R_{22}^2 + \epsilon_z R_{13}^2 R_{23}^2 \\
&\quad + \delta_x R_{12} R_{22} R_{13} R_{23} + \delta_y R_{13} R_{11} R_{23} R_{21} + \delta_z R_{11} R_{21} R_{12} R_{22} \\
&\quad + \chi_x R_{11} R_{21} D_{31} + \chi_y R_{12} R_{22} D_{32} + \chi_z R_{13} R_{23} D_{33} \\
&\quad + \epsilon_{15} R_{11} R_{21} D_{32} + \epsilon_{16} R_{11} R_{21} D_{33} + \epsilon_{24} R_{12} R_{22} D_{31} \\
&\quad + \epsilon_{26} R_{12} R_{22} D_{33} + \epsilon_{34} R_{13} R_{23} D_{31} + \epsilon_{35} R_{13} R_{23} D_{32}] \\
&\quad + \gamma_x (R_{12} R_{23} - R_{13} R_{22})^2 + \gamma_y (R_{11} R_{23} - R_{13} R_{21})^2 + \gamma_z (R_{11} R_{22} - R_{12} R_{21})^2 \\
&\quad + \epsilon_{45} (R_{11} R_{23} - R_{13} R_{21})(R_{12} R_{23} - R_{13} R_{22}) + \epsilon_{46} (R_{11} R_{22} - R_{12} R_{21})(R_{12} R_{23} - R_{13} R_{22}) \\
&\quad + \epsilon_{56} (R_{11} R_{22} - R_{12} R_{21})(R_{11} R_{23} - R_{13} R_{21}),
\end{aligned}$$

$$\begin{aligned}
\epsilon'_{45} &= \alpha^2/\beta^2 [2\epsilon_x R_{21} R_{31}^2 R_{11} + 2\epsilon_y R_{22} R_{32}^2 R_{12} + 2\epsilon_z R_{23} R_{33}^2 R_{13} \\
&\quad + \delta_x R_{32} R_{33} D_{31} + \delta_y R_{31} R_{33} D_{32} + \delta_z R_{31} R_{32} D_{33} \\
&\quad + \chi_x (R_{31} R_{33} D_{33} + R_{31} R_{32} D_{32}) + \chi_y (R_{22} R_{32} D_{22} + R_{12} R_{32} D_{12}) + \chi_z (R_{23} R_{33} D_{23} + R_{13} R_{33} D_{13}) \\
&\quad + \epsilon_{15} (R_{21} R_{31} D_{22} + R_{11} R_{31} D_{12}) + \epsilon_{16} (R_{21} R_{31} D_{23} + R_{11} R_{31} D_{13}) + \epsilon_{24} (R_{22} R_{32} D_{21} + R_{12} R_{32} D_{11}) \\
&\quad + \epsilon_{26} (R_{22} R_{32} D_{23} + R_{12} R_{32} D_{13}) + \epsilon_{34} (R_{23} R_{33} D_{21} + R_{13} R_{33} D_{11}) + \epsilon_{35} (R_{23} R_{33} D_{22} + R_{13} R_{33} D_{12})]
\end{aligned}$$

$$\begin{aligned}
& +2\gamma_x(R_{22}R_{33}-R_{23}R_{32})(R_{12}R_{33}-R_{13}R_{32})+2\gamma_y(R_{11}R_{33}-R_{13}R_{31})(R_{21}R_{33}-R_{23}R_{31}) \\
& \quad +2\gamma_z(R_{11}R_{32}-R_{12}R_{31})(R_{21}R_{32}-R_{22}R_{31}) \\
& +\epsilon_{45}[(R_{22}R_{33}-R_{23}R_{32})(R_{11}R_{33}-R_{13}R_{31})+(R_{13}R_{32}-R_{12}R_{33})(R_{23}R_{31}-R_{21}R_{33})] \\
& +\epsilon_{46}[(R_{12}R_{33}-R_{13}R_{32})(R_{22}R_{31}-R_{21}R_{32})+(R_{32}R_{11}-R_{12}R_{31})(R_{23}R_{32}-R_{22}R_{33})] \\
& +\epsilon_{56}[(R_{21}R_{33}-R_{23}R_{31})(R_{11}R_{32}-R_{12}R_{31})+(R_{21}R_{32}-R_{22}R_{31})(R_{11}R_{33}-R_{13}R_{31})]
\end{aligned}$$

$$\begin{aligned}
\epsilon'_{46} = & \alpha^2/\beta^2[2\epsilon_x R_{11}R_{21}^2R_{31} + 2\epsilon_y R_{12}R_{22}^2R_{32} + 2\epsilon_z R_{13}R_{23}^2R_{33} \\
& + \delta_x R_{22}R_{23}D_{21} + \delta_y R_{21}R_{23}D_{22} + \delta_z R_{21}R_{22}D_{23} \\
& + \chi_x(R_{21}R_{31}D_{31}+R_{11}R_{21}D_{11})+\chi_y(R_{22}R_{32}D_{32}+R_{12}R_{22}D_{12})+\chi_z(R_{23}R_{33}D_{33}+R_{13}R_{23}D_{13}) \\
& +\epsilon_{15}(R_{21}R_{31}D_{32}+R_{11}R_{21}D_{12})+\epsilon_{16}(R_{21}R_{31}D_{33}+R_{11}R_{21}D_{13})+\epsilon_{24}(R_{22}R_{32}D_{31}+R_{12}R_{22}D_{11}) \\
& +\epsilon_{26}(R_{22}R_{32}D_{33}+R_{12}R_{22}D_{13})+\epsilon_{34}(R_{23}R_{33}D_{31}+R_{13}R_{23}D_{11})+\epsilon_{35}(R_{23}R_{33}D_{32}+R_{13}R_{23}D_{12})] \\
& +2\gamma_x(R_{22}R_{33}-R_{23}R_{32})(R_{12}R_{23}-R_{13}R_{22})+2\gamma_y(R_{21}R_{33}-R_{23}R_{31})(R_{11}R_{23}-R_{13}R_{21}) \\
& \quad +2\gamma_z(R_{11}R_{22}-R_{12}R_{21})(R_{21}R_{32}-R_{22}R_{31}) \\
& +\epsilon_{45}[(R_{13}R_{22}-R_{12}R_{23})(R_{21}R_{33}-R_{23}R_{31})+(R_{13}R_{21}-R_{11}R_{23})(R_{22}R_{33}-R_{23}R_{32})] \\
& +\epsilon_{46}[(R_{22}R_{33}-R_{23}R_{32})(R_{11}R_{22}-R_{12}R_{21})+(R_{21}R_{32}-R_{22}R_{31})(R_{12}R_{23}-R_{13}R_{22})] \\
& +\epsilon_{56}[(R_{11}R_{22}-R_{12}R_{21})(R_{23}R_{31}-R_{21}R_{33})+(R_{21}R_{32}-R_{22}R_{31})(R_{13}R_{21}-R_{11}R_{23})]
\end{aligned}$$

$$\begin{aligned}
\epsilon'_{56} = & \alpha^2/\beta^2[2\epsilon_x R_{21}R_{11}^2R_{31} + 2\epsilon_y R_{22}R_{12}^2R_{32} + 2\epsilon_z R_{23}R_{13}^2R_{33} \\
& + \delta_x R_{12}R_{13}D_{11} + \delta_y R_{11}R_{13}D_{12} + \delta_z R_{11}R_{12}D_{13} \\
& + \chi_x(R_{11}R_{31}D_{31}+R_{11}R_{21}D_{21})+\chi_y(R_{12}R_{32}D_{31}+R_{12}R_{22}D_{22})+\chi_z(R_{13}R_{33}D_{33}+R_{13}R_{23}D_{23}) \\
& +\epsilon_{15}(R_{11}R_{31}D_{32}+R_{11}R_{21}D_{22})+\epsilon_{16}(R_{11}R_{31}D_{33}+R_{11}R_{21}D_{23})+\epsilon_{24}(R_{12}R_{32}D_{31}+R_{12}R_{22}D_{21}) \\
& +\epsilon_{26}(R_{12}R_{32}D_{33}+R_{12}R_{22}D_{23})+\epsilon_{34}(R_{13}R_{33}D_{31}+R_{13}R_{23}D_{21})+\epsilon_{35}(R_{13}R_{33}D_{32}+R_{13}R_{23}D_{22})] \\
& +2\gamma_x(R_{12}R_{23}-R_{13}R_{22})(R_{12}R_{33}-R_{13}R_{32})+2\gamma_y(R_{11}R_{23}-R_{13}R_{21})(R_{11}R_{33}-R_{13}R_{31}) \\
& \quad +2\gamma_z(R_{11}R_{22}-R_{12}R_{21})(R_{11}R_{32}-R_{12}R_{31}) \\
& +\epsilon_{45}[(R_{11}R_{23}-R_{13}R_{21})(R_{12}R_{33}-R_{13}R_{32})+(R_{12}R_{23}-R_{13}R_{22})(R_{11}R_{33}-R_{13}R_{31})] \\
& +\epsilon_{46}[(R_{11}R_{22}-R_{12}R_{21})(R_{12}R_{33}-R_{13}R_{32})+(R_{12}R_{23}-R_{13}R_{22})(R_{11}R_{32}-R_{12}R_{31})] \\
& +\epsilon_{56}[(R_{11}R_{22}-R_{12}R_{21})(R_{11}R_{33}-R_{13}R_{31})+(R_{11}R_{23}-R_{13}R_{21})(R_{11}R_{32}-R_{12}R_{31})], \quad (9)
\end{aligned}$$

where

$$\begin{aligned}
D_{11} &= R_{22}R_{33} + R_{23}R_{32}, \quad D_{12} = R_{21}R_{33} + R_{23}R_{31}, \quad D_{13} = R_{22}R_{31} + R_{21}R_{32}, \\
D_{21} &= R_{12}R_{33} + R_{13}R_{32}, \quad D_{22} = R_{11}R_{33} + R_{13}R_{31}, \quad D_{23} = R_{12}R_{31} + R_{11}R_{32}, \\
D_{31} &= R_{12}R_{23} + R_{13}R_{22}, \quad D_{32} = R_{13}R_{21} + R_{11}R_{23}, \quad D_{33} = R_{11}R_{22} + R_{12}R_{21}. \quad (10)
\end{aligned}$$

We could rewrite equations (9) in the matrix form. 21 WA parameters in the local coordinate system would transform to 21 WA parameters in the global coordinate system using the  $21 \times 21$  matrix with interesting properties. The first 15 non-primed WA parameters from equation (1) transform into the same 15 primed WA parameters. Remaining 6 WA parameters,  $\gamma_1$ ,  $\gamma_2$ ,  $\gamma_3$ ,  $\epsilon_{45}$ ,  $\epsilon_{46}$  and  $\epsilon_{56}$  are not involved in such a transformation. For the transformation of these 6 WA parameters, all 21 WA parameters are, however, involved.

## Conclusions

Several transformation formulae for WA parameters are presented. Perhaps the most important one is the set of equations allowing transformation of WA parameters from one Cartesian coordinate system to another. These formulae will certainly find applications in modelling of tilted higher-symmetry anisotropies as, for example, tilted transversely isotropic (TTI) media or tilted orthorhombic (TOR) media. In such cases, the transformation formulae will simplify considerably because only a few of non-primed WA parameters will be non-zero on the right sides of transformation equations (9).

In this contribution, we considered an isotropic medium with P- and S-wave velocities  $\alpha$  and  $\beta$  as a reference medium. It is not difficult to introduce WA parameters , whose reference medium might be VTI or orthorhombic with symmetry planes parallel to coordinate planes.

## Acknowledgement

I am grateful to the Research Project 16-05237S of the Grant Agency of the Czech Republic and the project "Seismic waves in complex 3-D structures" (SW3D) for support.

## References

- Bond, W., 1943. The mathematics of the physical properties of crystals. *Bell System Technical Journal*, **22**, 1-72, doi: 10.1002/j.1538-7305.1943.tb01304.x.
- Chapman, C. H., 2004. Fundamentals of seismic wave propagation. Cambridge University Press.
- Farra, V. and Pšenčík, I., 2003. Properties of the zero-, first- and higher-order approximations of attributes of elastic waves in weakly anisotropic media. *J. Acoust. Soc. Am.*, **114**, 1366–1378.
- Farra, V. and Pšenčík, I., 2017. Reflection moveout approximation for converted P-SV wave in a moderately anisotropic homogeneous VTI layer. In: *Seismic waves in Complex 3-D Structures*, Report 10, 51–58. Charles Univ. in Prague, Faculty of Mathematics and Physics, Department of Geophysics Praha, online at "<http://sw3d.cz>".

Jakobsen, M., Pšenčík, I., Iversen, E. and Ursin, B., 2017. On the parameterization of seismic anisotropy for ray-Born inversion. In: *Seismic waves in Complex 3-D Structures*, Report 10, 37–49. Charles Univ. in Prague, Faculty of Mathematics and Physics, Department of Geophysics Praha, online at “<http://sw3d.cz>”.

Mensch, T., and Rasolofosaon, P., 1997. Elastic-wave velocities in anisotropic media of arbitrary symmetry – generalization of Thomsens parameters  $\epsilon$ ,  $\delta$  and  $\gamma$ . *Geophys. J. Int.*, **128**, 43–64.

Pšenčík, I., and Gajewski, D., 1998. Polarization, phase velocity and NMO velocity of  $qP$  waves in arbitrary weakly anisotropic media. *Geophysics*, **63**, 1754–1766.

Pšenčík, I., Růžek, B., Lokajíček, T. and Svitek, T., 2017. Determination of rock-sample anisotropy from P- and S-wave traveltimes. In: *Seismic waves in Complex 3-D Structures*, Report 10, 15–35. Charles Univ. in Prague, Faculty of Mathematics and Physics, Department of Geophysics Praha, online at “<http://sw3d.cz>”.

Thomsen, L., 1986. Weak elastic anisotropy. *Geophysics*, **51**, 1954–1966.