Reflection moveout approximation for a converted P-SV wave in a moderately anisotropic homogeneous VTI layer

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Summary

We present and test an approximate formula for the reflection moveout of converted waves in a homogeneous VTI (transversely isotropic with the vertical axis of symmetry) layer. For its derivation, we use the weak-anisotropy approximation, i.e., we expand the square of the reflection traveltime in terms of weak-anisotropy (WA) parameters. Travel times are calculated along reference rays of converted waves reflected in reference isotropic media. This requires the determination of the point of reflection of the reference ray, at which the corresponding wave converts. Presented tests indicate that the accuracy of the formula is comparable with the accuracy of formulae derived in a similar way for unconverted waves. The tests also indicate that the formula can be applied not only to weakly, but also to moderately anisotropic VTI media.

Introduction

As in our previous papers, we are presenting reflection moveout formula based on the combined use of the weak-anisotropy approximation and weak-anisotropy (WA) parameters. In this article, we concentrate on converted waves in a homogeneous layer of transverse isotropy with vertical axis of symmetry (VTI). The derived approximate formula holds for both P-SV and SV-P converted waves. We, however, perform the derivation and concentrate on the converted P-SV wave only.

For the derivation of moveout formulae, Farra and Pšenčík (2013) or Farra, Pšenčík and Jílek (2016) used an actual ray of an unconverted P or SV wave reflected from a horizontal reflector, which coincided with one symmetry plane of the overlying anisotropic medium. Pšenčík and Farra (2016) and Farra and Pšenčík (2017) showed that it is possible to derive simple and still sufficiently accurate moveout formulae even without knowledge of actual rays, and extended their previous work to weakly anisotropic media of arbitrary symmetry and orientation. The basic step of their procedure was the replacement of actual rays by reference rays in reference isotropic media. For unconverted waves, the reference rays are symmetric with respect to the reflector and, therefore, it is straightforward how to construct them.

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In contrast to unconverted reflected waves, for which construction of a symmetric reference ray is easy, construction of a reference ray of a converted reflected wave in an anisotropic medium is a more complicated task. Fortunately, there were successful attempts to find the conversion point (the point of reflection, at which the mode conversion occurs) in the past. Probably the first were Tessmer and Behle (1988). Improved version of their formula proposed by Thomsen (1999) is used in this paper.

Determination of the conversion point in the reference medium

We consider a homogeneous *isotropic* layer underlaid by a horizontal reflector. On the surface of the layer, we consider source S and receiver R. These two points are connected by the ray of a converted P-S wave with the conversion point C at the reflector. In the following, we follow closely derivation by Thomsen (1999), but use slightly different notation corresponding to the notation, which we used in our previous studies. We denote the P- and S-wave velocities α and β, respectively. As Thomsen (1999), we denote their ratio by $\gamma = \alpha/\beta$, and angles of incidence and reflection of a ray of the converted P-S wave by θ_P and θ_S. By x, we denote the source-receiver offset and by x_C the offset of the conversion point. The depth of the layer is denoted H. Elementary trigonometry considerations for $\sin\theta_P$ and $\sin\theta_S$ yield:

$$\sin\theta_P = \frac{x_C}{\sqrt{x_C^2 + H^2}} , \qquad \sin\theta_S = \frac{x - x_C}{\sqrt{(x - x_C)^2 + H^2}} . \tag{1}$$

Combination of the Snell law and equation (1) leads to the equation

$$(x - x_C)^2 (x_C^2 + H^2)\gamma^2 = x_C^2[(x - x_C)^2 + H^2] . \tag{2}$$

Equation (2) is equivalent to equation (14) of Thomsen (1999). Normalizing equation (2) by H, introducing the normalized offset $\bar{x}$ and normalized offset of the conversion point $\bar{x}_C$,

$$\bar{x} = \frac{x}{H}, \qquad \bar{x}_C = \frac{x_C}{H} \tag{3}$$

and rearranging equation (2) into the form of a polynomial equation, we get

$$\bar{x}_C^4 - 2\bar{x}\bar{x}_C^3 + (1 + \bar{x}^2)\bar{x}_C^2 - \frac{2\gamma^2\bar{x}\bar{x}_C}{\gamma^2 - 1} + \frac{\gamma^2\bar{x}^2}{\gamma^2 - 1} = 0 . \tag{4}$$

This is a quartic polynomial equation for the normalized offset of the conversion point $\bar{x}_C$. It can be solved analytically using, for example, the so-called Ferrari procedure. It can also be solved numerically. Tessmer and Behle (1988) derived an approximate explicit formula for the determination of the offset of the conversion point, which was improved by Thomsen (1999). Taking into account the normalization specified in equation (3), we use here the approximate formula of Thomsen (1999). It reads

$$\bar{x}_C \sim \bar{x}(C_0 + C_2\frac{\bar{x}^2}{1 + C_3\bar{x}^2}) , \tag{5}$$

where

$$C_0 = \frac{\gamma}{1+\gamma}, \qquad C_2 = \frac{\gamma}{2}\frac{\gamma - 1}{(1+\gamma)^3} , \qquad C_3 = \frac{\gamma}{2}\frac{\gamma - 1}{(1+\gamma)^2} , \tag{6}$$

For the detailed study of accuracy of the expression (5), see Thomsen (1999).

Traveltime formula

We consider the Cartesian coordinate system, whose x_1- and x_2-axes are horizontal, the x_3-axis is vertical and positive downwards. The system is right-handed. As in the previous section, we consider a homogeneous layer underlaid, at the depth H, by a horizontal reflector. The layer is now, however, not isotropic, but *transversely isotropic with the axis of symmetry vertical (VTI)*. In this layer we consider a converted P-SV wave, it is a wave, which propagates as a P wave from the source S to the conversion point C, and as an SV wave from C to the receiver R. Without loss of generality, we can consider the profile along the x_1-axis, which means that the SV wave is polarized in the vertical (x_1, x_3) plane. The traveltime along the ray of the converted wave from S to R via C is $T = T_P + T_{SV}$, where T_P is the traveltime along the P-wave leg of the ray and T_{SV} along the SV-wave ray leg. From the geometry of the ray of the converted wave, we have the following expressions for the squares of traveltimes along the P- and SV-wave ray legs:

$$T_P^2 = \frac{x_C^2 + H^2}{v_P^2(\mathbf{N}^P)}, \qquad T_{SV}^2 = \frac{(x - x_C)^2 + H^2}{v_{SV}^2(\mathbf{N}^{SV})} \ . \tag{7}$$

Here x_C is the offset of the conversion point of the converted P-SV wave in the VTI medium. In equation (7), v_P and v_{SV} denote P- and SV-wave ray velocities. The vectors $\mathbf{N}^P$ and $\mathbf{N}^{SV}$ denote unit vectors parallel to the P- and SV-wave ray legs of the converted wave. We call the vectors $\mathbf{N}^P$ and $\mathbf{N}^{SV}$ ray vectors. Before proceeding further, let us rewrite equation (7) by using normalized quantities in it. In addition to quantities in equation (3), we also use

$$T_{0_P} = \frac{H}{\alpha}, \qquad T_{0_{SV}} = \frac{H}{\beta} \ , \tag{8}$$

where α and β are P- and S-wave velocities of the reference isotropic medium. Equation (7) now reads:

$$T_P^2 = T_{0_P}^2 \alpha^2 \frac{1 + \bar{x}_C^2}{v_P^2(\mathbf{N}^P)}, \qquad T_{SV}^2 = T_{0_{SV}}^2 \beta^2 \frac{1 + (\bar{x} - \bar{x}_C)^2}{v_{SV}^2(\mathbf{N}^{SV})} \ . \tag{9}$$

If $\bar{x}_C$ is exact, then the traveltime $T = T_P + T_{SV}$, where T_P and T_{SV} are given in equation (9), is also exact. At this point, we shall make two approximations, which we did also in our previous studies.

The first approximation is related to the fact that the actual ray of the converted wave is unknown. As, for example, Pšenčík and Farra (2016) or Farra and Pšenčík (2017), we replace the actual ray by the reference ray of the converted wave in the reference isotropic medium with P- and S-wave velocities α and β. The normalized offset of the conversion point is now considered in the reference isotropic medium. In this way, we replace the traveltime calculated along the actual ray by its first-order approximation calculated along a reference ray in the reference isotropic medium (Fermat's principle). The vectors $\mathbf{N}^P$ and $\mathbf{N}^{SV}$ are taken as vectors parallel to the P- and S-wave legs of the reference ray. We keep the same notation for $\mathbf{N}^P$ as in the case of an actual P-wave ray, but use $\mathbf{N}^S$ instead of $\mathbf{N}^{SV}$. Components of the ray vectors $\mathbf{N}^P$ and $\mathbf{N}^S$ are specified by the same formulae as those used in the above-mentioned references or Farra et al. (2016). The components of the vector $\mathbf{N}^P$ in the plane (x_1, x_3) read:

$$N_1^P = \frac{\bar{x}_C}{\sqrt{1+\bar{x}_C^2}} \; , \qquad N_2^P = 0 \; , \qquad N_3^P = \frac{1}{\sqrt{1+\bar{x}_C^2}} \; . \tag{10}$$

The components of the vector $\mathbf{N}^S$ in the same plane read:

$$N_1^S = \frac{\bar{x}-\bar{x}_C}{\sqrt{1+(\bar{x}-\bar{x}_C)^2}} \; , \qquad N_2^S = 0 \; , \qquad N_3^S = -\frac{1}{\sqrt{1+(\bar{x}-\bar{x}_C)^2}} \; . \tag{11}$$

The negative sign in the expression for N_3^S indicates upgoing character of the S-wave ray leg.

The second approximation consists in the replacement of exact squares of ray velocities in equation (9) by their approximations. As Pšenčík and Farra (2016) or Farra and Pšenčík (2017), we approximate squares of ray velocities by the first-order approximations of squares of phase velocities in the corresponding directions $\mathbf{N}$. Using equations (7) and (29) of Farra and Pšenčík (2013), we have, in the notation of this paper:

$$\widetilde{v_P^2}(\mathbf{N}^P) \sim \widetilde{c_P^2}(\mathbf{N}^P) \sim \alpha^2[1+2(\epsilon_x(N_1^P)^4+\delta_y(N_1^P)^2(N_3^P)^2+\epsilon_z(N_3^P)^4)] \tag{12}$$

and

$$\widetilde{v_{SV}^2}(\mathbf{N}^S) \sim \widetilde{c_{SV}^2}(\mathbf{N}^S) \sim \beta^2[1+2\gamma_y+2\gamma^2(\epsilon_x+\epsilon_z-\delta_y)(N_1^S)^2(N_3^S)^2] \; . \tag{13}$$

Tilde above the quantities indicates that they are approximate, specifically of the first order in WA parameters. Symbols c_P and c_{SV} denote P- and SV-wave phase velocities. Symbols ϵ_x, ϵ_z, δ_y and γ_y are WA parameters defined as follows:

$$\epsilon_x = \frac{A_{11}-\alpha^2}{2\alpha^2} \; , \quad \epsilon_z = \frac{A_{33}-\alpha^2}{2\alpha^2} \; , \quad \delta_y = \frac{A_{13}+2A_{55}-\alpha^2}{\alpha^2} \; , \quad \gamma_y = \frac{A_{55}-\beta^2}{2\beta^2} \; , \tag{14}$$

where $A_{\alpha\beta}$ denote density-normalized elastic moduli in the Voigt notation.

Inserting equations (10) and (11) into equations (12) and (13), we obtain approximate expressions for squares of ray velocities expressed in terms of the normalized offset $\bar{x}$. These expressions inserted to the traveltime formulae (9) yield

$$T_P^2 = T_{0_P}^2 \frac{(1+\bar{x}_C^2)^3}{P_P(\bar{x}_C)}, \qquad T_{SV}^2 = T_{0_SV}^2 \frac{[1+(\bar{x}-\bar{x}_C)^2]^3}{P_{SV}(x-\bar{x}_C)} \; . \tag{15}$$

Symbols $P_P(x)$ and $P_{SV}(x)$ represent polynomials

$$P_P(x) = (1+x^2)^2 + 2\delta_y x^2 + 2\epsilon_x x^4 + 2\epsilon_z \tag{16}$$

and

$$P_{SV}(x) = (1+x^2)^2(1+2\gamma_y) + 2\gamma^2(\epsilon_x+\epsilon_z-\delta_y)x^2 \; . \tag{17}$$

For the total time $T = T_P + T_{SV}$, we thus get:

$$T(\bar{x}) = T_{0P} \frac{(1+\bar{x}_C^2)^{3/2}}{P_P^{1/2}(\bar{x}_C)} + T_{0SV} \frac{[1+(\bar{x}-\bar{x}_C)^2]^{3/2}}{P_{SV}^{1/2}(\bar{x}-\bar{x}_C)} \; . \tag{18}$$

By differentiating equations (18), it is possible to obtain expressions for the normal moveout velocity v_{NMO} and the quartic term A_4 of the Taylor expansion of the squared traveltime T^2 with respect to the squared offset x^2.

Tests of accuracy

We test equation (18) for converted P-SV waves in the limestone model, whose P- and SV-wave anisotropy are $\sim 8\%$ and $\sim 5\%$, respectively, and the Mesaverde mudshale and the hard shale models with P-wave anisotropy $\sim 6\%$ and $\sim 25\%$, respectively, and SV-wave anisotropy of $\sim 12\%$. The anisotropy strength is defined as $2(c_{max} - c_{min})/(c_{max} + c_{min}) \times 100\%$, where c denotes corresponding phase velocity. The parameters of all three models are given in Table 1.

Model	α(km/s)	β(km/s)	ϵ_x	δ_y	ϵ_z	γ_y
Limestone	3.0	1.707	0.076	0.133	0.	0.
Mesaverde mudshale	4.53	2.703	0.034	0.184	0.	0.
Hard shale	3.0	1.914	0.252	0.034	0.	0.

Table 1: Parameters of the models used. α and β - P- and S-wave reference velocities, ϵ_x, δ_y, ϵ_z and γ_y - WA parameters.

In the following figures, we present plots of relative errors $(T - T_{ex})/T_{ex} \times 100\%$. Here T is the traveltime calculated from equation (18) and T_{ex} is the traveltime calculated using the package ANRAY (Gajewski and Pšenčík, 1990), which we consider exact. Each figure contains two curves obtained from equation (18). They differ by the way, in which the conversion point is estimated. The black curve is obtained from the approximate equation (5), the red curve is obtained by solving numerically the quartic equation (4).

In Figure 1, we show results for the weakly anisotropic limestone model. We can see that $\bar{x}_C$ determined by solving numerically quartic equation (4) leads to relative traveltime errors less than 0.1% for normalized offsets from 0. to 8. The use of the normalized conversion offset $\bar{x}_C$ determined from the approximate equation (5) leads to slightly larger errors, but still below 0.2% for normalized offsets between 0. and 8. Relative traveltime errors are thus comparable with relative errors of the first-order formula for unconverted P waves, but less than errors of the first-order formula for unconverted SV waves. Compare Figure 1 of this paper with Figures 1 and 3 of Farra and Pšenčík (2013).

Figure 2 shows relative traveltime errors of equation (18) applied to Mesaverde mudshale model. The errors are slightly larger than in Figure 1 (S-wave anisotropy is stronger), but they do not exceed 0.5% for the normalized offsets between 0. and 8. These errors are substantially smaller than errors of the first-order formula for unconverted SV, compare Figure 2 of this paper with Figure 4 of Farra and Pšenčík (2013).

Because of stronger anisotropy of the hard shale, relative traveltime errors of equation (18) in Figure 3 are larger. They nearly reach 2% around the normalized offset $\bar{x} \sim 2$. For the remaining offsets, the errors are smaller. In this case, the accuracy of equation (18) is is even higher than the accuracy of the first-order formula for unconverted SV wave, see Figure 5 of Farra and Pšenčík (2013).

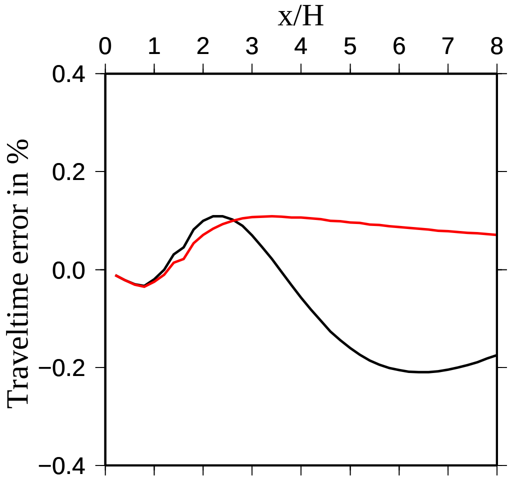

Figure 1: P-SV-wave moveout in the limestone model, P-wave anisotropy $\sim 8\%$, SV-wave anisotropy $\sim 5\%$. Variation with the normalized offset $\bar{x} = x/H$ of the relative traveltime error of approximate equation (18). Black curve - the conversion point estimated approximately from equation (5), red curve - the conversion point determined by the numerical solution of equation (4).

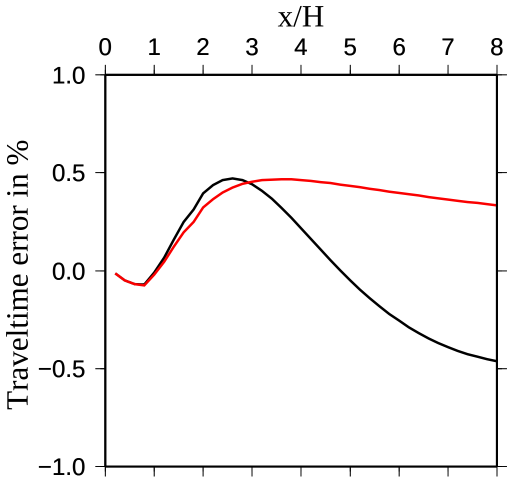

Figure 2: P-SV-wave moveout in the Mesaverde mudshale model, P-wave anisotropy $\sim$ 6%, SV-wave anisotropy $\sim 12\%$. Variation with the normalized offset $\bar{x} = x/H$ of of the relative traveltime error of approximate equation (18). Black curve - the conversion point estimated approximately from equation (5), red curve - the conversion point determined by the numerical solution of equation (4).

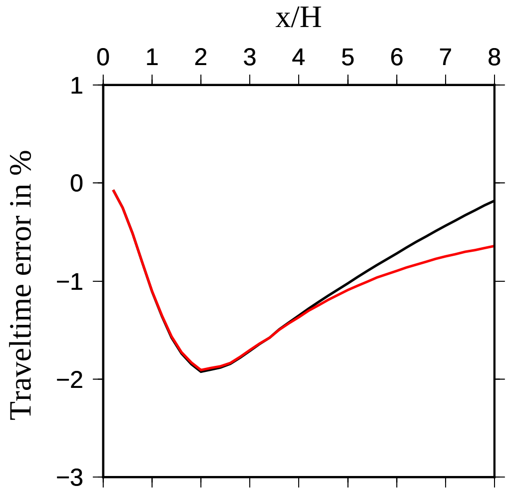

Figure 3: P-SV-wave moveout in the hard shale model, P-wave anisotropy $\sim 25\%$, SV-wave anisotropy $\sim 12\%$. Variation with the normalized offset $\bar{x} = x/H$ of of the relative traveltime error of approximate equation (18). Black curve - the conversion point estimated approximately from equation (5), red curve - the conversion point determined by the numerical solution of equation (4).

Conclusions

We derived an approximate, explicit and relatively simple reflection-moveout formula for a converted wave in a weakly or moderately anisotropic homogeneous VTI layer. The formula relates, in a simple and transparent way, traveltimes to the parameters of the medium represented by WA parameters. Along a profile, the formula depends on 4 WA parameters.

Although the derivations and tests were performed for the converted P-SV wave, due to its kinematic reciprocity, the formula holds also for the converted SV-P wave.

Performed tests indicate that the accuracy of the moveout formula is close to the accuracy of formulae derived in a similar way earlier for unconverted P or SV waves. The tests also show that the formula can be used not only for weakly, but also for moderately anisotropic media.

The derived formulae offer several possible extensions and generalizations. It is straightforward to use the formula for the derivation of expressions for the NMO velocity and the quartic coefficient of the Taylor expansion of the squared traveltime of a converted wave with respect to the offset. These quantities could be used for the generalization of the present formula, which holds for a single homogeneous layer, for the stack of horizontal layers. It also seems that using the concept of a common S wave, the formula could be generalized for an anisotropic medium of arbitrary symmetry and orientation.

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