Componental specification of plane waves in isotropic and anisotropic viscoelastic media

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Summary

Time-harmonic, homogeneous and inhomogeneous plane waves propagating in isotropic and anisotropic viscoelastic media are investigated. The componental specification of the slowness vector $\mathbf{p}$ is used, in which the slowness vector $\mathbf{p}$ is computed from its known projection $\mathbf{p}^\Sigma$ to an arbitrarily chosen plane Σ. The vectors $\mathbf{p}$ and $\mathbf{p}^\Sigma$ are, in general, complex-valued. The most important step in the procedure consists in the determination of the component σ of the slowness vector $\mathbf{p}$ to the normal $\mathbf{n}^\Sigma$ to Σ. For general anisotropic viscoelastic media, the component σ is a root of an algebraic equation of the sixth degree, with complex-valued coefficients. For isotropic viscoelastic media, the algebraic equation of the sixth degree factorizes to simple quadratic equations. For SH plane waves propagating in the plane of symmetry of a monoclinic (orthorhombic, hexagonal) viscoelastic medium it also factorizes providing a quadratic equation for SH waves. The componental specification of the slowness vector plays an important role in the solution of the problem of the reflection/transmission of plane waves at a plane interface between two viscoelastic anisotropic media.

Key words: attenuation, seismic anisotropy, theoretical seismology, viscoelasticity.

1 Introduction

In this paper, we study homogeneous and inhomogeneous time-harmonic plane waves, propagating in isotropic or anisotropic, viscoelastic or elastic media. We consider a homogeneous or inhomogeneous plane wave with the complex-valued slowness vector $\mathbf{p}$, specified by the so-called componental specification with respect to some arbitrarily chosen plane Σ. The complete slowness vector $\mathbf{p}$, however, is not known, only its complex-valued projection $\mathbf{p}^\Sigma$ to the plane Σ is given. The goal is to determine the complete complex-valued slowness vector $\mathbf{p}$ of the plane wave under consideration from the vector

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$\mathbf{p}^{\Sigma}$ situated in the plane Σ. The most important step in the procedure is the determination of the unknown component σ of the slowness vector to the normal $\mathbf{n}^{\Sigma}$ to the plane Σ. This problem plays an important role in various seismological applications, particularly in the problem of reflection and transmission of plane waves at a planar interface and in the problem of displacement-stress propagator matrices.

The componental specification of the slowness vector was briefly explained by Červený and Pšenčík (2005a,b). In the componental specification, the complex-valued slowness vector $\mathbf{p}$ is given by the relation $\mathbf{p} = \sigma\mathbf{n}^{\Sigma} + \mathbf{p}^{\Sigma}$, where $\mathbf{n}^{\Sigma}$ is a real-valued unit normal to the plane Σ, and $\mathbf{p}^{\Sigma}$ is the projection of the slowness vector $\mathbf{p}$ to the plane Σ. The quantity σ, representing the component of the slowness vector to $\mathbf{n}^{\Sigma}$, is sought as a function of the known $\mathbf{p}^{\Sigma}$. Both σ and $\mathbf{p}^{\Sigma}$ may be, in general, complex-valued.

The componental specification of the slowness vector has been broadly used in the solution of reflection-transmission problem of plane waves at a plane interface **between two homogeneous perfectly elastic isotropic or anisotropic media**. For anisotropic media see, e.g., Fedorov (1968), Musgrave (1970), Gajewski and Pšenčík (1987), Helbig (1994). For a review, see Červený (2001, Section 5.4.7). It has also been used in the computation and application of displacement-stress propagator matrices in **perfectly elastic isotropic or anisotropic 1-D media**, see, e.g., Kennett (1983, 2001), Chapman (1994, 2004), Thomson (1996a,b). A very similar approach, called usually the Stroh formalism (Stroh, 1962), has been used in applied mathematics and mechanics (Shuvalov, 2001).

For isotropic and anisotropic viscoelastic media, the mixed specification of the slowness vector has been also used (Červený and Pšenčík, 2005a,b). The mixed specification corresponds to the componental specification, for which the known vector $\mathbf{p}^{\Sigma}$ is purely imaginary. The choice of purely imaginary $\mathbf{p}^{\Sigma}$ implies that the plane Σ represents the wavefront of the plane wave under consideration. It was proved that the slowness vector $\mathbf{p}$ of any inhomogeneous or homogeneous plane wave propagating in an isotropic or anisotropic, viscoelastic or perfectly elastic medium without interfaces (from now on, we refer to such media as homogeneous) may be described by this specification. Analogously, any purely imaginary vector $\mathbf{p}^{\Sigma}$ yields an inhomogeneous or homogeneous (if $\mathbf{p}^{\Sigma} = 0$) plane wave propagating in a medium under consideration.

It is possible to show that any plane wave, either inhomogeneous or homogeneous, can be alternatively described by the componental or mixed specification of the slowness vector. The mixed specification, in which the plane Σ represents the wavefront of the plane wave under consideration, is more suitable for the investigation of basic properties of homogeneous and inhomogeneous plane waves in unbounded media. From the computational point of view, it is simpler and more straightforward, as it is not connected with problems of the specification of correct signs of square roots in the complex plane. The important physical quantities of the plane wave, like phase velocity, attenuation angle, etc., are obtained easily. The advantage of componental specification is that it gives directly the component σ of the slowness vector $\mathbf{p}$ to the normal of an arbitrarily chosen plane Σ, not necessarily coinciding with the wavefront. This is the main reason why the componental specification is more useful in the solution of the reflection/transmission problem at an arbitrarily situated and oriented planar interface Σ.

In this paper, we discuss the determination of the component σ of the complex-valued slowness vector $\mathbf{p}$ to the normal of an arbitrarily chosen plane Σ in an anisotropic or isotropic, viscoelastic or perfectly elastic medium, assuming the vector $\mathbf{p}^{\Sigma}$ situated in the plane Σ being known. In Section 2, we describe briefly the properties of homogeneous and inhomogeneous plane waves, propagating in anisotropic viscoelastic media, and give expressions for their phase velocity $\mathcal{C}$ and the attenuation angle γ (also called the inhomogeneity angle). We describe briefly the mixed specification of the slowness vector. In Section 3, we use the componental specification of the slowness vector, and derive expressions for the component σ of the slowness vector $\mathbf{p}$ to the normal of the plane Σ. We also present expressions for the unit vectors $\mathbf{n}^P$ and $\mathbf{n}^A$, specifying directions of the propagation and attenuation vectors. Further, we present expressions for the phase velocity $\mathcal{C}$ and the attenuation angle γ of the relevant homogeneous or inhomogeneous plane wave specified on the plane Σ. We also present relations between componental and mixed specifications in a homogeneous medium. In Section 4, we discuss certain special choices of the complex-valued vector $\mathbf{p}^{\Sigma}$ in the componental specification, such as the real-valued $\mathbf{p}^{\Sigma}$, the imaginary-valued $\mathbf{p}^{\Sigma}$, and the so-called coplanar case. By the "coplanar case", we understand here the case, in which the unit normal $\mathbf{n}^{\Sigma}$ to the plane Σ and the vectors $\mathrm{Re}\mathbf{p}$ and $\mathrm{Im}\mathbf{p}$ are all situated in one plane. If this is not the case, we speak of non-coplanar case. In Section 5, we pay attention to isotropic viscoelastic media and to the case of SH plane waves propagating in a plane of symmetry of a monoclinic (including orthorhombic and hexagonal) medium, for which all expressions can be written in a simple analytical form. In Section 6, we discuss applications of the componental and mixed specification of the slowness vector in the problem of reflection/transmission of waves at a plane interface separating anisotropic or isotropic viscoelastic media. In Section 7, we offer some concluding remarks.

2 Inhomogeneous plane waves in anisotropic viscoelastic media

We discuss time-harmonic plane waves propagating in homogeneous anisotropic viscoelastic media, and describe them in the following way:

$$u_i(x_j,t) = aU_j \exp[-\mathrm{i}\omega(t - p_n x_n)] \ . \tag{1}$$

Here u_i, U_i, and p_i are Cartesian components of the complex-valued displacement vector $\mathbf{u}$, the polarization vector $\mathbf{U}$ and the slowness vector $\mathbf{p}$. The symbol a denotes the complex-valued scalar amplitude. The anisotropic viscoelastic medium under consideration is specified by complex-valued density-normalized viscoelastic moduli a_{ijkl}, satisfying the symmetry relations

$$a_{ijkl} = a_{jikl} = a_{ijlk} = a_{klij} \ . \tag{2}$$

We use the notation

$$a_{ijkl} = a^R_{ijkl} - \mathrm{i}a^I_{ijkl} \ , \tag{3}$$

with the sign "$-$" of the imaginary part. This sign is related to the minus sign in the exponential factor of (1). In the Voigt notation, the moduli $a^R_{ijkl} - \mathrm{i}a^I_{ijkl}$ are replaced by the elements $A^R_{\alpha\beta} - \mathrm{i}A^I_{\alpha\beta}$ of the 6×6 complex-valued symmetric matrix $\mathbf{A}^R - \mathrm{i}\mathbf{A}^I$. We assume that the real-valued matrix $\mathbf{A}^R$ is positive definite and $\mathbf{A}^I$ is positive definite or zero. The density-normalized viscoelastic moduli a_{ijkl} and the elements $A_{\alpha\beta}$ are frequency dependent. We, however, do not write $a_{ijkl}(\omega)$ or $A_{\alpha\beta}(\omega)$ with the argument ω, because we consider arbitrary, but fixed frequency $\omega > 0$.

Assuming the validity of the correspondence principle (Carcione, 2007), the plane wave (1) must satisfy the time-harmonic equation of motion. This allows us to determine U_j and p_j. The equation of motion yields three linear algebraic equations for U_i,

$$\Gamma_{ik} U_k = U_i \ , \qquad i = 1, 2, 3 \ , \tag{4}$$

where $\mathbf{\Gamma}$ is the 3×3 generalized Christoffel matrix, with elements

$$\Gamma_{ik} = a_{ijkl} p_j p_l \ . \tag{5}$$

The condition of solvability of the system of equations (4) for U_1, U_2, U_3 is as follows:

$$\det[a_{ijkl} p_j p_l - \delta_{ik}] = 0 \ . \tag{6}$$

The slowness vector $\mathbf{p}$ satisfying the constraint relation (6) can be used to solve the system of linear equations (4) and determine the corresponding polarization vector $\mathbf{U}$. The complex-valued scalar amplitude a may be chosen freely.

Instead of the slowness vector $\mathbf{p}$, the wave vector $\mathbf{k} = \omega\mathbf{p}$ has been often used in seismological literature. As we consider only time-harmonic waves with the constant circular frequency ω, the difference between $\mathbf{p}$ and $\mathbf{k}$ is only formal, and we use only $\mathbf{p}$ in the following.

The complex-valued slowness vector $\mathbf{p}$ can be expressed in the following form:

$$\mathbf{p} = \mathbf{P} + \mathrm{i}\mathbf{A} \ , \tag{7}$$

where the vectors $\mathbf{P} = \mathrm{Re}\mathbf{p}$ and $\mathbf{A} = \mathrm{Im}\mathbf{p}$ are real-valued; $\mathbf{P}$ is called the propagation vector and $\mathbf{A}$ the attenuation vector. We introduce real-valued unit vectors $\mathbf{n}^P$ and $\mathbf{n}^A$ along $\mathbf{P}$ and $\mathbf{A}$ as

$$\mathbf{n}^P = \mathbf{P}/|\mathbf{P}| \ , \qquad \mathbf{n}^A = \mathbf{A}/|\mathbf{A}| \ . \tag{8}$$

The plane wave is called homogeneous if $\mathbf{n}^P = \mathbf{n}^A$, and inhomogeneous if $\mathbf{n}^P \neq \mathbf{n}^A$. We define the attenuation angle γ (also often called the inhomogeneity angle) as a non-oriented, non-negative angle made by real-valued unit vectors $\mathbf{n}^P$ and $\mathbf{n}^A$:

$$\cos\gamma = \mathbf{n}^P \cdot \mathbf{n}^A = \mathbf{P} \cdot \mathbf{A}/|\mathbf{P}||\mathbf{A}| \ . \tag{9}$$

Consequently, the plane wave is homogeneous for $\gamma = 0$, and inhomogeneous for $\gamma > 0$. For inhomogeneous plane waves, the plane specified by $\mathbf{n}^P$ and $\mathbf{n}^A$ is called the propagation-attenuation plane. The phase velocity $\mathcal{C}$ of a homogeneous or inhomogeneous plane wave is defined as

$$\mathcal{C} = 1/|\mathbf{P}| \ , \tag{10}$$

see Červený and Pšenčík (2005a).

Any plane wave, homogeneous or inhomogeneous, propagating in a homogeneous anisotropic viscoelastic medium is fully determined if its complex-valued slowness vector $\mathbf{p}$, complex-valued polarization vector $\mathbf{U}$ and the complex-valued scalar amplitude a are known. As the scalar amplitude a may be chosen arbitrarily and the polarization vector $\mathbf{U}$ can be simply calculated from (4) once $\mathbf{p}$ is known, the decisive role in the computation of plane waves is played by the slowness vector $\mathbf{p}$ satisfying the constraint relation (6). Several methods can be used for this purpose. One of them is the mixed specification of the slowness vector (Červený and Pšenčík, 2005a,b).

The mixed specification of the slowness vector has the form

$$\mathbf{p} = \sigma^W \mathbf{n} + \mathrm{i}D\mathbf{m} \qquad \text{with} \quad \mathbf{n} \cdot \mathbf{m} = 0 \; . \tag{11}$$

Here $\mathbf{n}$ and $\mathbf{m}$ are two mutually perpendicular unit vectors, and D is a real-valued scalar, $D \in (-\infty, \infty)$, called the inhomogeneity parameter. For $\mathbf{n}$, $\mathbf{m}$ and D given, the complex-valued scalar σ^W can be determined from equation

$$\det[a_{ijkl}(\sigma^W n_j + \mathrm{i}Dm_j)(\sigma^W n_l + \mathrm{i}Dm_l) - \delta_{ik}] = 0 \; . \tag{12}$$

Eq.(12) follows from (6), into which we inserted (11). The physical meaning of parameters $\mathbf{n}$, $\mathbf{m}$ and D is as follows: The unit vector $\mathbf{n}$ is perpendicular to the wavefront, i.e., parallel to the propagation vector and identical with $\mathbf{n}^P$. The unit vector $\mathbf{m}$, perpendicular to $\mathbf{n}$, defines (together with $\mathbf{n}$) the propagation-attenuation plane, in which vectors $\mathbf{n}^P$ and $\mathbf{n}^A$ are situated. It is, however, different from $\mathbf{n}^A$, as it is perpendicular to $\mathbf{n}$. The inhomogeneity parameter D is a measure of inhomogeneity of the plane wave in the propagation-attenuation plane. For $D = 0$, the plane wave is homogeneous, and for $D \neq 0$ inhomogeneous. The complex-valued scalar σ^W represents the component of the slowness vector $\mathbf{p}$ to the vector $\mathbf{n}$.

The slowness vector $\mathbf{p}$ of any inhomogeneous plane wave can be uniquely specified by unit vectors $\mathbf{n}$ and $\mathbf{m}$ and by the inhomogeneity parameter $D \neq 0$. Other way round, any parameters $\mathbf{n}$, $\mathbf{m}$ and $D \neq 0$ specify uniquely one slowness vector of an inhomogeneous plane wave.

The mixed specification of the slowness vector can be used for the investigation of homogeneous and inhomogeneous plane waves propagating in unbounded viscoelastic media without interfaces. For plane waves specified at an arbitrary plane Σ by the vector $\mathbf{p}^\Sigma$, it is more suitable to use the componental specification of the slowness vector.

3 Componental specification of the slowness vector

3.1 Definition and properties of the componental specification

The componental specification of the slowness vector is particularly useful in the solution of reflection/transmission problems. Let us emphasize that any slowness vector expressed in terms of the componental specification at an arbitrary plane Σ can be also expressed in terms of the mixed specification and vice versa.

In the componental specification of the complex-valued slowness vector $\mathbf{p}$ at an arbitrarily chosen plane Σ, we proceed as follows. We denote the real-valued unit normal to Σ by $\mathbf{n}^\Sigma$, and specify $\mathbf{p}$ in the following way:

$$\mathbf{p} = \sigma \mathbf{n}^\Sigma + \mathbf{p}^\Sigma \ , \qquad \text{with} \ \ \mathbf{p}^\Sigma \cdot \mathbf{n}^\Sigma = 0 \ . \tag{13}$$

Here $\mathbf{p}^\Sigma$ is a complex-valued vector, situated in the plane Σ. The unit vector $\mathbf{n}^\Sigma$ and the vector $\mathbf{p}^\Sigma$ are assumed to be known, but the complex-valued scalar quantity σ is to be determined from the constraint relation (6). Inserting (13) into (6) yields a polynomial equation of the sixth degree for σ with complex-valued coefficients, which reads:

$$\det[a_{ijkl}(\sigma n_j^\Sigma + p_j^\Sigma)(\sigma n_l^\Sigma + p_l^\Sigma) - \delta_{ik}] = 0 \ . \tag{14}$$

This is a basic equation of this paper. It has always six generally complex-valued roots σ. These roots correspond to P, S1 and S2 waves, propagating in opposite directions. In some special cases, the equation may factorize into two equations, one of the second degree and one of the fourth degree or to three equations of the second degree.

Once the roots σ of equation (14) are found, we can determine the propagation vector $\mathbf{P}$ and the attenuation vector $\mathbf{A}$ from:

$$\mathbf{P} = \mathbf{n}^\Sigma \mathrm{Re}\sigma + \mathrm{Re}\mathbf{p}^\Sigma \ , \qquad \mathbf{A} = \mathbf{n}^\Sigma \mathrm{Im}\sigma + \mathrm{Im}\mathbf{p}^\Sigma \ . \tag{15}$$

The magnitudes of these vectors are:

$$\begin{aligned} |\mathbf{P}| &= [(\mathrm{Re}\sigma)^2 + (\mathrm{Re}\mathbf{p}^\Sigma)(\mathrm{Re}\mathbf{p}^\Sigma)]^{1/2} \ , \\ |\mathbf{A}| &= [(\mathrm{Im}\sigma)^2 + (\mathrm{Im}\mathbf{p}^\Sigma)(\mathrm{Im}\mathbf{p}^\Sigma)]^{1/2} \ . \end{aligned} \tag{16}$$

The directions of vectors $\mathbf{P}$ and $\mathbf{A}$ are given by real-valued unit vectors $\mathbf{n}^P$ and $\mathbf{n}^A$, see (8),

$$\begin{aligned} \mathbf{n}^P &= (\mathbf{n}^\Sigma \mathrm{Re}\sigma + \mathrm{Re}\mathbf{p}^\Sigma)/[(\mathrm{Re}\sigma)^2 + (\mathrm{Re}\mathbf{p}^\Sigma)(\mathrm{Re}\mathbf{p}^\Sigma)]^{1/2} \ , \\ \mathbf{n}^A &= (\mathbf{n}^\Sigma \mathrm{Im}\sigma + \mathrm{Im}\mathbf{p}^\Sigma)/[(\mathrm{Im}\sigma)^2 + (\mathrm{Im}\mathbf{p}^\Sigma)(\mathrm{Im}\mathbf{p}^\Sigma)]^{1/2} \ . \end{aligned} \tag{17}$$

The phase velocity $\mathcal{C}$ of the plane wave under consideration is given by the relation $\mathcal{C} = 1/|\mathbf{P}|$, see (10):

$$\mathcal{C} = 1/[(\mathrm{Re}\sigma)^2 + (\mathrm{Re}\mathbf{p}^\Sigma)(\mathrm{Re}\mathbf{p}^\Sigma)]^{1/2} \ . \tag{18}$$

The attenuation angle γ is given by the relation $\cos\gamma = \mathbf{n}^P \cdot \mathbf{n}^A$, so that

$$\cos\gamma = \frac{(\mathrm{Re}\sigma)(\mathrm{Im}\sigma) + (\mathrm{Re}\mathbf{p}^\Sigma)(\mathrm{Im}\mathbf{p}^\Sigma)}{[(\mathrm{Re}\sigma)^2 + (\mathrm{Re}\mathbf{p}^\Sigma)(\mathrm{Re}\mathbf{p}^\Sigma)]^{1/2}[(\mathrm{Im}\sigma)^2 + (\mathrm{Im}\mathbf{p}^\Sigma)(\mathrm{Im}\mathbf{p}^\Sigma)]^{1/2}} \ . \tag{19}$$

3.2 Relations between componental and mixed specifications of the slowness vector

In a homogeneous medium, the componental and mixed specifications of the slowness vector $\mathbf{p}$ can be used alternatively; one of them can be expressed in terms of the other.

The plane Σ used in the componental specification can be selected arbitrarily. The relevant transformation does not require any solution of the algebraic equation of the sixth degree.

We use the slowness vector $\mathbf{p}$ specified by the componental specification in the form (13) with σ being a solution of the algebraic equation (14), and by the mixed specification in the form (11) with σ^W being a solution of the algebraic equation (12). We thus require that the expressions for the slowness vector $\mathbf{p}$ in both specifications are equal:

$$\sigma^W \mathbf{n} + \mathrm{i} D \mathbf{m} = \sigma \mathbf{n}^\Sigma + \mathbf{p}^\Sigma \ . \tag{20}$$

In the following, we shall use this equation to find transformations from mixed to componental specification and back.

a) **Transformation from componental to mixed specification**

In this case, the complex-valued scalar quantity σ, real-valued unit vector $\mathbf{n}^\Sigma$ and complex-valued vector $\mathbf{p}^\Sigma$ are assumed to be known. Complex-valued scalar quantity σ^W, mutually perpendicular, real-valued unit vectors $\mathbf{n}$ and $\mathbf{m}$, and the real-valued scalar D are to be determined.

The unit vector $\mathbf{n}$ specifies the orientation of the propagation vector and it thus reads

$$\mathbf{n} = \mathbf{n}^P \ , \tag{21}$$

where $\mathbf{n}^P$ is given in (17). Multiplying (20) by $\mathbf{n}^P$, we obtain σ^W,

$$\sigma^W = (\sigma \mathbf{n}^\Sigma + \mathbf{p}^\Sigma) \cdot \mathbf{n}^P \ . \tag{22}$$

Equation (20) then yields

$$D\mathbf{m} = \mathrm{Im}(\sigma \mathbf{n}^\Sigma - \sigma^W \mathbf{n}^P + \mathbf{p}^\Sigma) \ . \tag{23}$$

From (23), we can determine both $\mathbf{m}$ and D separately:

$$\begin{aligned} \mathbf{m} &= \mathrm{Im}(\sigma \mathbf{n}^\Sigma - \sigma^W \mathbf{n}^P + \mathbf{p}^\Sigma) / |\mathrm{Im}(\sigma \mathbf{n}^\Sigma - \sigma^W \mathbf{n}^P + \mathbf{p}^\Sigma)| \ , && (24) \\ D &= |\mathrm{Im}(\sigma \mathbf{n}^\Sigma - \sigma^W \mathbf{n}^P + p^\Sigma)| \ . && (25) \end{aligned}$$

Equations (21)–(25) allow to express the slowness vector (13) in mixed specification. In (24)–(25), we can change signs of both $\mathbf{m}$ and D, without affecting the slowness vector.

As we can see in (22), we do not need to solve the algebraic equation of the sixth degree for σ^W if we wish to transform the componental specification to the mixed specification of the slowness vector, for which σ is known. Moreover, we do not need to seek the unit vector $\mathbf{n}$ perpendicular to the wavefront; we obtain it using (21).

b) **Transformation from mixed to componental specification**

In this case, the complex-valued scalar quantity σ^W, mutually perpendicular, real-valued unit vectors $\mathbf{n}$ and $\mathbf{m}$, and real-valued scalar D are assumed to be known. In addition, we must know the real-valued unit vector $\mathbf{n}^\Sigma$, which specifies the plane Σ, to which $\mathbf{n}^\Sigma$ is perpendicular. The complex-valued scalar quantity σ and the complex-valued vector $\mathbf{p}^\Sigma$ are to be computed.

The transformation is now simpler than in the previous cases. Multiplying (20) by known $\mathbf{n}^\Sigma$, we obtain

$$\sigma = (\sigma^W \mathbf{n} + \mathrm{i}D\mathbf{m}) \cdot \mathbf{n}^\Sigma \ . \tag{26}$$

From (20), we further obtain

$$\mathbf{p}^\Sigma = \sigma^W \mathbf{n} - \sigma \mathbf{n}^\Sigma + \mathrm{i}D\mathbf{m} \ . \tag{27}$$

Equations (26) and (27) allow to express the slowness vector (11) in the componental form. Equations for the componental specification of the slowness vector strongly depend on the position of the plane Σ, specified by the unit vector $\mathbf{n}^\Sigma$ perpendicular to Σ.

c) **Transformation from one componental to another componental specification**

We consider two planes Σ_1 and Σ_2, with unit normals $\mathbf{n}^1$ and $\mathbf{n}^2$, and a slowness vector in the componental specifications related to these planes. We wish to transform the componental specifications from the plane Σ_1 to the plane Σ_2 for the given fixed slowness vector. Then

$$\sigma_1 \mathbf{n}^1 + \mathbf{p}^{\Sigma 1} = \sigma_2 \mathbf{n}^2 + \mathbf{p}^{\Sigma 2} \ . \tag{28}$$

Here $\sigma_1, \mathbf{n}^1$ and $\mathbf{p}^{\Sigma 1}$ corresponds to the plane Σ_1, $\sigma_2, \mathbf{n}^2$ and $\mathbf{p}^{\Sigma 2}$ to the plane Σ_2.

Assume that the quantities σ_1, $\mathbf{n}_1$ and $\mathbf{p}^{\Sigma 1}$ are known. To determine the relevant quantities for $\Sigma 2$, we must also give $\mathbf{n}^2$. From (28), we obtain

$$\sigma_2 = (\sigma_1 \mathbf{n}^1 + \mathbf{p}^{\Sigma 1}) \cdot \mathbf{n}^2 \ . \tag{29}$$

Eq. (28) also yields

$$\mathbf{p}^{\Sigma 2} = \sigma_1 \mathbf{n}^1 - \sigma_2 \mathbf{n}^2 + \mathbf{p}^{\Sigma 1} \ . \tag{30}$$

4 Special choices of the vector $\mathbf{p}^\Sigma$

Equations (4), (6), (13)-(19) are general and valid for any anisotropic or isotropic, viscoelastic or perfectly elastic medium. They are also valid for arbitrarily chosen plane Σ, and for arbitrary complex-, real- or imaginary-valued vector $\mathbf{p}^\Sigma$, specified at Σ. In this section, we consider an arbitrary medium, but special choices of the vector $\mathbf{p}^\Sigma$.

4.1 Real-valued $\mathbf{p}^\Sigma$

We denote by $\mathbf{e}$ a real-valued unit vector arbitrarily situated in the plane Σ, and by S a real-valued apparent slowness along $\mathbf{e}$. We define the vector $\mathbf{p}^\Sigma$ by the relation:

$$\mathbf{p}^\Sigma = S\mathbf{e} \ . \tag{31}$$

Using (13), the slowness vector $\mathbf{p}$ reads

$$\mathbf{p} = \sigma \mathbf{n}^\Sigma + S\mathbf{e} \ , \qquad \text{with} \ \ \mathbf{n}^\Sigma \cdot \mathbf{e} = 0 \ . \tag{32}$$

Here σ is a generally complex-valued root of the algebraic equation of the sixth degree with complex-valued coefficients

$$\det[a_{ijkl}(\sigma n_j^\Sigma + Se_j)(\sigma n_l^\Sigma + Se_l) - \delta_{ik}] = 0 \ . \tag{33}$$

The propagation and attenuation vectors $\mathbf{P}$ and $\mathbf{A}$ read

$$\mathbf{P} = \mathbf{n}^\Sigma \mathrm{Re}\sigma + S\mathbf{e} \ , \qquad \mathbf{A} = \mathbf{n}^\Sigma \mathrm{Im}\sigma \ . \tag{34}$$

An important property of the choice of the real-valued $\mathbf{p}^\Sigma$ is that the attenuation vector $\mathbf{A}$ is **always** perpendicular to the plane Σ (parallel to $\mathbf{n}^\Sigma$). The attenuation vector $\mathbf{A}$ is perpendicular to Σ always when $\mathrm{Im}\mathbf{p}^\Sigma = 0$. In all other cases, the attenuation vector $\mathbf{A}$ deviates from $\mathbf{n}^\Sigma$. This holds for arbitrary medium, viscoelastic or perfectly elastic, anisotropic or isotropic, and for $\mathbf{e}$ arbitrarily situated in Σ. Note that $\mathrm{Im}\sigma$ may be different from zero even in perfectly elastic media (evanescent waves).

The magnitudes of vectors $\mathbf{P}$ and $\mathbf{A}$ and the expressions for $\mathbf{n}^P$ and $\mathbf{n}^A$ follow directly from inserting $\mathrm{Im}\mathbf{p}^\Sigma = \mathbf{0}$ and $\mathrm{Re}\mathbf{p}^\Sigma = S\mathbf{e}$ to (16) and (17). For $\mathbf{n}^A$ we get from (17), $\mathbf{n}^A = \epsilon_A \mathbf{n}^\Sigma$, where $\epsilon_A = \mathrm{Im}\sigma/|\mathrm{Im}\sigma|$. The phase velocity $\mathcal{C}$ and the attenuation angle γ read

$$\mathcal{C} = 1/[(\mathrm{Re}\sigma)^2 + S^2]^{1/2} \ , \tag{35}$$

$$\cos\gamma = \epsilon_A \ \frac{\mathrm{Re}\sigma}{[(\mathrm{Re}\sigma)^2 + S^2]^{1/2}} \ . \tag{36}$$

4.2 Imaginary-valued $\mathbf{p}^\Sigma$

We again denote by $\mathbf{e}$ a real-valued unit vector situated in Σ, and by D an arbitrary real-valued scalar. Then the imaginary-valued vector $\mathbf{p}^\Sigma$ has the form:

$$\mathbf{p}^\Sigma = \mathrm{i}D\mathbf{e} \ . \tag{37}$$

Using (13), the complex-valued slowness vector $\mathbf{p}$ reads

$$\mathbf{p} = \sigma\mathbf{n}^\Sigma + \mathrm{i}D\mathbf{e} \ , \qquad \text{with} \ \ \mathbf{n}^\Sigma \cdot \mathbf{e} = 0 \ . \tag{38}$$

Here σ is a generally complex-valued root of the algebraic equation of the sixth degree with the complex-valued coefficients:

$$\det[a_{ijkl}(\sigma n_j^\Sigma + \mathrm{i}De_j)(\sigma n_l^\Sigma + \mathrm{i}De_l) - \delta_{ik}] = 0 \ . \tag{39}$$

In equation (39), a_{ijkl}, D, $\mathbf{n}^\Sigma$ and $\mathbf{e}$ are given, a_{ijkl} is real-valued or complex-valued, D, $\mathbf{n}^\Sigma$ and $\mathbf{e}$ are real-valued.

Once σ is determined, the propagation and attenuation vectors $\mathbf{P}$ and $\mathbf{A}$ read

$$\mathbf{P} = \mathbf{n}^\Sigma \mathrm{Re}\sigma \ , \qquad \mathbf{A} = \mathbf{n}^\Sigma \mathrm{Im}\sigma + D\mathbf{e} \ . \tag{40}$$

An important property of this specification is that the propagation vector $\mathbf{P}$ is perpendicular to Σ, and the plane Σ represents a wavefront of the plane wave under consideration. The real-valued travel time is constant on Σ.

The expressions for magnitudes of vectors $\mathbf{P}$ and $\mathbf{A}$ and for the unit vectors $\mathbf{n}^P$ and $\mathbf{n}^A$ follow simply from (16), and (17). For $\mathbf{n}^P$ we get from (17), $\mathbf{n}^P = \epsilon_P \mathbf{n}^\Sigma$, where $\epsilon_P = \mathrm{Re}\sigma/|\mathrm{Re}\sigma|$. The phase velocity $\mathcal{C}$ reads

$$\mathcal{C} = 1/|\mathrm{Re}\sigma| \; , \tag{41}$$

and the attenuation angle γ is given by the relation

$$\cos\gamma = \epsilon_P \quad \frac{\mathrm{Im}\sigma}{[(\mathrm{Im}\sigma)^2 + D^2]^{1/2}} \; . \tag{42}$$

Actually, for $\mathbf{p}^\Sigma$ given by (37), the componental specification of the slowness vector represents the mixed specification of the slowness vector. The mixed specification of the slowness vector was used intensively in a number of papers by Červený and Pšenčík (2005a,b, 2008, 2011). See also Sections 2 and 3.2.

4.3 Coplanar case: $\mathrm{Re}\mathbf{p}^\Sigma$ parallel to $\mathrm{Im}\mathbf{p}^\Sigma$

We again denote by $\mathbf{e}$ a real-valued unit vector along Σ and consider the vector $\mathbf{p}^\Sigma$ in the following form:

$$\mathbf{p}^\Sigma = Z\mathbf{e} \; . \tag{43}$$

Here Z is an arbitrary complex-valued scalar. Both real and imaginary parts of $\mathbf{p}^\Sigma$ are parallel, but they may have opposite orientation. The slowness vector $\mathbf{p}$ is given by the relation, see (13):

$$\mathbf{p} = \sigma\mathbf{n}^\Sigma + Z\mathbf{e} \; , \qquad \text{with} \;\; \mathbf{n}^\Sigma \cdot \mathbf{e} = 0 \; . \tag{44}$$

The generally complex-valued quantity σ is a root of the algebraic equation of the sixth degree:

$$\det[a_{ijkl}(\sigma n_j^\Sigma + Ze_j)(\sigma n_l^\Sigma + Ze_l) - \delta_{ik}] = 0 \; . \tag{45}$$

The propagation and attenuation vectors $\mathbf{P}$ and $\mathbf{A}$ are then given by the relations

$$\mathbf{P} = \mathbf{n}^\Sigma \mathrm{Re}\sigma + \mathbf{e}\mathrm{Re}Z \; , \qquad \mathbf{A} = \mathbf{n}^\Sigma \mathrm{Im}\sigma + \mathbf{e}\mathrm{Im}Z \; . \tag{46}$$

The magnitudes of $\mathbf{P}$ and $\mathbf{A}$ read

$$|\mathbf{P}| = [(\mathrm{Re}\sigma)^2 + (\mathrm{Re}Z)^2]^{1/2} \; , \quad |\mathbf{A}| = [(\mathrm{Im}\sigma)^2 + (\mathrm{Im}Z)^2]^{1/2} \; . \tag{47}$$

It follows immediately from (46) that we deal with the coplanar case. An important property of the coplanar case is that all quantities are fully confined to the plane specified by real-valued unit vectors $\mathbf{n}^\Sigma$ and $\mathbf{e}$. The phase velocity $\mathcal{C}$ and the attenuation angle γ are given by relations

$$\mathcal{C} \;=\; 1/[(\mathrm{Re}\sigma)^2 + (\mathrm{Re}Z)^2]^{1/2} \; , \tag{48}$$

$$\cos\gamma \;=\; [(\mathrm{Re}\sigma)(\mathrm{Im}\sigma) + (\mathrm{Re}Z)(\mathrm{Im}Z)]/|\mathbf{P}||\mathbf{A}| \; . \tag{49}$$

Equations derived in this section for the coplanar case are generalizations of those derived in Section 4.1 for the real-valued $\mathbf{p}^\Sigma$, and in Section 4.2 for the imaginary valued $\mathbf{p}^\Sigma$. Actually, we can define Z as $Z = S + \mathrm{i}D$.

4.4 Cartesian components of $\mathbf{p}^\Sigma$

We introduce two mutually perpendicular real-valued unit vectors $\mathbf{e}_1, \mathbf{e}_2$ in the plane Σ in such a way that the three vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 = \mathbf{n}^\Sigma$ form a right-handed triplet of unit vectors. Then, we can specify $\mathbf{p}^\Sigma$ as

$$\mathbf{p}^\Sigma = p_1^\Sigma \mathbf{e}_1 + p_2^\Sigma \mathbf{e}_2 \ , \tag{50}$$

where p_1^Σ and p_2^Σ are, in general, complex-valued. Consequently, the slowness vector $\mathbf{p}$ can be expressed as follows, see (13),

$$\mathbf{p} = \sigma \mathbf{n}^\Sigma + p_1^\Sigma \mathbf{e}_1 + p_2^\Sigma \mathbf{e}_2 \ . \tag{51}$$

Equation (14) for σ then reads

$$\det[a_{ijkl}(\sigma n_j^\Sigma + p_1^\Sigma e_{1j} + p_2^\Sigma e_{2j})(\sigma n_l^\Sigma + p_1^\Sigma e_{1l} + p_2^\Sigma e_{2l}) - \delta_{ik}] = 0 \ . \tag{52}$$

Here $\mathbf{n}^\Sigma, \mathbf{e}_1, \mathbf{e}_2$ are known real-valued unit vectors, p_1^Σ and p_2^Σ are the given complex-valued components of $\mathbf{p}^\Sigma$. Quantity σ is a generally complex-valued root of the algebraic equation (52) of the sixth degree with complex-valued coefficients.

Once σ is determined, the propagation and attenuation vectors $\mathbf{P}$ and $\mathbf{A}$ are given by relations

$$\begin{aligned} \mathbf{P} &= \mathbf{n}^\Sigma \mathrm{Re}\sigma + \mathbf{e}_1 \mathrm{Re} p_1^\Sigma + \mathbf{e}_2 \mathrm{Re} p_2^\Sigma \ , \\ \mathbf{A} &= \mathbf{n}^\Sigma \mathrm{Im}\sigma + \mathbf{e}_1 \mathrm{Im} p_1^\Sigma + \mathbf{e}_2 \mathrm{Im} p_2^\Sigma \ . \end{aligned} \tag{53}$$

This yields

$$\begin{aligned} |\mathbf{P}| &= [(\mathrm{Re}\sigma)^2 + (\mathrm{Re} p_1^\Sigma)^2 + (\mathrm{Re} p_2^\Sigma)^2]^{1/2} \ , \\ |\mathbf{A}| &= [(\mathrm{Im}\sigma)^2 + (\mathrm{Im} p_1^\Sigma)^2 + (\mathrm{Im} p_2^\Sigma)^2]^{1/2} \ . \end{aligned} \tag{54}$$

The unit vectors $\mathbf{n}^P$ and $\mathbf{n}^A$ along $\mathbf{P}$ and $\mathbf{A}$ are given by relations (17) with $\mathbf{p}^\Sigma$ given by (50). The relations for the phase velocity $\mathcal{C}$ and attenuation angle γ read

$$\mathcal{C} = 1/[(\mathrm{Re}\sigma)^2 + (\mathrm{Re} p_1^\Sigma)^2 + (\mathrm{Re} p_2^\Sigma)^2]^{1/2} \ , \tag{55}$$

$$\cos\gamma = [(\mathrm{Re}\sigma)(\mathrm{Im}\sigma) + (\mathrm{Re} p_1^\Sigma)(\mathrm{Im} p_1^\Sigma) + (\mathrm{Re} p_2^\Sigma)(\mathrm{Im} p_2^\Sigma)]/|\mathbf{P}||\mathbf{A}| \ . \tag{56}$$

It may be useful to choose the direction of the vector $\mathbf{e}_1$ so that it specifies the orientation of the vector $\mathrm{Re}\mathbf{p}^\Sigma$. This implies $\mathrm{Re} p_2^\Sigma = 0$. Then the propagation vector $\mathbf{P}$ is confined to the plane $(\mathbf{n}^\Sigma, \mathbf{e}_1)$, and $\mathbf{p}^\Sigma$ is given by the relation

$$\mathbf{p}^\Sigma = p_1^\Sigma \mathbf{e}_1 + \mathrm{i}\mathrm{Im} p_2^\Sigma \mathbf{e}_2 \ . \tag{57}$$

Here p_1^Σ is a complex-valued scalar, $p_1^\Sigma = S + \mathrm{i}D_1$, and p_2^Σ is a purely imaginary scalar, $p_2^\Sigma = \mathrm{i}D_2$. Here S is the apparent slowness along the vector $\mathbf{e}_1$, D_1 and D_2 are the appropriate inhomogeneity parameters along $\mathbf{e}_1$ and $\mathbf{e}_2$. The slowness vector $\mathbf{p}$ is then given by the relation

$$\mathbf{p} = \sigma \mathbf{n}^\Sigma + p_1^\Sigma \mathbf{e}_1 + \mathrm{i}D_2 \mathbf{e}_2 \ . \tag{58}$$

The complex-valued quantity σ is a root of equation (52), in which we put $\mathrm{Re}p_2^\Sigma = 0$. All other relevant equations are given by (53)-(56) with $\mathrm{Re}p_2^\Sigma = 0$.

We remind the reader that the propagation vector $\mathbf{P}$ is confined to the plane $(\mathbf{n}^\Sigma, \mathbf{e}_1)$, but the attenuation vector $\mathbf{A}$ points generally outside this plane. For this reason, we speak of the non-coplanar case. The coplanar case occurs when $D_2 = 0$.

5 Special cases of media

As mentioned above, equations of Sections 2, 3 and 4 are valid for any anisotropic or isotropic, viscoelastic or perfectly elastic media. In general, the determination of the slowness vector $\mathbf{p}$ requires numerical solution of an algebraic equation of the sixth degree with complex-valued coefficients to determine the generally complex-valued quantity σ.

In some special cases, the computations simplify, and the solution of the algebraic equation of the sixth degree can be found in a simple analytic form. Important examples are the isotropic viscoelastic media, where all formulae are particularly simple. Analytic expressions can be also obtained for SH waves propagating in a plane of symmetry of a monoclinic (orthorhombic, hexagonal) viscoelastic medium. In both cases, we can give simple analytic expressions for all computed quantities.

5.1 Isotropic viscoelastic media

For isotropic viscoelastic media, a_{ijkl} are given by the well-known relation

$$a_{ijkl} = \frac{\lambda}{\rho}\delta_{ij}\delta_{kl} + \frac{\mu}{\rho}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \ . \tag{59}$$

Here λ/ρ and μ/ρ are density-normalized complex-valued Lamé's viscoelastic moduli. For a_{ijkl} given by (59), equation (6) factorizes:

$$\det[a_{ijkl}p_jp_l - \delta_{ik}] = (\alpha^2 p_ip_i - 1)(\beta^2 p_kp_k - 1)^2 = 0 \ , \tag{60}$$

where

$$\alpha^2 = \frac{\lambda + 2\mu}{\rho} \ , \qquad \beta^2 = \frac{\mu}{\rho} \ . \tag{61}$$

Here α is the complex-valued velocity of P waves, β is the complex-valued velocity of S waves. Consequently, we can write the constraint relation for both P and S waves in the same form

$$p_ip_i = 1/V^2 \ , \tag{62}$$

where

$$\begin{aligned} V^2 &= \alpha^2 \qquad \text{for P wave} \ , \\ V^2 &= \beta^2 \qquad \text{for S wave} \ . \end{aligned} \tag{63}$$

We now consider an arbitrary plane Σ, and an arbitrarily chosen complex-valued vector $\mathbf{p}^\Sigma$, situated in the plane Σ. Inserting the componental specification (13) into (62), we obtain

$$\sigma^2 + \mathbf{p}^\Sigma \cdot \mathbf{p}^\Sigma = 1/V^2 \ . \tag{64}$$

This yields a very simple expression for σ:

$$\sigma = \pm[1/V^2 - \mathbf{p}^\Sigma \cdot \mathbf{p}^\Sigma]^{1/2} \ . \tag{65}$$

It is common to express $1/V^2$ in terms of $\mathrm{Re}V^2$ and of the quality factor Q given by the relation

$$Q^{-1} = -\mathrm{Im}(V^2)/\mathrm{Re}(V^2) \ . \tag{66}$$

From positive definiteness of the 6×6 matrix $\mathbf{A}^R$ and positive definiteness or zero of the 6×6 matrix $\mathbf{A}^I$, which form the matrix $\mathbf{A}^R - \mathrm{i}\mathbf{A}^I$ of viscoelastic moduli in the Voigt notation, we have $\mathrm{Re}V^2 > 0$ and $\mathrm{Im}V^2 \leq 0$. Thus the quality factor is a real-valued positive scalar, which can became infinite if $\mathrm{Im}V^2 = 0$. For $V = \alpha$, we get the quality factor for P waves, for $V = \beta$ for S waves. Using Q, we can express V^2 in the following way:

$$V^2 = \mathrm{Re}(V^2)(1 - \mathrm{i}/Q) \tag{67}$$

which finally yields:

$$\frac{1}{V^2} = \frac{1 + \mathrm{i}Q^{-1}}{\mathrm{Re}V^2(1 + Q^{-2})} \ . \tag{68}$$

Inserting (65) into (13), we obtain the complete analytical expression for the slowness vector $\mathbf{p}$:

$$\mathbf{p} = \pm(1/V^2 - \mathbf{p}^\Sigma \cdot \mathbf{p}^\Sigma)^{1/2}\mathbf{n}^\Sigma + \mathbf{p}^\Sigma \ , \tag{69}$$

where $1/V^2$ is given by (68), and $\mathbf{p}^\Sigma$ by various expression given in Section 4.

The derivation of other relevant quantities is easy. For the propagation and attenuation vectors $\mathbf{P}$ and $\mathbf{A}$ we get from (15):

$$\begin{aligned} \mathbf{P} &= \pm\mathrm{Re}(1/V^2 - \mathbf{p}^\Sigma \cdot \mathbf{p}^\Sigma)^{1/2}\mathbf{n}^\Sigma + \mathrm{Re}\mathbf{p}^\Sigma \ , \\ \mathbf{A} &= \pm\mathrm{Im}(1/V^2 - \mathbf{p}^\Sigma \cdot \mathbf{p}^\Sigma)^{1/2}\mathbf{n}^\Sigma + \mathrm{Im}\mathbf{p}^\Sigma \ . \end{aligned} \tag{70}$$

Magnitudes of vectors $\mathbf{P}$ and $\mathbf{A}$ read:

$$\begin{aligned} |\mathbf{P}| &= [(\mathrm{Re}(1/V^2 - \mathbf{p}^\Sigma \cdot \mathbf{p}^\Sigma)^{1/2})^2 + (\mathrm{Re}\mathbf{p}^\Sigma)^2]^{1/2} \ , \\ |\mathbf{A}| &= [(\mathrm{Im}(1/V^2 - \mathbf{p}^\Sigma \cdot \mathbf{p}^\Sigma)^{1/2})^2 + (\mathrm{Im}\mathbf{p}^\Sigma)^2]^{1/2} \ . \end{aligned} \tag{71}$$

The phase velocity $\mathcal{C}$ and the attenuation angle $\cos\gamma$ are given by equations (10) and (9), respectively.

We shall now consider several important examples of the specification of $\mathbf{p}^\Sigma$ for isotropic viscoelastic media

a) **Real-valued $\mathbf{p}^\Sigma$.**

We consider $\mathbf{p}^\Sigma = S\mathbf{e}$, where S and $\mathbf{e}$ have the same meaning as in (31). Then $\sigma = \pm(1/V^2 - S^2)^{1/2}$ and

$$\mathbf{p} = \pm(1/V^2 - S^2)^{1/2}\mathbf{n}^\Sigma + S\mathbf{e} \ . \tag{72}$$

The determination of other expressions is straightforward. The attenuation vector $\mathbf{A} = \pm\mathrm{Im}(\mathbf{1/V^2} - \mathbf{S^2})^{\mathbf{1/2}}\mathbf{n}^{\mathbf{\Sigma}}$ is always perpendicular to the plane Σ. It may be non-vanishing even in perfectly elastic isotropic media when $1/V^2$ is real-valued. In this case, we obtain $\mathbf{A} = \pm(S^2 - 1/V^2)^{1/2}\mathbf{n}^\Sigma$ for $1/V^2 < S^2$ and speak of evanescent waves. For $1/V^2 > S^2$, we obtain $\mathbf{A} = \mathbf{0}$.

b) **Imaginary-valued $\mathbf{p}^\Sigma$**

We consider $\mathbf{p}^\Sigma = \mathrm{i}D\mathbf{e}$, where D and $\mathbf{e}$ have the same meaning as in (37). Then $\sigma = \pm(1/V^2 + D^2)^{1/2}$ and the slowness vector $\mathbf{p}$ is given by the relation

$$\mathbf{p} = \pm(1/V^2 + D^2)^{1/2}\mathbf{n}^\Sigma + \mathrm{i}D\mathbf{e} \ . \tag{73}$$

In this case the plane Σ again represents the wavefront of the plane wave under consideration. This specification of $\mathbf{p}^\Sigma$ corresponds to the mixed specification of the slowness vector, which was studied in detail by Červený and Pšenčík (2005a,b).

c) **Coplanar case**

We use $\mathbf{p}^\Sigma = Z\mathbf{e}$, where Z is an arbitrary complex-valued scalar, see (43). Then $\sigma = \pm(1/V^2 - Z^2)^{1/2}$ and the slowness vector $\mathbf{p}$ is given by the relation

$$\mathbf{p} = \pm(1/V^2 - Z^2)^{1/2}\mathbf{n}^\Sigma + Z\mathbf{e} \ . \tag{74}$$

In this and previous cases, both the propagation and attenuation vectors $\mathbf{P}$ and $\mathbf{A}$ are confined to the plane specified by unit vectors $\mathbf{n}^\Sigma$ and $\mathbf{e}$, and thus we deal with the coplanar case.

d) **Homogeneous plane wave**

Very interesting results are obtained for $\mathbf{p}^\Sigma$ given by the relation $\mathbf{p}^\Sigma = Z\mathbf{e}$, with

$$Z = aV^{-1} \ , \tag{75}$$

where a is a real valued scalar, $0 \le a^2 \le 1$, and V is the complex-valued velocity given in (68). The quantity σ is then given by the relation

$$\sigma = \pm[\frac{1}{V^2} - a^2\frac{1}{V^2}]^{1/2} = \pm\frac{1}{V}\sqrt{1-a^2} \ . \tag{76}$$

The slowness vector $\mathbf{p} = \sigma\mathbf{n}^\Sigma + \mathbf{p}^\Sigma$ can be expressed as follows:

$$\mathbf{p} = \frac{1}{V}\mathbf{N}^H \ , \tag{77}$$

with

$$\mathbf{N}^H = \pm\sqrt{1-a^2}\mathbf{n}^\Sigma + a\mathbf{e} \ . \tag{78}$$

Here $\mathbf{N}^H$ is a real-valued unit vector, situated in the plane given by $\mathbf{n}^\Sigma$ and $\mathbf{e}$. The propagation and attenuation vectors $\mathbf{P}$ and $\mathbf{A}$ read

$$\mathbf{P} = \mathrm{Re}(1/V)\mathbf{N}^H \ , \qquad \mathbf{A} = \mathrm{Im}(1/V)\mathbf{N}^H \ . \tag{79}$$

Equations (79) show that vectors $\mathbf{P}$ and $\mathbf{A}$ are parallel. In other words, the plane wave under consideration is homogeneous.

To conclude: When we take $\mathbf{p}^\Sigma = aV^{-1}\mathbf{e}$ on the plane Σ, we obtain a homogeneous plane wave propagating from Σ. The direction of propagation $\mathbf{N}^H$ of this homogeneous plane wave is controlled by the real-valued scalar a, see (78). For $a = 0$, $\mathbf{N}^H = \pm\mathbf{n}^\Sigma$; for $a = \pm 1$, $\mathbf{N}^H = \pm\mathbf{e}$.

e) **Non-coplanar case**

Similarly as in Section 4.4, we can introduce two mutually perpendicular real-valued unit vectors $\mathbf{e}_1, \mathbf{e}_2$ in the plane Σ in such a way that the three vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 = \mathbf{n}^\Sigma$ form a right-handed triplet of unit vectors. We can introduce $\mathbf{p}^\Sigma$ in the form of (50). All equations then remain the same as in Section 4.4, only the expression for σ can be now written explicitly:

$$\sigma = [\frac{1}{V^2} - \mathbf{p}_1^\Sigma \cdot \mathbf{p}_1^\Sigma - \mathbf{p}_2^\Sigma \cdot \mathbf{p}_2^\Sigma]^{1/2} \ . \tag{80}$$

Both $\mathbf{p}_1^\Sigma$ and $\mathbf{p}_2^\Sigma$ may be chosen arbitrarily and may be complex-, real-, imaginary-valued or zero. For example, for $\mathbf{p}_2^\Sigma = \mathbf{0}$, we obtain the coplanar case in the plane $(\mathbf{n}^\Sigma, \mathbf{e}_1)$.

It may be useful to generalize slightly the coplanar case, considering general complex-valued $\mathbf{p}_1^\Sigma$, but purely imaginary $\mathbf{p}_2^\Sigma$,

$$\mathbf{p}_2^\Sigma = \mathrm{i}D_2\mathbf{e}_2 \ . \tag{81}$$

This yields the formula (57) for $\mathbf{p}^\Sigma$. For σ we have in this case:

$$\sigma = [1/V^2 - \mathbf{p}_1^\Sigma \cdot \mathbf{p}_1^\Sigma + D_2^2]^{1/2} \tag{82}$$

and $\mathbf{p}$ is given by (58). The propagation vector $\mathbf{P}$ is fully confined to the plane $(\mathbf{n}^\Sigma, \mathbf{e}_1)$:

$$\mathbf{P} = \mathrm{Re}\sigma\, \mathbf{n}^\Sigma + S\mathbf{e}_1 \ , \tag{83}$$

but the attenuation vector $\mathbf{A}$ may point outside this plane:

$$\mathbf{A} = \mathrm{Im}\sigma\, \mathbf{n}^\Sigma + D_1\mathbf{e}_1 + D_2\mathbf{e}_2 \ . \tag{84}$$

We speak of the non-coplanar case.

5.2 SH waves in a plane of symmetry of monoclinic viscoelastic media

The simplest case of anisotropic media, which can be solved analytically, is the case of SH waves propagating in a plane of symmetry of a monoclinic (orthorhombic, hexagonal) viscoelastic medium. This case is very useful for simple illustration of differences between anisotropic and isotropic viscoelastic media.

We choose the Cartesian coordinate system x_1, x_2, x_3 in such a way that the plane of symmetry Σ^S corresponds to the coordinate plane x_1, x_3. Both real-valued and imaginary-valued parts of the polarization vector $\mathbf{U}$ of the SH plane wave are perpendicular to the plane Σ^S. In the plane of symmetry, the constraint relation reads (Červený and Pšenčík, 2005a; Carcione, 2007):

$$A_{66}p_1^2 + A_{44}p_3^2 + 2A_{46}p_1p_3 = 1 \ . \tag{85}$$

Here A_{66}, A_{44} and A_{46} are the complex-valued density-normalized viscoelastic moduli, in the Voigt notation. For $A_{46} = 0$, the monoclinic viscoelastic medium reduces to the orthorhombic or hexagonal medium.

We now introduce an arbitrary straight-line l in the plane Σ^S, and the projection of the slowness vector $\mathbf{p}^\Sigma$ along l. We assume that $\mathbf{p}^\Sigma$ is known; it may be complex-valued, real-valued or imaginary valued.

The componental specification (13) of the slowness vector in the plane (x_1, x_3) yields:

$$p_1 = \sigma n_1^\Sigma + p_1^\Sigma \ , \qquad p_3 = \sigma n_3^\Sigma + p_3^\Sigma \ . \tag{86}$$

Inserting (86) into (85) yields a quadratic equation for σ. Its solution yields two roots $\sigma_{1,2}$:

$$\sigma_{1,2} = -E_{22}/\Gamma_{22} \pm [1/\Gamma_{22} - (p_1^\Sigma n_3^\Sigma - n_1^\Sigma p_3^\Sigma)^2 \Delta/\Gamma_{22}^2]^{1/2} \ , \tag{87}$$

where

$$\begin{aligned} \Gamma_{22} &= A_{66}(n_1^\Sigma)^2 + A_{44}(n_3^\Sigma)^2 + 2A_{46}n_1^\Sigma n_3^\Sigma \ , \\ E_{22} &= A_{66}n_1^\Sigma p_1^\Sigma + A_{44}n_3 p_3^\Sigma + A_{46}(n_1^\Sigma p_3^\Sigma + n_3^\Sigma p_1^\Sigma) \ , \\ F_{22} &= A_{66}(p_1^\Sigma)^2 + A_{44}(p_3^\Sigma)^2 + 2A_{46}p_1^\Sigma p_3^\Sigma \ , \end{aligned} \tag{88}$$

and

$$\Delta = A_{44}A_{66} - A_{46}^2 \ . \tag{89}$$

Both propagation vector $\mathbf{P}$ and attenuation vector $\mathbf{A}$ are situated in the plane of symmetry Σ^S,

$$\begin{aligned} P_1 &= \mathrm{Re}\sigma \, n_1^\Sigma + \mathrm{Re}p_1^\Sigma \ , \quad P_3 = \mathrm{Re}\sigma \, n_3^\Sigma + \mathrm{Re}p_3^\Sigma \ , \\ A_1 &= \mathrm{Im}\sigma \, n_1^\Sigma + \mathrm{Im}p_1^\Sigma \ , \quad A_3 = \mathrm{Im}\sigma \, n_3^\Sigma + \mathrm{Im}p_3^\Sigma \ . \end{aligned} \tag{90}$$

The phase velocity $\mathcal{C}$ of the plane wave under consideration reads

$$\mathcal{C} = 1/[(\mathrm{Re}\sigma)^2 + (\mathrm{Re}p_1^\Sigma)^2 + (\mathrm{Re}p_3^\Sigma)^2]^{1/2} \ , \tag{91}$$

and the attenuation angle γ can be calculated using the relation

$$\cos\gamma = [\mathrm{Re}\sigma\mathrm{Im}\sigma + \mathrm{Re}p_1^\Sigma\mathrm{Im}p_1^\Sigma + \mathrm{Re}p_3^\Sigma\mathrm{Im}p_3^\Sigma]/|\mathbf{P}||\mathbf{A}| \ . \tag{92}$$

6 Application of mixed and componental specifications of the slowness vector in the reflection/transmission process

In this section, we discuss the application of both mixed and componental specifications of the slowness vector in the process of reflection and transmission of plane waves at a planar interface Σ separating two homogeneous viscoelastic media.

It is useful to specify the slowness vector $\mathbf{p}^{inc}$ of the incident wave by the mixed specification. In the mixed specification, the slowness vector $\mathbf{p}^{inc}$ is specified with respect to a plane wavefront Σ^W of the incident wave. We denote by $\mathbf{n}^W$ the unit normal to the plane Σ^W and by $\mathbf{p}^W$ the projection of the slowness vector $\mathbf{p}^{inc}$ to Σ^W. The vector $\mathbf{p}^W$ is imaginary valued, because the travel time along Σ^W is constant and thus the real part of $\mathbf{p}^W$ is zero.

Consider now a plane wave with the slowness vector $\mathbf{p}^{inc}$, incident at a planar interface Σ. The vector $\mathbf{p}^\Sigma$tangent to Σ can be then computed by projecting $\mathbf{p}^{inc}$ into the plane Σ. The slowness vector $\mathbf{p}^{rt}$ of a generated (reflected, transmitted) wave is then determined from $\mathbf{p}^\Sigma$ using the componental specification.

In the following, we work with quantities σ^{inc} and σ^{rt}. The former is always related to the mixed specification, the latter to the componental specification of the slowness vector.

6.1 Anisotropic viscoelastic media

We denote by a^{inc}_{ijkl} the complex-valued density-normalized viscoelastic moduli in the halfspace in which the incident wave propagates, and by a^{rt}_{ijkl} the viscoelastic moduli in the halfspace in which the selected R/T wave propagates. For reflected waves, the moduli a^{rt}_{ijkl} are identical with a^{inc}_{ijkl}. As mentioned above, the vector $\mathbf{p}^W$ is purely imaginary. In the mixed specification, used for the incident wave, it is given by a simple relation

$$\mathbf{p}^W = \mathrm{i}D\mathbf{m} \ . \tag{93}$$

The real-valued unit vector $\mathbf{m}$ is perpendicular to $\mathbf{n}^W$, but otherwise it can be arbitrarily oriented in the plane Σ^W. The real-valued scalar D, called the inhomogeneity parameter, is a projection of the attenuation vector $\mathbf{A} = \mathrm{Im}\mathbf{p}^{inc}$ to the plane Σ^W. Then the mixed specification of the slowness vector $\mathbf{p}^{inc}$ is given by the relation, see (11),

$$\mathbf{p}^{inc} = \sigma^{inc}\mathbf{n}^W + \mathrm{i}D\mathbf{m} \ . \tag{94}$$

Here σ^{inc} is a root of an algebraic equation of the sixth degree with complex-valued coefficients, see (12):

$$\det[a^{inc}_{ijkl}(\sigma^{inc}n^W_j + \mathrm{i}Dm_j)(\sigma^{inc}n^W_l + \mathrm{i}Dm_l) - \delta_{ik}] = 0 \ . \tag{95}$$

Note that the parameters of incident plane waves, which should be given, are the real-valued unit vectors $\mathbf{n}^W$ and $\mathbf{m}$, and the real-valued inhomogeneity parameter D: $-\infty < D < \infty$.

Consider now a planar interface Σ, defined by its normal $\mathbf{n}^\Sigma$. We introduce two real-valued vectors $\mathbf{N}^\Sigma$ and $\mathbf{M}^\Sigma$, given by the relations

$$\begin{aligned}\mathbf{N}^\Sigma &= \mathbf{n}^\Sigma \times (\mathbf{n}^W \times \mathbf{n}^\Sigma) = \mathbf{n}^W - \mathbf{n}^\Sigma(\mathbf{n}^W \cdot \mathbf{n}^\Sigma) \ , \\ \mathbf{M}^\Sigma &= \mathbf{n}^\Sigma \times (\mathbf{m} \times \mathbf{n}^\Sigma) = \mathbf{m} - \mathbf{n}^\Sigma(\mathbf{m} \cdot \mathbf{n}^\Sigma) \ . \end{aligned} \quad (96)$$

These vectors represent projections of vectors $\mathbf{n}^W$ and $\mathbf{m}$ into the plane Σ. Note that the vectors $\mathbf{N}^\Sigma$ and $\mathbf{M}^\Sigma$ are generally neither unit nor mutually perpendicular.

The projection of the complex-valued slowness vector $\mathbf{p}^{inc}$ of the incident wave into the planar interface Σ is given by the relation

$$\mathbf{p}^\Sigma = \sigma^{inc}\mathbf{N}^\Sigma + \mathrm{i}D\mathbf{M}^\Sigma \ . \quad (97)$$

The slowness vector $\mathbf{p}^{rt}$ of the reflected/transmitted generated at the planar interface Σ is then given by the relation

$$\mathbf{p}^{rt} = \sigma^{rt}\mathbf{n}^\Sigma + \mathbf{p}^\Sigma \ , \quad (98)$$

where $\mathbf{p}^\Sigma$ is given by (97) and σ^{rt} is a root of an algebraic equation of the sixth degree:

$$\det[a^{rt}_{ijkl}(\sigma^{rt}n^\Sigma_j + p^\Sigma_j)(\sigma^{rt}n^\Sigma_l + p^\Sigma_l) - \delta_{ik}] = 0 \ . \quad (99)$$

The above equations can be expressed in many alternative forms. They are valid for arbitrary density-normalized viscoelastic parameters a^{inc}_{ijkl} and a^{rt}_{ijkl}, both complex-valued and/or real-valued), for coplanar and non-coplanar cases, for homogeneous and inhomogeneous plane waves, for arbitrarily chosen incident plane wave and for arbitrarily oriented planar interface Σ.

Once the slowness vector of the generated wave is determined, we can simply calculate the propagation and attenuation vectors, and the phase velocity $\mathcal{C}$ and the attenuation angle γ of this wave.

The advantage of the described approach is that it does not use the attenuation angle γ for the specification of the slowness vector of the incident wave. The plane waves corresponding to a particular attenuation angle γ may not exist (Červený and Pšenčík, 2011). The approach based on the inhomogeneity parameter D of the incident wave fully avoids this problem.

6.2 Isotropic viscoelastic media

In viscoelastic isotropic media, the procedure remains the same as in viscoelastic anisotropic media, but solutions of algebraic equations of sixth degree can be given in an analytic form. The complex-valued velocities of plane waves are denoted by V, as indicated in (63) with (61). We use again the superscripts "inc" and "rt" to denote quantities related to incident and R/T waves.. Thus, V^{inc} is the complex-valued velocity of the incident wave, and V^{rt} the complex-valued velocity of selected R/T wave. The velocities V^{inc} and V^{rt} may, of course, correspond to P or S waves, according to the problem under consideration.

All equations of Section 6.1 remain exactly the same even for isotropic media, only the algebraic equations of the sixth degree (95) for incident wave and (99) for R/T wave are replaced by their analytical solutions. For a selected incident wave (P or S), the solution of (95) is (Červený and Pšenčík, 2011; Eq.23):

$$\sigma^{inc} = \pm[(1/V^{inc})^2 + D^2]^{1/2} \ . \tag{100}$$

This expression is indeed very simple. For a selected reflected/transmitted wave, the solution of (99) following from (65) and (97) reads:

$$\sigma^{rt} = \pm[(1/V^{rt})^2 - (\sigma^{inc}\mathbf{N}^\Sigma + \mathrm{i}D\mathbf{M}^\Sigma)^2]^{1/2} \ . \tag{101}$$

In our problem, only one of the signs used in front of the complex-valued square roots in (100) and (101) has a physical meaning. The determination of the proper sign is a very complicated problem in the study of reflection/transmission coefficients of plane waves at a planar interface separating two isotropic viscoelastic media. There is a broad literature devoted to it, see, e.g., Krebes (1983), Ruud (2006), Krebes and Daley (2007), Červený (2007), Sidler, Carcione and Holliger (2008), etc. This problem, however, is not a subject of this article.

The expressions for σ^{rt} and $\mathbf{p}^{rt}$ may be given in many alternative forms. For example, we can express the vectors $\mathbf{N}^\Sigma$ and $\mathbf{M}^\Sigma$ in terms of simpler vectors $\mathbf{n}^\Sigma$, $\mathbf{n}^W$ and $\mathbf{m}$. From (96) we get:

$$\begin{aligned} \mathbf{N}^\Sigma\mathbf{N}^\Sigma &= 1 - (\mathbf{n}^\Sigma \cdot \mathbf{n}^W)^2 \ , \\ \mathbf{M}^\Sigma\mathbf{M}^\Sigma &= 1 - (\mathbf{n}^\Sigma \cdot \mathbf{m})^2 \ , \\ \mathbf{N}^\Sigma\mathbf{M}^\Sigma &= -(\mathbf{n}^\Sigma \cdot \mathbf{n}^W)(\mathbf{n}^\Sigma \cdot \mathbf{m}) \ . \end{aligned} \tag{102}$$

This yields:

$$(\sigma^{inc}\mathbf{N}^\Sigma + \mathrm{i}D\mathbf{M}^\Sigma)^2 = (1/V^{inc})^2 - [\sigma^{inc}(\mathbf{n}^\Sigma \cdot \mathbf{n}^W) + \mathrm{i}D(\mathbf{n}^\Sigma \cdot \mathbf{m})]^2 \ . \tag{103}$$

Inserting (103) into (101), we obtain

$$\sigma^{rt} = \pm\{(1/V^{rt})^2 - (1/V^{inc})^2 + [\sigma^{inc}(\mathbf{n}^\Sigma \cdot \mathbf{n}^W) + \mathrm{i}D(\mathbf{n}^\Sigma \cdot \mathbf{m})]^2\}^{1/2} \ . \tag{104}$$

The slowness vector $\mathbf{p}^{rt}$ of the reflected/transmitted wave is given by the relation, see eq. (98):

$$\mathbf{p}^{rt} = \sigma^{rt}\mathbf{n}^\Sigma + \sigma^{inc}\mathbf{N}^\Sigma + \mathrm{i}D\mathbf{M}^\Sigma \ . \tag{105}$$

Here σ^{rt} is given by (104), σ^{inc} by (100), $\mathbf{N}^\Sigma$ and $\mathbf{M}^\Sigma$ by (96).

The expression (104) simplifies considerably for monotypic reflected waves, as $V^{rt} = V^{inc}$ in this case, and (104) yields:

$$\sigma^{rt} = \pm\{[\sigma^{inc}(\mathbf{n}^\Sigma \cdot \mathbf{n}^W) + \mathrm{i}D(\mathbf{n}^\Sigma \cdot \mathbf{m})]^2\}^{1/2} \ . \tag{106}$$

Another useful form of expressions for σ^{rt} and $\mathbf{p}^{rt}$ is based on a slightly modified specification of the vector $\mathbf{p}^W$. Instead of (100), we use

$$\mathbf{p}^W = \mathrm{i}D\mathbf{m} = \mathrm{i}D_1\mathbf{m}_1 + \mathrm{i}D_2\mathbf{m}_2 \ , \tag{107}$$

where

$$\begin{aligned} \mathbf{m}_2 &= (\mathbf{n}^\Sigma \times \mathbf{n}^W)/|\mathbf{n}^\Sigma \times \mathbf{n}^W| \\ \mathbf{m}_1 &= \mathbf{n}^W \times \mathbf{m}_2 \ . \end{aligned} \tag{108}$$

The mutually perpendicular unit vectors $\mathbf{m}_1$ and $\mathbf{m}_2$ are situated in the plane Σ^W. The vector $\mathbf{m}_2$ is parallel to the intersection of the planes Σ^W and Σ, i.e., it is situated in both planes Σ^W and Σ. The vectors $\mathbf{m}_1$, $\mathbf{m}_2$ and $\mathbf{n}^W$ form a triplet of mutually perpendicular unit vectors. The quantity σ^{inc} is then given by the relation

$$\sigma^{inc} = \pm[(1/V^{inc})^2 + D_1^2 + D_2^2]^{1/2} \ , \tag{109}$$

and the expression for the slowness vector $\mathbf{p}^{inc}$ of the incident wave reads:

$$\mathbf{p}^{inc} = \sigma^{inc}\mathbf{n}^W + \mathrm{i}D_1\mathbf{m}_1 + \mathrm{i}D_2\mathbf{m}_2 \ . \tag{110}$$

Projecting $\mathbf{n}^W$, $\mathbf{m}_1$ and $\mathbf{m}_2$ on the plane Σ, we get

$$\mathbf{p}^\Sigma = \sigma^{inc}\mathbf{N}^\Sigma + \mathrm{i}D_1\mathbf{M}_1^\Sigma + \mathrm{i}D_2\mathbf{M}_2^\Sigma \ , \tag{111}$$

where $\mathbf{N}^\Sigma$ is given by (96), and $\mathbf{M}_1^\Sigma$, $\mathbf{M}_2^\Sigma$ by analogous relations

$$\begin{aligned} \mathbf{M}_1^\Sigma &= \mathbf{n}^\Sigma \times (\mathbf{m}_1 \times \mathbf{n}^\Sigma) = \mathbf{m}_1 - \mathbf{n}^\Sigma(\mathbf{m}_1 \cdot \mathbf{n}^\Sigma) \ , \\ \mathbf{M}_2^\Sigma &= \mathbf{n}^\Sigma \times (\mathbf{m}_2 \times \mathbf{n}^\Sigma) = \mathbf{m}_2 - \mathbf{n}^\Sigma(\mathbf{m}_2 \cdot \mathbf{n}^\Sigma) = \mathbf{m}_2 \ . \end{aligned} \tag{112}$$

The second relation $\mathbf{M}_2^\Sigma = \mathbf{m}_2$ in (112) is important. It is a consequence of the fact that $\mathbf{m}_2$ is introduced as vector perpendicular to vectors $\mathbf{n}^W$ and $\mathbf{n}^\Sigma$.

Now we rewrite σ^{rt} given by (104), using the new specification of $\mathrm{i}D\mathbf{m}$, see (107). For $\mathrm{i}D(\mathbf{n}^\Sigma\mathbf{m})$ we obtain

$$\mathrm{i}D(\mathbf{n}^\Sigma \cdot \mathbf{m}) = \mathrm{i}D_1(\mathbf{n}^\Sigma \cdot \mathbf{m}_1) + \mathrm{i}D_2(\mathbf{n}^\Sigma \cdot \mathbf{m}_2) = \mathrm{i}D_1(\mathbf{n}^\Sigma \cdot \mathbf{m}_1) \ . \tag{113}$$

Inserting (113) into (104) yields the final expression for σ^{rt}:

$$\sigma^{rt} = \pm\{(1/V^{rt})^2 - (1/V^{inc})^2 + [\sigma^{inc}(\mathbf{n}^\Sigma \cdot \mathbf{n}^W) + \mathrm{i}D_1(\mathbf{n}^\Sigma \cdot \mathbf{m}_1)]^2\}^{1/2} \ . \tag{114}$$

The corresponding slowness vector of the R/T wave is given by the relation, see (105):

$$\mathbf{p}^{rt} = \sigma^{rt}\mathbf{n}^\Sigma + \sigma^{inc}\mathbf{N}^\Sigma + \mathrm{i}D_1\mathbf{M}_1^\Sigma + \mathrm{i}D_2\mathbf{M}_2^\Sigma \ . \tag{115}$$

Here $\mathbf{N}^\Sigma$, $\mathbf{M}_1^\Sigma$ and $\mathbf{M}_2^\Sigma$ can be expressed in terms of $\mathbf{n}^\Sigma$, $\mathbf{n}^W$, $\mathbf{m}_1$ and $\mathbf{m}_2$ using (96) and (112). It is interesting to note that the expression (114) for σ^{rt} does not explicitly depend on D_2. The quantity D_2 is included only in expression for σ^{inc}, given by (109).

Let us summarize the final results for the non-coplanar case. Consider the slowness vector $\mathbf{p}^{inc}$ of the incident wave, specified by relation (110), with σ^{inc} given by (109). The quantities D_1 and D_2 are the inhomogeneity parameters in the plane specified by vectors $\mathbf{n}^\Sigma$, $\mathbf{n}^W$, and in the direction perpendicular to this plane, respectively. The inhomogeneity parameter D_2 controls the degree of non-complanarity of the incident wave. The R/T plane wave is also non-complanar in this case. The slowness vector $\mathbf{p}^{rt}$ of the R/T wave is given by the relation (115) with σ^{rt} given by (114) and σ^{inc} by (109).

All the above equations simplify considerably for **the coplanar case**, which we obtain by specifying $D_2 = 0$ in the equations for the non-coplanar case. For the slowness vector of the incident wave we get from (110):

$$\mathbf{p}^{inc} = \sigma^{inc}\mathbf{n}^W + \mathrm{i}D_1\mathbf{m}_1 \ , \tag{116}$$

where

$$\sigma^{inc} = \pm[(1/V^{inc})^2 + D_1^2]^{1/2} \ , \tag{117}$$

see (109). For the slowness vector of the R/T wave we get from (115):

$$\mathbf{p}^{rt} = \sigma^{rt}\mathbf{n}^\Sigma + \sigma^{inc}\mathbf{N}^\Sigma + \mathrm{i}D_1\mathbf{M}_1^\Sigma \ , \tag{118}$$

where σ^{rt} is given by (114). Note that the slowness vector of the R/T wave is coplanar if the slowness vector of the incident wave is coplanar.

Finally, we consider a **homogeneous incident wave**. In this case $D_1 = D_2 = 0$. For the slowness vector $\mathbf{p}^{inc}$ of the incident plane wave we get from (110) and (109):

$$\mathbf{p}^{inc} = \sigma^{inc}\mathbf{n}^W = \pm\mathbf{n}^W/V^{inc} \ . \tag{119}$$

For the slowness vector of the R/T wave we get from (115) and (109):

$$\mathbf{p}^{rt} = \sigma^{rt}\mathbf{n}^\Sigma \pm \mathbf{N}^\Sigma/V^{inc} \ , \tag{120}$$

where

$$\sigma^{rt} = \pm\{(1/V^{rt})^2 - (1/V^{inc})^2[1 - (\mathbf{n}^\Sigma \cdot \mathbf{n}^W)^2]\}^{1/2} \ . \tag{121}$$

Note that $\mathbf{n}^\Sigma \cdot \mathbf{n}^W$ corresponds to cos i, where i is the angle of incidence. Consequently, $1 - (\mathbf{n}^\Sigma \cdot \mathbf{n}^W)^2 = \sin^2 i$.

It should be emphasized that an incident homogeneous plane wave does not necessarily generate a homogeneous R/T plane wave.

7 Concluding remarks

The derived equations for the component σ of the slowness vector $\mathbf{p}$ into the normal $\mathbf{n}^\Sigma$ to a plane Σ can be used for homogeneous and inhomogeneous time-harmonic plane waves, propagating in anisotropic or isotropic, viscoelastic or perfectly elastic media. The position of the plane Σ may be arbitrary and the known vector $\mathbf{p}^\Sigma$ situated in the plane Σ may be also arbitrarily oriented in the plane Σ and may be complex-valued, real-valued or imaginary-valued.

The described formalism, based on the componental specification of the slowness vector, can be suitably used to determine the reflection/transmission coefficients of plane waves at a plane interface Σ between two viscoelastic anisotropic halfspaces. For the incident wave, the general mixed specification (11) of the slowness vector $\mathbf{p}^{inc}$ may be used.

The vector $\mathbf{p}^{\Sigma}$ is then determined by projecting $\mathbf{p}^{inc}$ into the plane Σ. The slowness vectors of generated waves are then calculated from $\mathbf{p}^{\Sigma}$ using the componental specification. From appropriate interface conditions, we then determine reflection/transmission coefficients. The detailed algorithms with relevant computations would increase the length of this paper inadmissibly, and will be discussed elsewhere.

The computation of reflection/transmission coefficients of plane waves at a plane interface between two viscoelastic halfspaces in the frequency domain, however, relies on the validity of the correspondence principle. In the correspondence principle, the complex-valued viscoelastic moduli are used in the frequency domain. For a more detailed discussion of the correspondence principle, see Bland (1960), Carcione (2007), Borcherde (2009), and Morozov (2011). Morozov (2011) claims that the correspondence principle is rigorously applicable only to boundless uniform media and that it should be used with caution when applied to heterogeneous cases. We believe that the application of componental specification of the slowness vector would be useful in further illumination of the problem.

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