Weak-anisotropy moveout approximations for P waves in homogeneous TTI layers

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Summary

We offer approximate nonhyperbolic P-wave moveout formulae applicable to weakly or moderately anisotropic media of arbitrary anisotropy. Instead of commonly used Taylor expansion of the square of the reflection traveltime in terms of the square of the offset, we expand the traveltime in terms of weak-anisotropy (WA) parameters. We specify the general formulae for the TI media with arbitrary tilt of the axis of symmetry. Resulting formulae depend on three WA parameters specifying the TI symmetry and two angles specifying the orientation of the axis of symmetry. Tests of the accuracy of the more accurate of approximate formulae indicate that maximum relative errors are not greater than 2.5% for P-wave anisotropy of, approximately, 26%.

Introduction

Nonhyperbolic moveout analysis has been in the centre of attention of exploration seismologists for many years. It plays an especially important role in studies of anisotropic media. We refer to Tsvankin (2001) and Tsvankin and Grechka (2011) as basic publications, in which many details and an extensive list of references on nonhyperbolic moveout can be found. The moveout formulae are mostly based on the Taylor expansion of the square of reflection traveltime in terms of the square of the source-receiver offset. In our recent papers (Farra and Pšenčík, 2013a, b; Farra, Pšenčík and Jílek, 2016), we used an alternative approach based on the weak-anisotropy approximation. We expanded the square of reflection traveltime in terms of weak-anisotropy (WA) pararameters. WA parameters represent a description of an anisotropic medium alternative to the description based on the Voigt notation. The WA parameters can be considered as a generalization of Thomsen's (1986) parameters, which were designed for VTI (transversely isotropic with vertical axis of symmetry) media, to anisotropic media of arbitrary symmetry and orientation. In the above-mentioned papers, we concentrated on the nonhyperbolic moveout of unconverted (mostly P) waves in a homogeneous anisotropic layer underlaid by a reflector coinciding with a symmetry plane of the anisotropic medium. The approximate formulae were relatively simple, and at the same time their accuracy even for moderately anisotropic media was comparable with standardly used formulae. The request of coin-

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cidence of the reflector with a plane of the symmetry of the overlying medium, however, ruled out, for example, media of triclinic symmetry, but, more importantly, also higher-symmetry media like transversely isotropic or orthorhombic media with tilted elements of their symmetries.

In this paper, we propose nonhyperbolic-moveout formulae for unconverted P waves, which can be used for the study of weak or moderate anisotropy of arbitrary symmetry and orientation. The formulae are again based on the weak-anisotropy approximation. For the specification of the medium, we use the WA parameters. Especially convenient for our task is the simple possibility to transform the WA parameters from one coordinate system to another. Although the basic formula is applicable to anisotropy of arbitrary symmetry, in this contribution we concentrate on the TTI media.

As mentioned above, the use of WA parameters allows their simple transformation between various coordinate systems. In problems like the one described in this paper, we need, generally, three coordinate systems. The first, which we call a *global system*, is the general Cartesian coordinate system with one axis vertical and two horizontal. In this system, P-wave propagation is described by 15 P-wave WA parameters. The second coordinate system, which we call *profile system*, shares the vertical axis with the global system and its horizontal axes make an angle with the horizontal axes of the global system. The source-receiver profile is parallel to one of the horizontal axes. The reason for the introduction of such a coordinate system is a substantial reduction of the number of WA parameters involved in the moveout formulae. Only 3 of 15 P-wave WA parameters are required in the profile coordinate system. The third coordinate system, which we call *crystal symmetry system* is related to the considered type of anisotropy. Since we concentrate here on TTI media, our crystal symmetry coordinate system has one axis parallel to the axis of symmetry of the studied TI medium, the other two axes might be chosen arbitrarily. P-wave propagation in this system is described by 3 TI P-wave WA parameters.

We choose the source-receiver profile along the x_1 axis of the global coordinate system. This means that the profile and global coordinate systems are identical, and the WA parameters defined with respect to these coordinate system are identical too. Only when deriving the azimuthal dependence of the NMO velocity, we use the profile coordinate system different from the global one.

We start the paper with the section, in which we present two formulae for nonhyperbolic moveout in media of arbitrary symmetry. For the one, which promises good accuracy, we also present the corresponding NMO velocity and quartic term. In addition, we discuss properties and accuracy of the two-way zero-offset traveltimes obtained from the moveout formulae. In the following section, we test the accuracy of both proposed formulae for varying mutual orientation of the source-receiver profile and the axis of symmetry of the TTI medium. We close the main part of the text by concluding remarks and indications of possible extensions of the presented formulae. In Appendix A, we review the first-order approximation of the square of the phase velocity expressed in terms of WA parameters. In Appendix B, necessary transformation formulae from the crystal symmetry to the profile (global) coordinate system are given.

The lower-case indices $i, j, k, l, ...$ take the values of 1,2,3, the upper-case indices $I, J, K, L, ...$ take the values of 1,2. Vectors and matrices are denoted by bold letters.

Moveout for a weakly or moderately anisotropic medium of arbitrary symmetry

Let us consider a global coordinate system x_i, with the x_3-axis vertical and positive downwards. The axes x_1 and x_2 are situated in the horizontal plane, and are chosen so that the whole system is right-handed. In this coordinate system, we use 15 P-wave weak-anisotropy (WA) parameters (A2), instead of 21 elastic moduli in Voigt notation, describing P-wave anisotropy of arbitrary symmetry. For more details on WA parameters, see Appendix A and Farra et al. (2016).

Further, let us consider a unit vector $\mathbf{t}$, arbitrarily oriented in the space, specifying the orientation of the axis of symmetry of a homogeneous transversely isotropic (TI) medium. If the vector $\mathbf{t}$ is vertical, we deal with the so-called VTI medium, if $\mathbf{t}$ is horizontal, the medium is called HTI, for arbitrary orientation of the vector $\mathbf{t}$, the medium is TTI (transversely isotropic with tilted axis of symmetry). The vector $\mathbf{t}$ can be specified by two angles, the azimuth angle φ and the polar angle θ, see equation (B1) of Appendix B. The angle φ is the angle made by the projection of the vector $\mathbf{t}$ to the horizontal plane (x_1, x_2) with the positive direction of the x_1-axis. It is measured positively from the x_1- towards the x_2-axis. The angle θ is the angle, which the vector $\mathbf{t}$ makes with the positive x_3-axis. Let us make the axis of symmetry of the TTI medium the x_3^{TI}-axis of the local (crystal symmetry) coordinate system, in which we can specify the P-wave attributes by three P-wave WA parameters, ϵ_x^{TI}, ϵ_z^{TI} and δ_y^{TI}. The other two axes of the local right-handed coordinate system can be associated with the vectors $\mathbf{i}_1$ and $\mathbf{i}_2$, defined in equation (B2) of Appendix B.

We consider a homogeneous TTI layer underlaid by a plane horizontal reflector Σ situated at the depth H. We assume that the source S and receivers R are located on the profile parallel to the x_1 axis and situated on the layer's surface. Between the source S and any of the receivers R we consider a reference ray of a wave reflected from the reflector Σ. The reference ray is situated in the reference isotropic medium, in the plane (x_1, x_3). It is symmetric with respect to the normal to the reflector Σ at the reflection point. Let us emphasize that the actual ray of the considered reflected P wave is, generally, neither coinciding with the reference ray nor symmetric. The reflection point of the actual ray differs from the reflection point of the reference ray (the so-called reflection-point dispersal, see, e.g., Pech, Tsvankin and Grechka, 2003). Only in the case of monoclinic and higher symmetry media, in which the plane of symmetry coincides with the reflector Σ, the actual ray coincides with the reference ray and it is thus symmetric, see Farra et al. (2016), and there is no reflection-point dispersal. Components of the ray vector $\mathbf{N}$, a unit vector parallel to the down-going or up-going part of the reference ray of the reflected wave, can be specified by the same formulae as those used by Farra et al. (2016). The components of the vector $\mathbf{N}$ in the plane (x_1, x_3) read:

$$N_1 = \frac{\bar{x}}{\sqrt{1+\bar{x}^2}} \ , \qquad N_2 = 0 \ , \qquad N_3 = \pm\frac{1}{\sqrt{1+\bar{x}^2}} \ . \tag{1}$$

Here, the sign $+$ in the expression for N_3 corresponds to the down-going part of the reference ray, and we denote the corresponding ray vector $\mathbf{N}^d$, and the sign $-$ corresponds to the up-going part of the reference ray and we denote the corresponding ray vector $\mathbf{N}^u$. The symbol $\bar{x}$ in equation (1) denotes the normalized offset. In the following, we also use the two-way zero-offset traveltime T_0 in the reference isotropic medium. Both $\bar{x}$ and T_0

are defined as:

$$\bar{x} = \frac{x}{2H} \ , \qquad T_0 = \frac{2H}{\alpha} \ . \tag{2}$$

In equation (2), x is the offset (distance of the source S from the receiver R), H is the depth of the reflector Σ, and α is the velocity of the reference isotropic medium. T_0 generally differs from the actual two-way zero-offset traveltime.

Moveout calculations for TTI media can be performed in various ways. In the most straightforward way, we could fix the vector $\mathbf{t}$ specifying the orientation of the axis of symmetry. From its orientation and knowledge of the three P-wave TI WA parameters we could determine the WA parameters in the global Cartesian coordinate system. We could then, as Farra et al. (2016), apply the moveout formulae along an arbitrary source-receiver profile on the surface. For this, it would be necessary to transform the WA parameters from the global coordinate system to the local (profile) coordinate system related to the selected source-receiver profile. In this contribution, when not indicated otherwise, we use a simpler procedure. We choose the profile coinciding with the x_1 coordinate axis of the global coordinate system, and vary the orientation of the axis of symmetry of the TTI medium with respect to the profile. In this case, the local coordinate system coincides with the global one, and the WA parameters are identical in both systems. We, therefore, work with the WA parameters specified in the global coordinate system. Thus for each orientation of the axis of symmetry, the transformation of the three TI WA parameters into the WA parameters, specified in the global coordinates, is necessary.

In the following, we present two first-order traveltime approximations of different accuracy. The first is based on the approximate formula for the phase velocity (Pšenčík and Gajewski, 1998), the other is based on the recent work of the authors, based on the approximate expression for the square of the phase velocity.

Traveltime approximation I

We use traveltime equations proposed by Pšenčík and Gajewski (1998) for weakly anisotropic media of arbitrary symmetry. Let us transform equations (A-6) with (A-8) of Pšenčík and Gajewski (1998) into the notation of Farra and Pšenčík (2013a,b) or Farra et al. (2016). We get:

$$T^2(\bar{x}) = T_0^2 \frac{(1+\bar{x}^2)}{2\alpha^2} \Big(4\alpha^2 - \mathcal{B} \Big) \ , \tag{3}$$

where

$$\mathcal{B} = B_{33}(\mathbf{N}^d) + B_{33}(\mathbf{N}^u) \ , \tag{4}$$

see the definitions of vectors $\mathbf{N}^d$ and $\mathbf{N}^u$ after equation (1). The quantity $B_{33}(\mathbf{N})$ represents the first-order approximation of the square of the P-wave phase velocity c. The expression for $B_{33}(\mathbf{N})$ in terms of 15 P-wave WA parameters (A2) is given in equation (A1) of Appendix A. The symbol $T = T(\bar{x})$ in equation (3) denotes the traveltime of the considered unconverted reflected P wave.

Use of the expression B_{33} given in equation (A1) in equation (4) yields:

$$\mathcal{B} = 2\alpha^2 + 4\alpha^2 N_3^2[(\delta_y - \epsilon_x - \epsilon_z)N_1^2 + (\delta_x - \epsilon_y - \epsilon_z)N_2^2 + 2\chi_z N_1 N_2 + \epsilon_z]$$

$$+4\alpha^2[\epsilon_x N_1^2 + \epsilon_y N_2^2 + (\delta_z - \epsilon_x - \epsilon_y)N_1^2 N_2^2 + 2\epsilon_{16} N_1^3 N_2 + 2\epsilon_{26} N_1 N_2^3] \ . \tag{5}$$

We can see that the expression $\mathcal{B}$ in equation (5) depends on only 9 of 15 P-wave WA parameters. These are the same parameters, which appear in B_{33} for monoclinic media, see Farra et al. (2016). Note that the expression (5) does not depend on 6 WA parameters χ_x, χ_y, ϵ_{15}, ϵ_{24}, ϵ_{34} and ϵ_{35}. The expression (5) is also independent of the velocity α of the reference isotropic medium. This is, however, not true for the moveout formula (3), which does depend on α.

The number of WA parameters, on which $\mathcal{B}$ depends, can be further reduced in the profile coordinate system. If we limit ourselves to the vertical plane containing the source-receiver profile, and choose, for example, the source-receiver profile along the x_1-axis of the global coordinate system (i.e., global and profile coordinate systems coincide and WA parameters are identical in both systems), the components of the ray vectors corresponding to the down- and up-going parts of the reference ray are given by (1), and equation (5) reduces to:

$$\mathcal{B} = 2\alpha^2[1 + 2(\delta_y - \epsilon_x - \epsilon_z)N_1^2 N_3^2 + 2\epsilon_x N_1^2 + 2\epsilon_z N_3^2] \ . \tag{6}$$

As we can see, in this case the factor $\mathcal{B}$ depends on 3 WA parameters only, specifically on ϵ_x, ϵ_z and δ_y. These are WA parameters necessary for the specification of VTI media. Equation (6) thus represents a confirmation of Rasolofosaon's (2003) idea that any P-wave kinematic algorithm for VTI symmetry can be used for moderately anisotropic media of arbitrary symmetry. If we use equation (1) in equation (6), and insert the result to equation (3), we obtain the traveltime formula:

$$T^2(\bar{x}) = T_0^2 \ \frac{2(1+\bar{x}^2)^2 - P(\bar{x})}{(1+\bar{x}^2)} \ , \tag{7}$$

where

$$P(\bar{x}) = (1+\bar{x}^2)^2 + 2\epsilon_x \bar{x}^4 + 2\delta_y \bar{x}^2 + 2\epsilon_z \ . \tag{8}$$

The polynomial $P(\bar{x})$ in equation (8) corresponds to the polynomial $P(\bar{x})$ given in equation (10) of Farra et al. (2016). The only difference is that in equation (8) above, we assume $\epsilon_z \neq 0$ while $\epsilon_z = 0$ was assumed in equation (10) of Farra et al. (2016).

For the zero offset, $\bar{x} = 0$, equation (7) yields

$$T^2(0) = T_0^2 \ (1 - 2\epsilon_z) = \frac{4H^2}{\alpha^4}(2\alpha^2 - A_{33}) \ , \tag{9}$$

where A_{33} is the density-normalized elastic modulus in the Voigt notation. We can see that the two-way zero-offset traveltime $T(0)$ depends on the choice of the reference velocity α. For the choice $\alpha^2 = A_{33}$, (i.e. $\epsilon_z = 0$), we find that

$$T^2(0) = T_0^2 = \frac{4H^2}{A_{33}} \ . \tag{10}$$

In this case, the approximate two-way zero-offset traveltime equals its counterpart T_0 in the reference medium. See more discussion on the two-way zero-offset traveltime in the following section.

Traveltime approximation II

Tests of the accuracy of formula (7) show that the accuracy of equation (7) is rather poor. We use therefore an alternative, more accurate, expression proposed by Farra et al. (2016). It is the first-order expression for the traveltime squared, in which the square of the actual ray velocity is replaced by the first-order approximation of the square of the phase velocity in the direction of the ray vector $\mathbf{N}$. Because of generally different phase velocities along the down-going and up-going parts of the reference ray, we have to split the derivation of the traveltime formula into two parts: the traveltime T_d along the down-going, and T_u along the up-going part of the reference ray:

$$T_d^2(\bar{x}) = \frac{T_0^2\alpha^2}{4\widetilde{c^2}(\mathbf{N}^d)}(1+\bar{x}^2) \ , \qquad T_u^2(\bar{x}) = \frac{T_0^2\alpha^2}{4\widetilde{c^2}(\mathbf{N}^u)}(1+\bar{x}^2) \ . \tag{11}$$

The first-order approximations of the squares of the phase velocities $\widetilde{c^2}(\mathbf{N}^d)$ and $\widetilde{c^2}(\mathbf{N}^u)$ can be determined from equation (A1), if we specify them for the vertical plane (x_1, x_3) and express the ray vectors $\mathbf{N}_d$ and $\mathbf{N}_u$ using equation (1):

$$\widetilde{c^2}(\mathbf{N}_d) = \frac{\alpha^2 P_d(\bar{x})}{(1+\bar{x}^2)^2} \ , \qquad \widetilde{c^2}(\mathbf{N}_u) = \frac{\alpha^2 P_u(\bar{x})}{(1+\bar{x}^2)^2} \ . \tag{12}$$

The symbols P_d and P_u denote the polynomials:

$$P_d(\bar{x}) = (1+\bar{x}^2)^2 + 2\epsilon_x\bar{x}^4 + 4\epsilon_{15}\bar{x}^3 + 2\delta_y\bar{x}^2 + 4\epsilon_{35}\bar{x} + 2\epsilon_z \ ,$$

$$P_u(\bar{x}) = (1+\bar{x}^2)^2 + 2\epsilon_x\bar{x}^4 - 4\epsilon_{15}\bar{x}^3 + 2\delta_y\bar{x}^2 - 4\epsilon_{35}\bar{x} + 2\epsilon_z \ . \tag{13}$$

By inserting equations (12) into equations (11) for T_d^2 and T_u^2, we obtain for $T^2 = (T_d + T_u)^2$ the following expression:

$$T^2(\bar{x}) = \frac{T_0^2}{4}(1+\bar{x}^2)^3[P_d^{-1/2}(\bar{x}) + P_u^{-1/2}(\bar{x})]^2 \ . \tag{14}$$

Equation (14) is a more accurate, but also more complicated, moveout equation than equation (7) with (8). In fact, equation (7) could be obtained from (14) by linearization. We can see that the two WA parameters ϵ_{15} and ϵ_{35}, appearing in equation (14) through equations (13), do not appear in equation (7) with (8). Moreover, the terms containing these WA parameters in equations (13) appear in it with opposite signs. It indicates that the effect of these parameters on T^2 is negligible in the first-order approximation. This is confirmed by the results of the tests presented in the following section. If we neglect the parameters ϵ_{15} and ϵ_{35} in equations (13), we arrive at the following relation:

$$P_d(\bar{x}) \sim P_u(\bar{x}) \sim P(\bar{x}) \ , \tag{15}$$

where $P(\bar{x})$ is the polynomial defined in equation (8).

With relation (15) in mind, we can rewrite equation (14) to the form:

$$T^2(\bar{x}) = T_0^2 \frac{(1+\bar{x}^2)^3}{P(\bar{x})} \ . \tag{16}$$

This equation is identical to equation (16) of Farra et al. (2016). In contrast to equation (7), equation (16) is independent of the velocity α of the reference isotropic medium.

For the zero-offset, $\bar{x} = 0$, equation (16) yields

$$T^2(0) = T_0^2/(1+2\epsilon_z) = \frac{4H^2}{A_{33}} \ . \tag{17}$$

We can see that, as expected, the approximate two-way zero-offset traveltime (17) does not depend on the choice of the reference velocity α. It is interesting to see that its value corresponds to the special choice of α, $\alpha^2 = A_{33}$, in equation (10) in the preceding section. Let us note that A_{33} in equations (10) and (17) represents the first-order approximation of the square of the vertical phase velocity, see equations (A1) and (A2). Thus, in contrast to results of Farra et al. (2016), $T(0)$ in this study represents only an approximation of the actual two-way zero-offset traveltime, which is $2H/c_v$, where c_v is the exact value of the vertical phase velocity. See also the results of numerical tests in the next section.

By differentiating equation (16) with respect to the square of the offset x, we obtain two important expressions, for the square of the inverse NMO velocity v_{NMO} and for the quartic coefficient A_4 of the Taylor expansion of $T^2(x)$:

$$v_{NMO}^{-2} = \alpha^{-2} \frac{1+6\epsilon_z - 2\delta_y}{(1+2\epsilon_z)^2} \ , \tag{18}$$

$$A_4 = 2\frac{\delta_y - \epsilon_x - \epsilon_z + 2(\delta_y - 2\epsilon_z)^2/(1+2\epsilon_z)}{\alpha^4 T_0^2 (1+2\epsilon_z)^2} \ . \tag{19}$$

Let us emphasize that both equations (18) and (19) are independent of the choice of α since equation (16), from which they are obtained, is independent of α too. If we choose the reference velocity α as Farra et al. (2016) did, then $\epsilon_z = 0$ and the above equations transform to equations of a simpler form:

$$v_{NMO}^{-2} = \alpha^{-2}(1-2\delta_y) \ , \tag{20}$$

$$A_4 = 2\frac{\delta_y - \epsilon_x + 2(\delta_y)^2}{\alpha^4 T_0^2} \ . \tag{21}$$

The form of the above equations for v_{NMO}^{-2} and A_4 is identical to that derived for the same quantities by Farra et al. (2016). We emphasize again that equations of Farra et al. (2016) were derived for anisotropic media upto monoclinic, whose one symmetry plane coincides with the reflector. The latter condition guaranteed the use of the exact rays in moveout calculations. For the evaluation of formulae (14)-(21), we, however, use reference rays, which might differ from exact rays, and their reflection points may show position dispersal.

Equations (18) and (20) can also be rewritten in the form of the P-wave normal-moveout-ellipse (Tsvankin, 2001; Tsvankin and Grechka, 2011). In the following, we

concentrate on equation (20), in which the reference velocity α is chosen as the vertical velocity so that $\epsilon_z = 0$. For a moment, let us consider the source-receiver profile not coinciding with the x_1-axis, as it was so far, but making an angle Φ with it. As Farra et al. (2016), we introduce a local (profile) coordinate system x'_i, whose x'_3 axis coincides with the x_3 axis of the global coordinate system and x'_1 axis coincides with the source-receiver profile. As Farra et al. (2016), we mark the WA parameters related to x'_i coordinates by the prime. The WA parameters ϵ'_z and δ'_y, which we now use in equations (18) and (20) relate to the WA parameters defined with respect to the global Cartesian coordinates as

$$\delta'_y = \delta_x \sin^2\Phi + \delta_y \cos^2\Phi + 2\chi_z \sin\Phi\cos\Phi \qquad \epsilon'_z = \epsilon_z \ . \tag{22}$$

The relations (22) follow from the transformations similar to those described in Appendix B of Farra et al. (2016). For their exact form, see equation (42) of Pšenčík and Gajewski (1998). If we insert δ'_y to equation (20), we obtain the well-known formula expressing the elliptical character of the P-wave NMO velocity:

$$v_{NMO}^{-2} = W_{11}\cos^2\Phi + 2W_{12}\cos\Phi\sin\Phi + W_{22}\sin^2\varphi \ . \tag{23}$$

The coefficients W_{IJ} in equation (23) have the form

$$W_{11} = \alpha^{-2}(1-2\delta_y) \ , \quad W_{12} = -2\alpha^{-2}\chi_z \ , \quad W_{22} = \alpha^{-2}(1-2\delta_x) \ . \tag{24}$$

Equations (24) are formally equivalent with equations (C-14) of Farra et al. (2016). But equations (C-14) were derived for anisotropic media of monoclinic or higher symmetry, whose one plane of symmetry coincided with the reflector while equations (24) hold for any symmetry.

All the above-presented formulae can be applied to weakly or moderately anisotropic media of *arbitrary* symmetry. In this paper, we study their accuracy for TTI media with *arbitrarily* oriented axis of symmetry. For this purpose, it is necessary to express the WA parameters used in the above formulae in terms of the WA parameters (B5) defined in the local coordinate system x_i^{TI}. The necessary transformation relations are given in equations (B6). The WA parameters in the global Cartesian coordinate system x_i are expressed in (B6) in terms of three WA parameters ϵ_x^{TI}, ϵ_z^{TI}, δ_y^{TI} and two angles specifying the unit vector $\mathbf{t}$, which controls the orientation of the axis of symmetry of the TTI medium in the global coordinate system.

If we insert equations (B6) to equations (24), we get the coefficients of the NMO-velocity ellipse, which represents a generalization of the ellipse presented in Sec. 2.4.1 of Tsvankin and Grechka (2011). In equations (23) and (24), an arbitrary orientation of the axis of symmetry is possible. If the axis of symmetry is situated either in the (x_1, x_3) plane (as in Tsvankin and Grechka, 2011), i.e., $t_2 = 0$, or in the (x_2, x_3) plane, i.e., $t_1 = 0$, then the WA parameter $\chi_z = 0$, see equations (B6), and the semiaxes of the NMO ellipse are aligned with the coordinates x_1 and x_2 of the global coordinate system. This makes a comparison with results of Tsvankin and Grechka (2011) possible. If we use the linearized version of equation (18), and choose the reference velocity α as Tsvankin and Grechka (2011), it is as the velocity along the axis of symmetry, we obtain coefficients W_{11} and W_{22}, which are comparable with the coefficients in equations (2.51)-(2.53) of Tsvankin and Grechka (2011). We must only take into account the differences between Thomsen's (1986) and TI WA parameters. The use of equation (20) offers slightly different expressions since instead of the specification of α as the velocity along the axis of symmetry ($\epsilon_z^{TI} = 0$), we use an approximation of the vertical velocity ($\epsilon_z = 0$) in equation (20).

Tests of accuracy

We test the accuracy of formula (7) and of formulae (14) and (16). We check the relative errors $(T - T_{ex})/T_{ex} \times 100\%$, where T denotes traveltimes calculated from the presented approximate formulae, and T_{ex} denotes the traveltime calculated using the package ANRAY (Gajewski and Pšenčík, 1990), which we consider exact. As a model we use the Greenhorn shale model with a P-wave anisotropy of $\sim 26\%$. The anisotropy strength is defined as $2(c_{max} - c_{min})/(c_{max} + c_{min}) \times 100\%$. The WA parameters of the Greenhorn shale model are $\epsilon_x^{TI} = 0.256$, $\epsilon_z^{TI} = 0$ and $\delta_y^{TI} = -0.0523$. The reference velocity α is chosen as $\alpha = 3.094$ km/s, which corresponds to the velocity along the axis of symmetry ($\epsilon_z^{TI} = 0$). In all the following figures, we present relative traveltime errors measured along the source-receiver profile parallel to the x_1 axis and for the varying orientation of the axis of symmetry of the TTI medium. Each figure corresponds to a selected azimuth angle φ with four values of the polar angle θ, $\theta = 0°$, $30°$, $60°$ and $90°$; $\theta = 0°$ corresponds to the VTI medium, $\theta = 90°$ corresponds to the HTI medium whose axis of symmetry makes the azimuthal angle φ with the axis x_1. Due to the symmetry of the model, it is sufficient to study only the interval of azimuthal angles $0° - 90°$.

We start with the test of accuracy of equation (7). Results for the axis of symmetry specified by the azimuth angle $\varphi = 45°$ and for two different choices of the reference P-wave velocity α are shown in Figure 1. We can see that the relative errors are quite large, especially for the VTI ($\theta = 0°$) and HTI ($\theta = 90°$) cases. They may reach 14%. Thus, the accuracy of the formula (7) is rather low. Moreover, accuracy of equation (7) depends on the choice of the reference velocity α. This is illustrated by the comparison of plots a and b in Figure 1. In Figure 1a, α is chosen as specified above, to make the WA parameter ϵ_z^{TI} zero, i.e., α=3.094 km/s. In Figure 1b, α is chosen so that $\alpha^2 = A_{33}$, where A_{33} represents, approximately, the square of the vertical P-wave phase velocity. For $\theta = 0°$, which corresponds to the VTI case, α is the same in both plots and, therefore, the corresponding curves are identical. With increasing value of the polar angle θ, the velocity α also increases to the value of 3.81 km/s for $\theta = 90°$, which corresponds to the HTI case. We can see a significant difference of curves corresponding to $\theta = 90°$ in the two plots. Relative errors slightly lower for $\theta = 60°$ and significantly lower for $\theta = 90°$ in Figure 1b indicate that the proper choice of the reference velocity α is its choice $\alpha^2 = A_{33}$. This is in agreement with the choice of α in equation (10) for the zero-offset traveltime.

The deviations of the approximate two-way zero-offset traveltimes from their actual values in Figure 1a are caused by the choice of α as the velocity along the axis of symmetry (α=3.094 km/s). For $\theta = 0°$ (VTI), α represents exact vertical velocity, and thus $T(0)$ obtained from equation (10) is also exact. For $\theta = 90°$ (HTI), α differs significantly from the vertical phase velocity (3.81 km/s). As a consequence, $T(0)$ obtained from equation (9) does not represent a good approximation of the actual two-way zero-offset traveltime.

In Figure 1b, α is chosen so that $\alpha^2 = A_{33}$, which represents a better approximation of the vertical velocity than the choice illustrated in Figure 1a. From this reason, the approximate values of $T(0)$ are much closer to the actual ones. Small deviations are caused by the fact discussed after equation (17). A_{33} represents the first-order approximation of the square of the vertical phase velocity corresponding to the actual two-way zero-offset traveltime.

In Figures 2 and 3, we compare accuracy of formulae (14) and (16), respectively. These

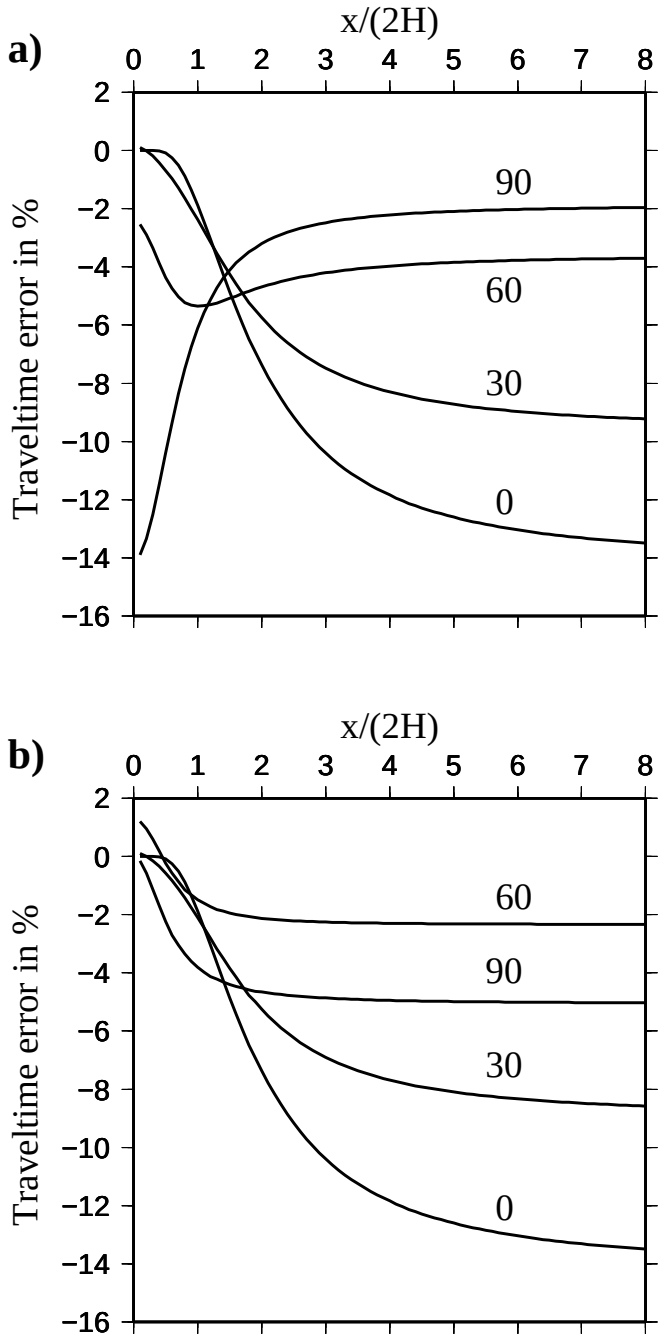

Figure 1: TTI Greenhorn shale model; anisotropy $\sim 26\%$. Variation of relative traveltime errors with the normalized offset $\bar{x} = x/(2H)$, calculated with the first-order approximation (7) for the azimuth $\varphi = 45^o$. Square of the reference velocity is (a) $\alpha^2 = 9.57(km/s)^2$ ($\epsilon_z^{TI} = 0$), (b) $\alpha^2 = A_{33}$ ($\epsilon_z = 0$). Relative errors are estimated for the angles θ made by the axis of symmetry with the vertical x_3-axis, $\theta = 0^o$, 30°, 60° and 90°.

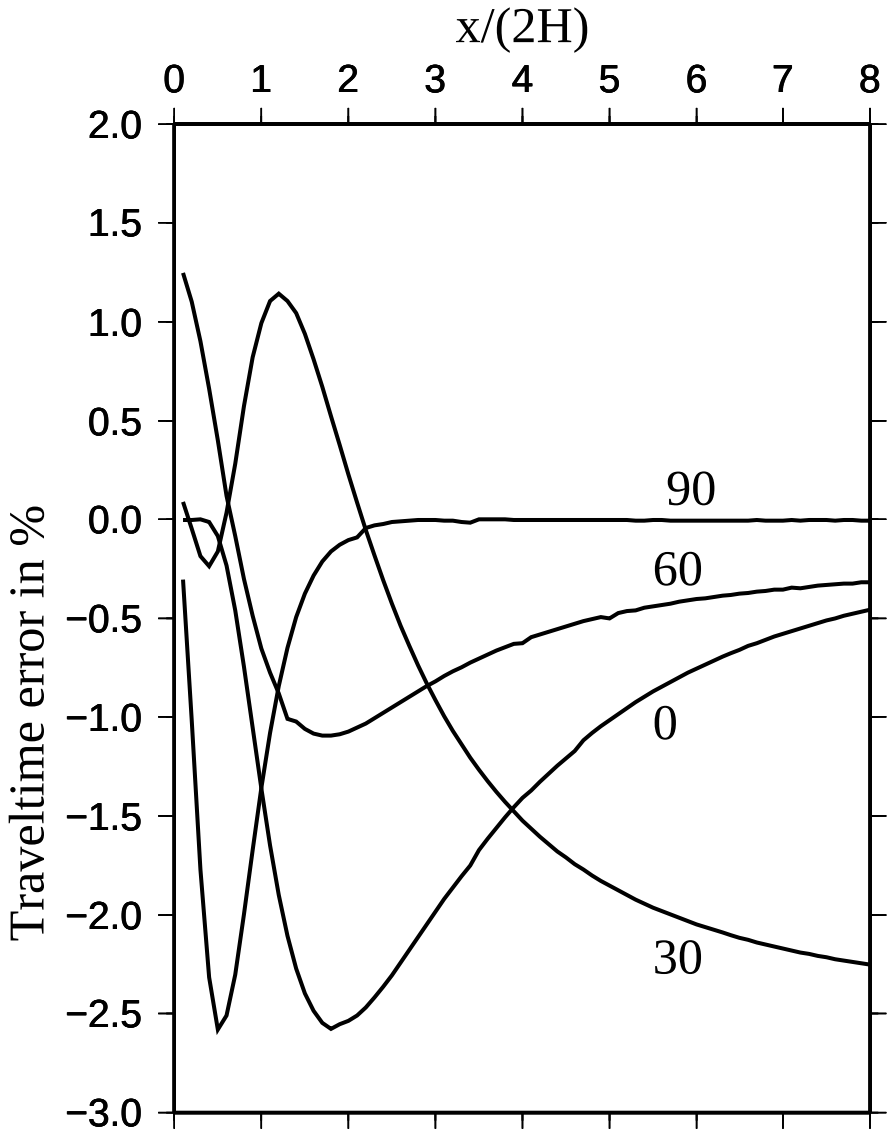

Figure 2: TTI Greenhorn shale model; anisotropy $\sim 26\%$. Variation of relative traveltime errors with the normalized offset $\bar{x} = x/(2H)$, calculated with the first-order approximation (14), for the azimuth $\varphi = 0^o$. Relative errors are estimated for the angles θ made by the axis of symmetry with the vertical x_3-axis, $\theta = 0^o$, 30°, 60° and 90°.

formulae are independent of the choice of α. The azimuth angle φ is specified as $\varphi = 0°$. First of all, we can see a dramatic reduction of the relative traveltime errors with respect to Figure 1. Maximum error is about 2.5%, which is comparable with the maximum error obtained with the corresponding P-wave moveout formula for VTI media proposed by

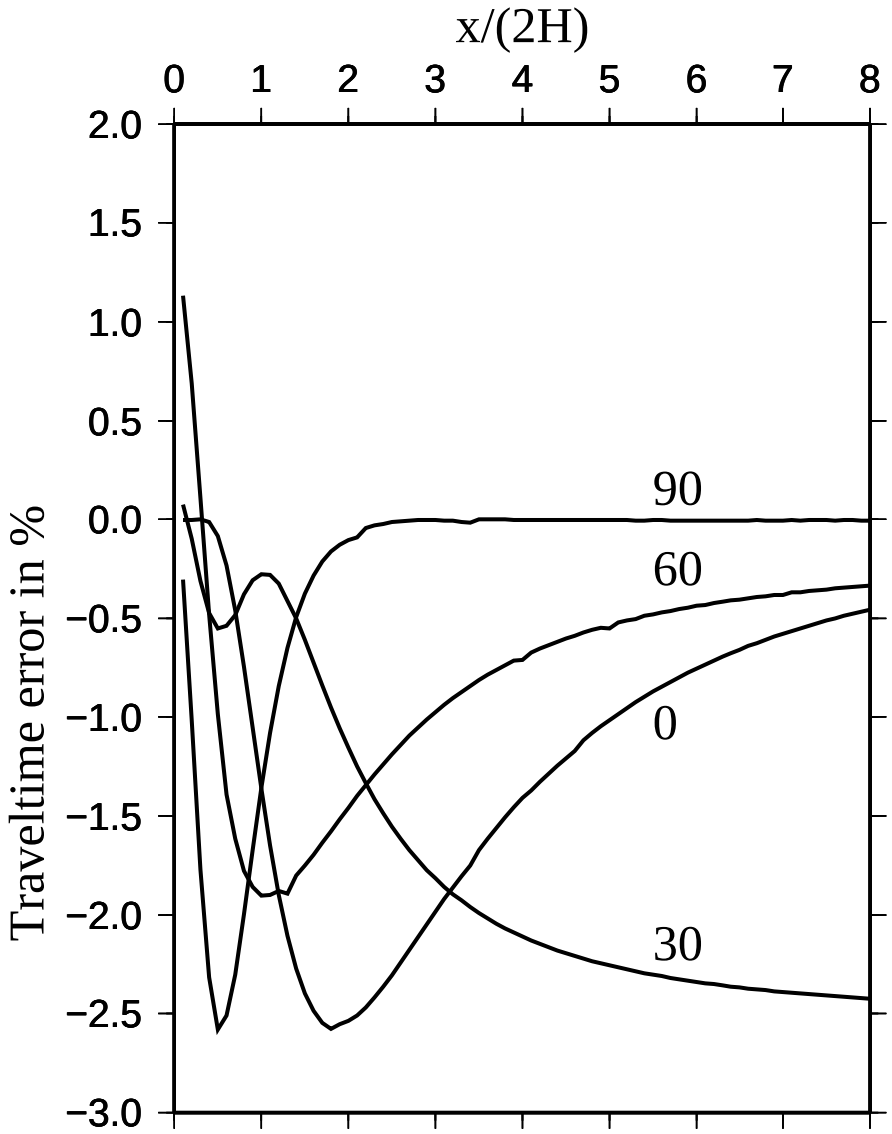

Figure 3: TTI Greenhorn shale model; anisotropy $\sim 26\%$. Variation of relative traveltime errors with the normalized offset $\bar{x} = x/(2H)$, calculated with the first-order approximation (16), for the azimuth $\varphi = 0^o$. Relative errors are estimated for the angles θ made by the axis of symmetry with the vertical x_3-axis, $\theta = 0^o$, 30°, 60° and 90°.

Farra and Pšenčík (2013a) and slightly worse than the long-spread moveout equation (Tsvankin, 2001; equation 4.23). In fact, the curve corresponding to $\theta = 0°$ is identical to the red curve in Figure 2d of Farra and Pšenčík (2013a). Similar maximum error of about 2.5% can be observed for $\theta = 90°$. For intermediate values of polar angles θ, the

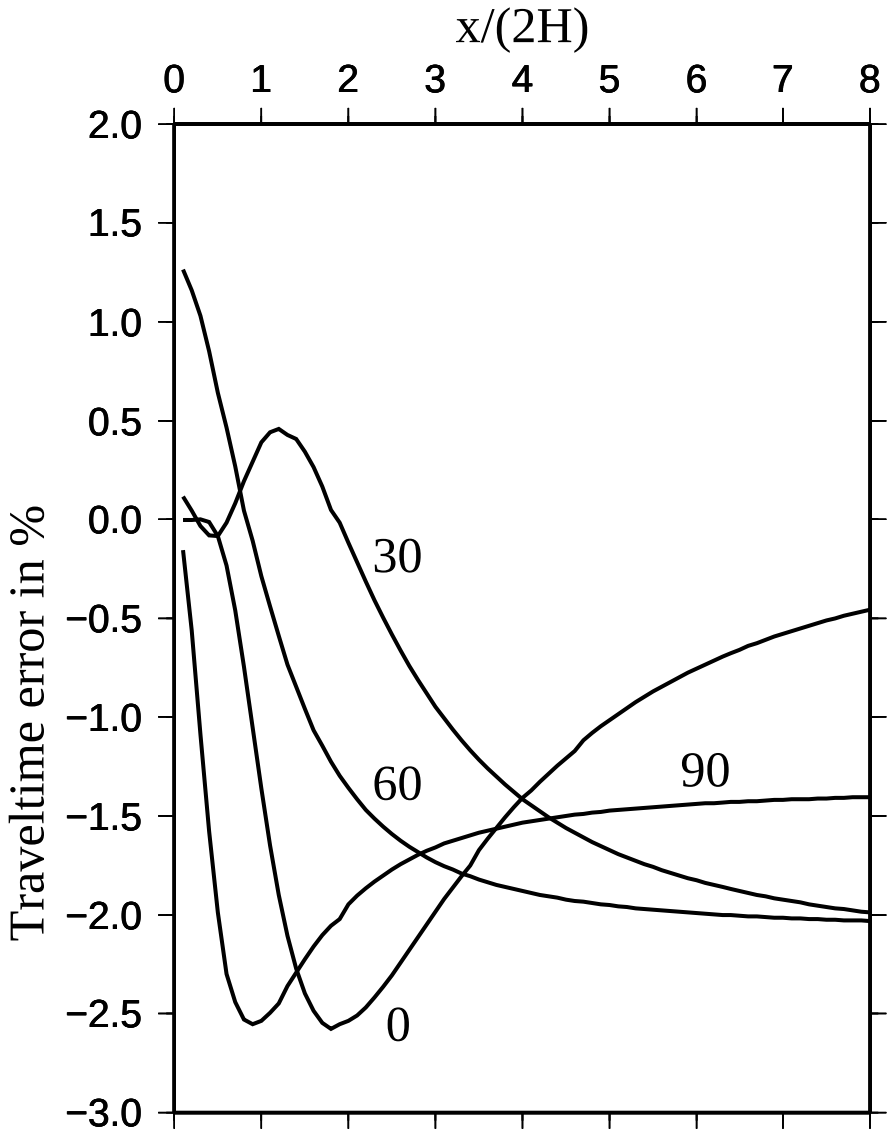

Figure 4: TTI Greenhorn shale model; anisotropy $\sim 26\%$. Variation of relative traveltime errors with the normalized offset $\bar{x} = x/(2H)$, calculated with the first-order approximation (14) for the azimuth $\varphi = 45^o$. Relative errors are estimated for the angles θ made by the axis of symmetry with the vertical x_3-axis, $\theta = 0^o$, 30°, 60° and 90°.

errors are slightly lower. Interesting is mutual comparison of Figures 2 and 3. For $\theta = 0°$ and $90°$, behaviour of the corresponding curves is identical. It is obvious, because these are VTI and HTI cases, for which $\epsilon_{15} = \epsilon_{35} = 0$. For this case the formulae (14) and (16) are identical and thus yield the same results. Different situation is for $\theta = 30°$ and $60°$.

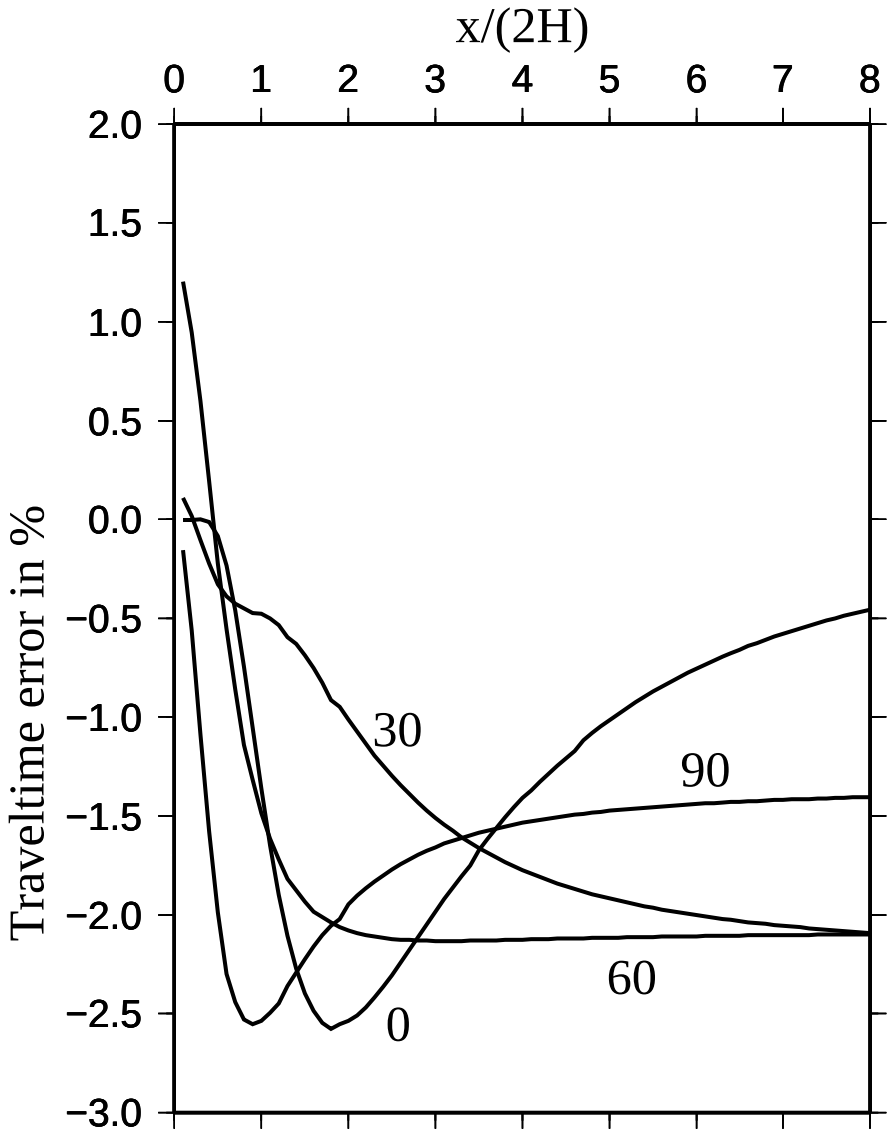

Figure 5: TTI Greenhorn shale model; anisotropy $\sim 26\%$. Variation of relative traveltime errors with the normalized offset $\bar{x} = x/(2H)$, calculated with the first-order approximation (16), for the azimuth $\varphi = 45^o$. Relative errors are estimated for the angles θ made by the axis of symmetry with the vertical x_3-axis, $\theta = 0^o$, 30°, 60° and 90°.

For these angles, ϵ_{15} and ϵ_{35} are non-zero. For $\theta = 30°$, $\epsilon_{15} = -0.178$ and $\epsilon_{35} = -0.044$ and for $\theta = 60°$, $\epsilon_{15} = -0.044$ and $\epsilon_{35} = -0.178$. We can observe a different behaviour of corresponding curves in Figures 2 and 3, but on both figures, the maximum errors are less than 2.5%. This means that we can use simpler equation (16) instead of (14) without loss of accuracy.

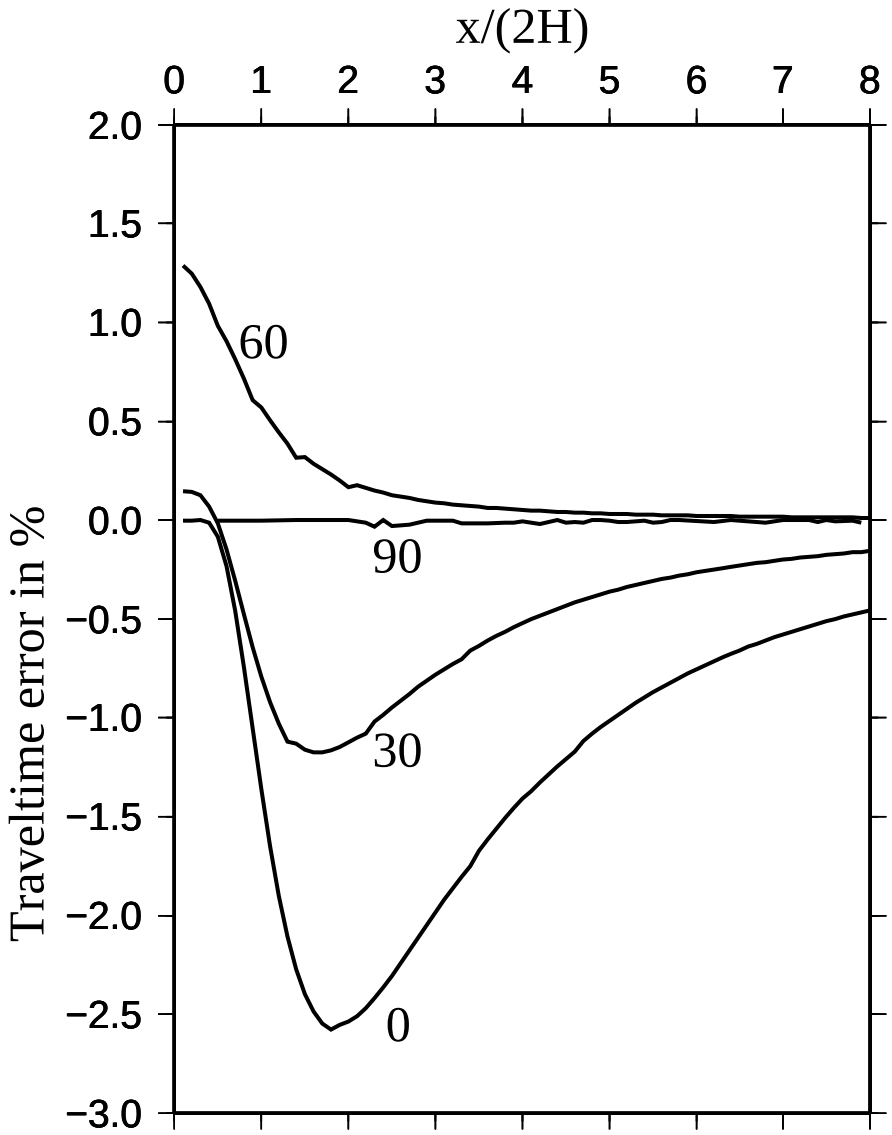

Figure 6: TTI Greenhorn shale model; anisotropy $\sim 26\%$. Variation of relative traveltime errors with the normalized offset $\bar{x} = x/(2H)$, calculated with the first-order approximation (16), for the azimuth $\varphi = 90^o$. Relative errors are estimated for the angles θ made by the axis of symmetry with the vertical x_3-axis, $\theta = 0^o$, 30°, 60° and 90°.

The square of the zero-offset traveltime $T^2(0)$ in Figures 2 and 3 is given by equation (17), which is independent of α. For $\theta = 0°$ and $90°$, equation (17) yields exact zero-offset traveltimes confirmed by the zero relative traveltime errors in Figures 2 and 3. For polar angles $\theta = 30°$ and $60°$, however, A_{33} in equation (17) represents only the first-order approximation of the square of the vertical phase velocity, which leads to traveltime errors

observable in Figures 2 and 3.

Many other tests have shown that one can use equation (16) instead of (14) without loss of accuracy of the approximate formula. One of such tests was for the azimuth of the axis of symmetry $\varphi = 45°$. Results are shown in Figures 4 and 5 for the equation (14) and (16), respectively. Because the polar angles $\theta = 0°$ and $90°$ correspond to the VTI and HTI cases, in which $\epsilon_{15} = \epsilon_{35} = 0$, the corresponding curves are again identical in both figures. For $\theta = 30°$, $\epsilon_{15} = -0.149$ and $\epsilon_{35} = -0.031$, and for $\theta = 60°$, $\epsilon_{15} = -0.102$ and $\epsilon_{35} = -0.126$. The corresponding curves generated by equations (14) and (16) differ, but their maximum error remains below 2.5%. In fact, the maximum error is about 2%.

For completeness, in Figure 6, we show results for the azimuth angle $\varphi = 90°$. In this case both equations (14) and (16) yield the same results because $\epsilon_{15} = \epsilon_{35} = 0$ for any polar angle θ, see equations (B1) and (B6). Maximum error of about 2.5% can be again observed on the curve corresponding to $\theta = 0°$. This case corresponds, as in all previous figures, to the VTI case. The errors for other values of polar angles θ are substantially smaller. The zero error for $\theta = 90°$ is the consequence of the fact that in this case the source-receiver profile is situated in the isotropy plane perpendicular to the axis of symmetry.

Conclusions

We present approximate P-wave moveout formulae for weakly or moderately anisotropic media of *arbitrary* symmetry. The formulae are based on the first-order weak-anisotropy approximation. They are specified by the P-wave WA parameters defined in the global Cartesian coordinate system. By expressing these WA parameters in terms of three P-wave WA parameters ϵ_x^{TI}, ϵ_z^{TI} and δ_y^{TI} and two angles specifying the orientation of the axis of symmetry of a TTI medium, we can obtain an approximate P-wave moveout formula for TTI media with arbitrarily oriented axis of symmetry. The WA parameters ϵ_x^{TI}, ϵ_z^{TI} and δ_y^{TI} are defined in the local (crystal symmetry) Cartesian coordinate system, whose one coordinate axis coincides with the axis of symmetry of the medium.

We test two kinds of formulae. One is based on the first-order approximation of the square of the phase slowness c^{-2}, the other is based on the first-order approximation of the square c^2 of the phase velocity. The former depends on the choice of the value of the reference velocity α, the latter does not. The tests performed indicate clear superiority of the latter approximation. Its accuracy is comparable with the accuracy of formulae proposed previously for anisotropic media of higher symmetry whose one plane of symmetry coincides with the reflector. This is surprising because in contrast to the moveout formulae for the aforementioned media of higher symmetry, the formulae presented in this paper are related to reference rays, which generally differ from the actual rays and whose reflection points are shifted with respect to the reflection points of actual rays (reflection-point dispersal). In our formulae, we make no corrections for the reflection-point dispersal.

The tests show that neglecting WA parameters ϵ_{15} and ϵ_{35}, which reduces presented moveout formulae to the form derived for anisotropic media upto monoclinic symmetry with the reflector coinciding with the symmetry plane, affects accuracy of the presented results negligibly. In weakly or moderately anisotropic media of arbitrary symmetry, it

is thus possible to use formulae depending on only 3 WA parameters in the local profile coordinate system x'_i. Due to our special choice of the source-receiver profile, the local profile and global coordinate systems coincide and, thus, the three needed WA parameters coincide too, $\epsilon_x = \epsilon'_x$, $\epsilon_z = \epsilon'_z$ and $\delta_y = \delta'_y$. For TTI media considered in this paper, these three WA parameters can be obtained, using simple transformation rules, from knowledge of the three TI P-wave WA parameters ϵ_x^{TI}, ϵ_z^{TI} and δ_y^{TI} and of the unit vector $\mathbf{t}$ specifying the orientation of the direction of the axis of symmetry of the TI medium.

The performed tests indicate that the more accurate of the proposed moveout formulae can be used not only in weakly, but also in moderately anisotropic media. The maximum relative traveltime errors of about 2.5% were observed for anisotropy of, approximately, 26%, which cannot be considered weak.

In case of higher-symmetry anisotropic media overlaying a reflector, which coincides with one of their symmetry planes, the more accurate nonhyperbolic moveout formula derived in this paper reduces to formulae derived recently for HTI, VTI, orthorhombic or monoclinic media satisfying the above condition.

The same procedure, which we used in this paper for TTI media, can be used to generalize the presented formulae for tilted orthorhombic or monoclinic media.

The formulae can be simply generalized for media with a dipping reflector Σ. The dip of the reflector and the orientation of the axis of symmetry of the TTI medium do not need to be related. It is, for example, not necessary to confine the symmetry axis to the dip plane of the reflector. In such a case, the approximate relation similar to equation (16), taking into account the dip, will be applicable only for small reflector dips. For larger dips, formula similar to equation (14) must be used.

Generalization to layered media can be carried out using effective medium parameters, following the approaches based on the approaches known from literature, resulting from the knowledge of the NMO velocity and the quartic coefficient of the moveout expansion.

The moveout formulae can be expressed in terms of WA parameters specified with respect to the global coordinate system, i.e., without being related to the orientation of the axis of symmetry, or in terms of WA parameters in the local (crystal symmetry) coordinate system related to the axis of symmetry.

The presented moveout formulae can also be generalized for either separate or coupled shear waves. The form of the moveout formulae holds up hopes for the generalization for converted waves.

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Appendix A

Phase velocity and WA parameters in global Cartesian coordinates

Farra and Pšenčík (2003) offer the first-order approximation of square of the phase velocity in the form:

$$\widetilde{c^2}(\mathbf{n}) = B_{33}(\mathbf{n}) = \alpha^2$$
$$+2\alpha^2[2n_3^3(\epsilon_{34}n_2 + \epsilon_{35}n_1) + n_3^2((\delta_y - \epsilon_x - \epsilon_z)n_1^2 + (\delta_x - \epsilon_y - \epsilon_z)n_2^2 + 2\chi_z n_1 n_2 + \epsilon_z)$$
$$+2n_3(\chi_x n_1^2 n_2 + \chi_y n_1 n_2^2 + \epsilon_{15}n_1^3 + \epsilon_{24}n_2^3)$$
$$+\epsilon_x n_1^2 + \epsilon_y n_2^2 + (\delta_z - \epsilon_x - \epsilon_y)n_1^2 n_2^2 + 2\epsilon_{16}n_1^3 n_2 + 2\epsilon_{26}n_1 n_2^3]. \qquad (A1)$$

Here n_i are components of a unit vector $\mathbf{n}$ specifying the direction, in which the evaluation of the square of the phase velocity is desired. The symbol α denotes the P-wave velocity in the reference isotropic medium. Greek symbols denote 15 P-wave weak-anisotropy parameters:

$$\epsilon_x = \frac{A_{11} - \alpha^2}{2\alpha^2}\ , \quad \epsilon_y = \frac{A_{22} - \alpha^2}{2\alpha^2}\ , \quad \epsilon_z = \frac{A_{33} - \alpha^2}{2\alpha^2}\ ,$$
$$\delta_x = \frac{A_{23} + 2A_{44} - \alpha^2}{\alpha^2}\ , \quad \delta_y = \frac{A_{13} + 2A_{55} - \alpha^2}{\alpha^2}\ , \quad \delta_z = \frac{A_{12} + 2A_{66} - \alpha^2}{\alpha^2}\ ,$$
$$\chi_x = \frac{A_{14} + 2A_{56}}{\alpha^2}\ , \quad \chi_y = \frac{A_{25} + 2A_{46}}{\alpha^2}\ , \quad \chi_z = \frac{A_{36} + 2A_{45}}{\alpha^2}\ ,$$
$$\epsilon_{15} = \frac{A_{15}}{\alpha^2}\ , \quad \epsilon_{16} = \frac{A_{16}}{\alpha^2}\ , \quad \epsilon_{24} = \frac{A_{24}}{\alpha^2}\ , \quad \epsilon_{26} = \frac{A_{26}}{\alpha^2}\ , \quad \epsilon_{34} = \frac{A_{34}}{\alpha^2}\ , \quad \epsilon_{35} = \frac{A_{35}}{\alpha^2}\ . \qquad (A2)$$

For more details on WA parameters, see Farra et al. (2016).

Appendix B

Transformation relations for selected WA parameters

We specify the axis of symmetry of a TTI medium by the vector $\mathbf{t}$. In the global coordinate system, it is specified by the azimuth angle φ and polar angle θ:

$$\mathbf{t} \equiv (\cos\varphi\sin\theta, \sin\varphi\sin\theta, \cos\theta). \qquad (B1)$$

The vector $\mathbf{t}$ is parallel to the x_3^{TI}-axis of the crystal coordinate system x_i^{TI} whose coordinate axes are specified by three unit vectors defined as

$$\mathbf{i}_1 \equiv D^{-1}(t_1t_3, t_2t_3, -D^2), \quad \mathbf{i}_2 \equiv D^{-1}(-t_2, t_1, 0), \quad \mathbf{i}_3 = \mathbf{t} \equiv (t_1, t_2, t_3), \qquad (B2)$$

where

$$D = (t_1^2 + t_2^2)^{1/2}, \quad t_1^2 + t_2^2 + t_3^2 = 1. \qquad (B3)$$

Vectors $\mathbf{i}_1$ and $\mathbf{i}_2$ are situated in the plane perpendicular to the axis of symmetry, the vector $\mathbf{i}_2$ being chosen horizontal. The above three vectors $\mathbf{i}_k$ represent columns of the transformation matrix from the local to global coordinates:

$$\begin{pmatrix} t_1t_3/D & -t_2/D & t_1 \\ t_2t_3/D & t_1/D & t_2 \\ -D & 0 & t_3 \end{pmatrix}. \qquad (B4)$$

In the local coordinate system x_i^{TI}, the P-wave WA parameters are specified as follows:

$$\epsilon_x^{TI} = \epsilon_y^{TI} = \frac{1}{2}\delta_z^{TI}, \quad \delta_x^{TI} = \delta_y^{TI},$$

$$\chi_x^{TI} = \chi_y^{TI} = \chi_z^{TI} = \epsilon_{15}^{TI} = \epsilon_{16}^{TI} = \epsilon_{24}^{TI} = \epsilon_{26}^{TI} = \epsilon_{34}^{TI} = \epsilon_{35}^{TI} = 0 \ . \qquad (B5)$$

It means that the only 3 non-zero WA parameters are ϵ_x^{TI}, ϵ_z^{TI} and δ_y^{TI}.

The moveout equation (14) requires knowledge of 5 WA parameters ϵ_x, ϵ_z, δ_y, ϵ_{15} and ϵ_{35}, equations (7) and (16) of 3 WA parameters ϵ_x, ϵ_z and δ_y in the global Cartesian coordinate system x_i. For completeness, we also add expressions for δ_x and χ_z required in equation (22). All the above WA parameters can be obtained from the WA parameters (B5) using transformation equations involving the components of the vector $\mathbf{t}$, see equation (B1):

$$\epsilon_x = \epsilon_x^{TI}(t_2^2 + t_3^2)^2 + \epsilon_z^{TI} t_1^4 + \delta_y^{TI} t_1^2(t_2^2 + t_3^2) \ ,$$

$$\epsilon_z = \epsilon_x^{TI}(t_1^2 + t_2^2)^2 + \epsilon_z^{TI} t_3^4 + \delta_y^{TI} t_3^2(t_1^2 + t_2^2) \ ,$$

$$\delta_x = 2\epsilon_x^{TI}(3t_2^2 t_3^2 + t_1^2) + 6\epsilon_z^{TI} t_2^2 t_3^2 + \delta_y^{TI}[(t_1^2 + t_3^2)t_3^2 + t_2^2(t_1^2 + t_2^2) - 4t_2^2 t_3^2] \ ,$$

$$\delta_y = 2\epsilon_x^{TI}(3t_1^2 t_3^2 + t_2^2) + 6\epsilon_z^{TI} t_1^2 t_3^2 + \delta_y^{TI}[(t_2^2 + t_3^2)t_3^2 + t_1^2(t_1^2 + t_2^2) - 4t_1^2 t_3^2] \ ,$$

$$\chi_z = 2\epsilon_x^{TI} t_1 t_2(3t_3^2 - 1) + 6\epsilon_z^{TI} t_1 t_2 t_3^2 + \delta_y^{TI} t_1 t_2(1 - 6t_3^2) \ ,$$

$$\epsilon_{15} = -2\epsilon_x^{TI} t_1 t_3(t_2^2 + t_3^2) + 2\epsilon_z^{TI} t_1^3 t_3 + \delta_y^{TI} t_1 t_3(1 - 2t_1^2) \ ,$$

$$\epsilon_{35} = -2\epsilon_x^{TI} t_1 t_3(t_1^2 + t_2^2) + 2\epsilon_z^{TI} t_1 t_3^3 + \delta_y^{TI} t_1 t_3(1 - 2t_3^2) \ . \qquad (B6)$$

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