Calculation of the spatial gradient of the independent parameter along geodesics for a general Hamiltonian function

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Summary

The Hamiltonian geometry is a generalization of the Finsler geometry, which is in turn a generalization of the Riemann geometry. The Hamiltonian geometry is based on the first–order partial differential Hamilton–Jacobi equations for the characteristic function which represents the distance between two points. The Hamiltonian equations of geodesic deviation may serve to calculate geodesic deviations, amplitudes of waves, and the second–order spatial derivatives of the characteristic function or action. The propagator matrix of geodesic deviation contains all the linearly independent solutions of the linear ordinary differential equations of geodesic deviation. The definition of the propagator matrix of geodesic deviation depends on the independent parameter along geodesics.

The previously derived relations between the propagator matrix of the Hamiltonian equations of geodesic deviation and the second–order spatial derivatives of the characteristic function contain the spatial gradients of the independent parameter along the geodesic calculated between two points. In this paper, we derive the equations for calculating the spatial gradients of the independent parameter along geodesics from the propagator matrix of geodesic deviation. These new equations enable us to derive the explicit expressions for the second–order spatial derivatives of the characteristic function in terms of the propagator matrix of geodesic deviation. All equations are derived for a general Hamiltonian function.

Keywords

Hamilton–Jacobi equation, geodesics (rays), geodesic deviation, characteristic function, wave propagation, Hamiltonian geometry, Finsler geometry.

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1. Introduction

The basics of a very general geometry of geodesics (rays) were formulated by Sir William Rowan Hamilton in 1832 (Hamilton, 1837). Hamilton's formulation is based on the first–order partial differential Hamilton–Jacobi equations for the characteristic function which represents the distance between points. The form of the Hamilton–Jacobi equation is specified in terms of the Hamiltonian function. Hamilton (1837) considered the Hamiltonian functions which are homogeneous functions of the first degree with respect to the spatial gradient of the characteristic function, but his theory is applicable to general Hamiltonian functions as well. Hamilton's formulation with a general Hamiltonian function represents a very useful generalization of the Finsler geometry, and describes the propagation of various waves (e.g., elastic, electromagnetic, Dirac) in the high–frequency approximation. A general Hamiltonian function is especially useful in describing waves which propagate with different velocities in opposite directions (e.g., electrons in an electromagnetic field, sound waves in flowing media). The Finsler geometry (Finsler, 1918) represents a special case of the Hamiltonian geometry, with Hamiltonian functions which are homogeneous functions of the second degree with respect to the spatial gradient of the characteristic function.

The non–linear ordinary differential equations of geodesics (Hamilton's equations) may serve to calculate geodesics, the characteristic function (point–to–point distance, two–point travel time) with its first–order spatial derivatives, or the action (distance for general initial conditions, travel time for general initial conditions) with its first–order spatial derivatives. The linear ordinary differential equations of geodesic deviation derived by Červený (1972) may serve to calculate geodesic deviations, amplitudes of waves, the second–order spatial derivatives of the characteristic function, or the second–order spatial derivatives of the action. Geodesic perturbations, higher–order geodesic deviations, and the perturbation derivatives and higher–order spatial derivatives of the characteristic function or of the action can be calculated by quadratures along geodesics (Klimeš, 2002; 2010).

The characteristic function and the propagator matrix of geodesic deviation have found many important applications in wave propagation (Červený, 2001). Klimeš (2013) derived the relations between the propagator matrix of geodesic deviation and the second–order spatial derivatives of the characteristic function for a general Hamiltonian function. The relations contain the spatial gradients of the independent parameter along the geodesic calculated between two points.

In this paper, we derive the equations for calculating the spatial gradients of the independent parameter along geodesics from the propagator matrix of geodesic deviation. These new equations enable us to derive the explicit expressions for the second–order spatial derivatives of the characteristic function in terms of the propagator matrix of geodesic deviation. All equations are derived for a general Hamiltonian function.

2. Hamiltonian geometry of geodesics (rays)

2.1. Hamiltonian function

We consider a smooth manifold (differentiable manifold), and coordinates x^i of its coordinate chart. At each point x^i, the tangent space contains contravariant vectors y^i, and the cotangent space contains covariant vectors y_i such as the gradients of functions. We consider Hamiltonian function $H(x^i, y_j)$, which is a real–valued function of coordinates x^i and of covariant vector y_j from the cotangent space at point x^i, and which is differentiable within its definition domain. The Hamiltonian function may be represented by any reasonably smooth function of x^i and y_j.

Hamilton (1837) called the subset of the "unit" vectors y_j in the cotangent space at point x^i, defined by equation

$$H(x^i, y_j) = C \quad , \tag{1}$$

the *surface of components of normal slowness*. Now it is often called the *phase–slowness surface* or briefly the *slowness surface*, sometimes the *index surface*. In the Finsler geometry, it is referred to as the *figuratrix*. Constant C is determined by the meaning of the Hamiltonian function.

2.2. Action, the characteristic function and the Hamilton–Jacobi equations

The Hamilton–Jacobi equation is a partial differential equation of the first order. The Hamilton–Jacobi equation for *action* (*distance for general initial conditions*, *travel time for general initial conditions*) $S(x^m)$ reads

$$H\big(x^i, \tfrac{\partial S}{\partial x^j}(x^m)\big) = C \quad . \tag{2}$$

Hamilton (1837) also defined the *characteristic function* (*point–to–point distance*, *two–point travel time*)

$$V(x^m, \tilde{x}^n) \tag{3}$$

from point $\tilde{x}^n$ to point x^m. Note that the characteristic function need not be reciprocal,

$$V(x^m, \tilde{x}^n) \neq V(\tilde{x}^n, x^m) \quad . \tag{4}$$

The characteristic function satisfies the Hamilton–Jacobi equations

$$H\big(x^m, \tfrac{\partial V}{\partial x^n}(x^a, \tilde{x}^b)\big) = C \tag{5}$$

and

$$H\big(\tilde{x}^m, -\tfrac{\partial V}{\partial \tilde{x}^n}(x^a, \tilde{x}^b)\big) = C \tag{6}$$

(Hamilton, 1837, eq. C). Note that one of equations (5) and (6) serves as the initial conditions for the other. The Hamilton–Jacobi equations express the requirement that the gradient of the action or of the characteristic function is "unit", see the definition (1) of unit covariant vectors.

2.3. Equations of geodesics

Hamilton's equations (*equations of geodesics, equations of rays, ray tracing equations*) read

$$\frac{\mathrm{d}}{\mathrm{d}\gamma}x^i = \frac{\partial H}{\partial y_i}(x^m, y_n) \quad , \tag{7}$$

$$\frac{\mathrm{d}}{\mathrm{d}\gamma}y_i = -\frac{\partial H}{\partial x^i}(x^m, y_n) \quad . \tag{8}$$

Hamilton (1837) referred to these equations as the *general equations of rays*. Hamilton's equations (7)–(8) can simply be derived by differentiating the Hamilton–Jacobi equation (2) or (5) with respect to coordinates x^j, and putting $y_i = \frac{\partial V}{\partial x^i}(x^a, \tilde{x}^b)$. If we differentiate the Hamilton–Jacobi equation (6) with respect to coordinates $\tilde{x}^j$ and put $\tilde{y}_i = -\frac{\partial V}{\partial \tilde{x}^i}(x^a, \tilde{x}^b)$, we obtain Hamilton's equations

$$\frac{\mathrm{d}}{\mathrm{d}\gamma}\tilde{x}^i = -\frac{\partial H}{\partial \tilde{y}_i}(\tilde{x}^m, \tilde{y}_n) \quad , \tag{9}$$

$$\frac{\mathrm{d}}{\mathrm{d}\gamma}\tilde{y}_i = \frac{\partial H}{\partial \tilde{x}^i}(\tilde{x}^m, \tilde{y}_n) \tag{10}$$

for initial point $\tilde{x}^j$. The meaning of the independent parameter γ along the geodesic and the sensitivity of the geodesic to the initial conditions depend on the form of the Hamiltonian function. Covariant vector y_i in (7)–(8), which represents the first–order partial derivatives of the characteristic function with respect to spatial coordinates, analogously as covariant vector $\tilde{y}_i$ in (9)–(10), was called the *normal slowness* by Hamilton (1837). Now it is usually called the *slowness vector*.

Hamilton's equations (7)–(8) and (9)–(10) directly yield expressions

$$\frac{\partial V}{\partial x^i} = y_i \quad , \tag{11}$$

$$\frac{\partial V}{\partial \tilde{x}^i} = -\tilde{y}_i \tag{12}$$

for the first–order spatial derivatives of the characteristic function (Hamilton, 1837).

Characteristic function $V(x^m, \tilde{x}^n)$ can be calculated by quadrature

$$V(x^m, \tilde{x}^n) = \int_0^\gamma y^r \frac{\mathrm{d}}{\mathrm{d}\gamma}x^r \,\mathrm{d}\gamma \tag{13}$$

along the geodesic obtained using Hamilton's equations (7)–(8), or by quadrature

$$V(x^m, \tilde{x}^n) = -\int_0^\gamma \tilde{y}^r \frac{\mathrm{d}}{\mathrm{d}\gamma}\tilde{x}^r \,\mathrm{d}\gamma \tag{14}$$

along the geodesic obtained using Hamilton's equations (9)–(10).

Hamilton's equations (7)–(8) and (9)–(10) also define function

$$\gamma(x^m, \tilde{x}^n) \tag{15}$$

from point $\tilde{x}^n$ to point x^m, with initial conditions $\gamma(\tilde{x}^m, \tilde{x}^n) = 0$. Note that this function need not be reciprocal,

$$\gamma(x^m, \tilde{x}^n) \neq \gamma(\tilde{x}^n, x^m) \quad . \tag{16}$$

We shall need function $\gamma(x^m, \tilde{x}^n)$ in the relations between the propagator matrix of geodesic deviation and the second–order spatial derivatives of the characteristic function.

2.4. Propagator matrix of geodesic deviation

The *Hamiltonian equations of geodesic deviation* (*paraxial ray equations*, *dynamic ray tracing equations*) derived by Červený (1972) are obtained by differentiating Hamilton's equations (7)–(8) with respect to the initial conditions for the geodesics.

The propagator matrix of geodesic deviation from point $\tilde{x}^b$ to point x^a is defined by equation

$$\mathbf{\Pi}(x^a, \tilde{x}^b) = \begin{pmatrix} \frac{\partial x^i}{\partial \tilde{x}^j} & \frac{\partial x^i}{\partial \tilde{y}_j} \\ \frac{\partial y_i}{\partial \tilde{x}^j} & \frac{\partial y_i}{\partial \tilde{y}_j} \end{pmatrix} \quad , \tag{17}$$

where the derivatives with respect to initial conditions $\tilde{x}^j$, $\tilde{y}_j$ for Hamilton's equations (7)–(8) are taken at fixed parameter γ along geodesics.

The Hamiltonian equations of geodesic deviation for the propagator matrix read

$$\frac{\mathrm{d}}{\mathrm{d}\gamma}\mathbf{\Pi}(x^a, \tilde{x}^b) = \begin{pmatrix} H^{,i}_{,j} & H^{,ij} \\ -H_{,ij} & -H^{,j}_{,i} \end{pmatrix} \mathbf{\Pi}(x^a, \tilde{x}^b) \quad , \tag{18}$$

with unit initial conditions. Here

$$H_{,ij} = \frac{\partial^2 H}{\partial x^i \partial x^j}(x^m, y_n) \quad , \tag{19}$$

$$H^{,i}_{,j} = \frac{\partial^2 H}{\partial y_i \partial x^j}(x^m, y_n) \quad , \tag{20}$$

$$H^{,ij} = \frac{\partial^2 H}{\partial y_i \partial y_j}(x^m, y_n) \quad . \tag{21}$$

Equations (18) of geodesic deviation may differ for different Hamiltonian functions corresponding to equivalent Hamilton–Jacobi equations.

The propagator matrix of geodesic deviation contains all linearly independent solutions of the equations of geodesic deviation, and may thus be used to calculate the geodesic deviation for any initial conditions.

2.5. Relation between the propagator matrix of geodesic deviation and the second–order spatial derivatives of the characteristic function

The linear ordinary differential Hamiltonian equations (18) of geodesic deviation can be used to calculate the propagator matrix (17) of geodesic deviation. The second–order spatial derivatives of the characteristic function can be obtained from the propagator matrix (17) of geodesic deviation.

The unique relations between the second–order spatial derivatives of characteristic function (3) and the propagator matrix (17) of geodesic deviation, derived by Klimeš (2013, eqs. 27–30), read

$$\left(\frac{\partial^2 V}{\partial x^i \partial x^j}+\frac{1}{\Gamma}\frac{\partial\gamma}{\partial x^i}\frac{\partial\gamma}{\partial x^j}\right)\frac{\partial x^j}{\partial\tilde{y}_k}=\frac{\partial y_i}{\partial\tilde{y}_k} \quad , \tag{22}$$

$$\left(\frac{\partial^2 V}{\partial \tilde{x}^i \partial x^j}+\frac{1}{\Gamma}\frac{\partial\gamma}{\partial \tilde{x}^i}\frac{\partial\gamma}{\partial x^j}\right)\frac{\partial x^j}{\partial\tilde{y}_k}=-\delta_i^k \quad , \tag{23}$$

$$\frac{\partial x^i}{\partial\tilde{y}_j}\left(\frac{\partial^2 V}{\partial \tilde{x}^j \partial \tilde{x}^k}+\frac{1}{\Gamma}\frac{\partial\gamma}{\partial \tilde{x}^j}\frac{\partial\gamma}{\partial \tilde{x}^k}\right)=\frac{\partial x^i}{\partial\tilde{x}^k} \quad , \tag{24}$$

where integral

$$\Gamma=\int_0^\gamma\left(\frac{\partial\gamma}{\partial x^r}H^{,rs}\frac{\partial\gamma}{\partial x^s}\right)\mathrm{d}\gamma \tag{25}$$

is calculated along the geodesic, and Kronecker delta δ_i^k represents the components of the identity matrix. Function $\gamma(x^m,\tilde{x}^n)$ is defined in Section 2.3. Relations (22)–(24) are not applicable if integral (25) is equal to zero, which may happen, e.g., if Hamiltonian function $H(x^i,y_j)$ is a homogeneous function of the first degree with respect to y_n.

Only three submatrices of propagator matrix (17) of geodesic deviation are used in relations (22)–(24). Note that the fourth submatrix of matrix (17) carries no additional information; it can be calculated from the three submatrices used in relations (22)–(24) thanks to the symplectic property of the propagator matrix (17) of geodesic deviation.

3. Calculation of the spatial gradient of the independent parameter along geodesics

3.1. Notation

Hereinafter, $\tilde{H}$ refers to Hamiltonian function $\tilde{H} = H(\tilde{x}^p, \tilde{y}_q)$ taken at initial point $\tilde{x}^p$, whereas H refers to Hamiltonian function $H = H(x^m, y_n)$ taken at point x^m. A subscript following a comma denotes the partial derivative with respect to coordinate x^i, e.g., $H_{,i} = \frac{\partial H}{\partial x^i}$ or $V_{,i} = \frac{\partial V}{\partial x^i}$. A subscript with a tilde following a comma denotes the partial derivative with respect to initial coordinate $\tilde{x}^a$, e.g., $\tilde{H}_{,\tilde{a}} = \frac{\partial \tilde{H}}{\partial \tilde{x}^a}$ or $V_{,\tilde{a}} = \frac{\partial V}{\partial \tilde{x}^a}$. A superscript following a comma denotes the partial derivative with respect to slowness vector component y_i or $\tilde{y}_a$, e.g., $H^{,i} = \frac{\partial H}{\partial y_i}$ or $\tilde{H}^{,\tilde{a}} = \frac{\partial \tilde{H}}{\partial \tilde{y}_i}$. The Einstein summation applies also to the pair of an index without a tilde and an equal index with a tilde.

3.2. Spatial gradient of the independent parameter along geodesics

We denote the submatrix of the propagator matrix (17), which is used in equation (23) by $X^{i\tilde{a}}$,

$$X^{i\tilde{a}} = \frac{\partial x^i}{\partial \tilde{y}_a} \quad . \tag{26}$$

We assume that matrix $X^{i\tilde{a}}$ is regular, and denote the inverse matrix by $X_{\tilde{a}k}$,

$$X^{i\tilde{a}} X_{\tilde{a}k} = \delta^i_k \quad . \tag{27}$$

We may then express relation (23) as

$$V_{,\tilde{a}j} + \Gamma^{-1} \gamma_{,\tilde{a}} \gamma_{,j} = -X_{\tilde{a}j} \quad . \tag{28}$$

We now produce four simple identities useful for our derivation. We differentiate the Hamilton–Jacobi equation (5) with respect to $\tilde{x}^a$ and obtain equation (Hamilton, 1837, eqs. U, I)

$$H^{,k} V_{,k\tilde{a}} = 0 \quad . \tag{29}$$

We differentiate the Hamilton–Jacobi equation (6) with respect to x^i and obtain equation (Hamilton, 1837, eqs. X, I)

$$\tilde{H}^{,\tilde{a}} V_{,\tilde{a}i} = 0 \quad . \tag{30}$$

The points x^i and $\tilde{x}^a$ of a geodesic calculated by Hamilton's equations (7)–(8) or (9)–(10) may be parametrized by independent parameter γ along a geodesic, $x^i = x^i(\gamma)$ or $\tilde{x}^a = \tilde{x}^a(\gamma)$. If we keep initial point $\tilde{x}^a$ of the geodesic fixed and differentiate function (15) with respect to γ, we obtain

$$\frac{\mathrm{d}\gamma}{\mathrm{d}\gamma} = \gamma_{,k} \frac{\mathrm{d}x^k}{\mathrm{d}\gamma} \quad . \tag{31}$$

Considering Hamilton's equation (7), identity (31) reads

$$\gamma_{,k} H^{,k} = 1 \quad . \tag{32}$$

If we keep point x^i of the geodesic fixed and differentiate function (15) with respect to γ, we obtain

$$\frac{\mathrm{d}\gamma}{\mathrm{d}\gamma} = \gamma_{,\tilde{a}} \frac{\mathrm{d}\tilde{x}^a}{\mathrm{d}\gamma} \quad . \tag{33}$$

Considering Hamilton's equation (9), identity (33) becomes

$$\gamma_{,\tilde{a}} \tilde{H}^{,\tilde{a}} = -1 \quad . \tag{34}$$

We multiply relation (28) from the left–hand side by contravariant vector $\tilde{H}^{,\tilde{a}}$ and consider identities (30) and (34). We then obtain relation

$$\Gamma^{-1} \gamma_{,j} = \tilde{H}^{,\tilde{a}} X_{\tilde{a}j} \quad , \tag{35}$$

which immediately yields expression

$$\boxed{\gamma_{,j} = \Gamma \, \tilde{H}^{,\tilde{a}} X_{\tilde{a}j}} \tag{36}$$

for the gradient of the independent parameter $\gamma(x^i, \tilde{x}^a)$ along geodesics with respect to point x^i. We multiply relation (28) from the right–hand side by contravariant vector $H^{,j}$ and consider identities (29) and (32). We then obtain relation

$$\Gamma^{-1} \gamma_{,\tilde{a}} = -X_{\tilde{a}j} H^{,j} \quad , \tag{37}$$

which immediately yields expression

$$\boxed{\gamma_{,\tilde{a}} = -\Gamma \, X_{\tilde{a}j} H^{,j}} \tag{38}$$

for the gradient of the independent parameter $\gamma(x^i, \tilde{x}^a)$ along geodesics with respect to point $\tilde{x}^a$. We multiply relation (36) by contravariant vector $H^{,j}$ and consider identity (32), or multiply relation (38) by contravariant vector $\tilde{H}^{,\tilde{a}}$ and consider identity (34), to obtain relation

$$1 = \Gamma \, \tilde{H}^{,\tilde{r}} X_{\tilde{r}s} H^{,s} \quad , \tag{39}$$

which immediately yields expression

$$\boxed{\Gamma = (\tilde{H}^{,\tilde{r}} X_{\tilde{r}s} H^{,s})^{-1}} \tag{40}$$

for integral (25).

Expressions (36), (38) and (40), derived from relation (23), represent the main result of this paper.

3.3. Second–order spatial derivatives of the characteristic function in terms of the propagator matrix of geodesic deviation

If we insert expressions (36) and (38) into relations (22)–(24), we obtain explicit expressions

$$\frac{\partial^2 V}{\partial \tilde{x}^a \partial x^j} = -X_{\tilde{a}j} + \Gamma \, X_{\tilde{a}r} H^{,r} \tilde{H}^{,\tilde{s}} X_{\tilde{s}j} \quad , \tag{41}$$

$$\frac{\partial^2 V}{\partial x^i \partial x^j} = \frac{\partial y_i}{\partial \tilde{y}_c} X_{\tilde{c}j} - \Gamma \, \tilde{H}^{,\tilde{r}} X_{\tilde{r}i} \tilde{H}^{,\tilde{s}} X_{\tilde{s}j} \quad , \tag{42}$$

$$\frac{\partial^2 V}{\partial \tilde{x}^a \partial \tilde{x}^b} = X_{\tilde{a}k} \frac{\partial x^k}{\partial \tilde{x}^b} - \Gamma \, X_{\tilde{a}r} H^{,r} X_{\tilde{b}s} H^{,s} \tag{43}$$

for the second–order spatial derivatives of the characteristic function in terms of the propagator matrix of geodesic deviation. Matrix $X_{\tilde{a}j}$ inverse to matrix $X^{i\tilde{a}}$ is defined by relations (26)–(27), and quantity Γ is given by expression (40).

4. Conclusions

The propagator matrix (17) of geodesic deviation contains all the linearly independent solutions of the equations of geodesic deviation. It can be calculated using the Hamiltonian equations (18) of geodesic deviation.

Relations (22)–(24) between the propagator matrix of geodesic deviation and the second–order spatial derivatives of the characteristic function contain the spatial gradients of the independent parameter along geodesics. The spatial gradients of the independent parameter along geodesics can be calculated from the propagator matrix of geodesic deviation using relations (36) and (38). Integral (25), which also appears in relations (22)–(24), can be calculated from the propagator matrix of geodesic deviation using relation (40).

The second–order spatial derivatives of the characteristic function can be calculated from the propagator matrix of geodesic deviation using relations (41)–(43). However, the propagator matrix of geodesic deviation cannot be calculated from the second–order spatial derivatives of the characteristic function without knowledge of the spatial gradients of the independent parameter along geodesics and of integral (25), see relations (22)–(24).

All equations are derived for a general Hamiltonian function. The derived relations are applicable to the high–frequency approximations of propagation of various waves (e.g., elastic, electromagnetic, Dirac), to the Finsler geometry, to the Riemann geometry, and to their various applications such as general relativity.

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