Transformation of paraxial matrices at a general interface between two general media

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Summary

Paraxial matrices are the derivatives of the phase–space coordinates of rays with respect to the initial conditions for Hamilton's equations of rays. In smooth media, the paraxial matrices satisfy the Hamiltonian equations of geodesic deviation, also called the paraxial ray tracing equations or the dynamic ray tracing equations.

We derive the explicit equations for transforming these paraxial matrices at a general smooth interface between two general media. The transformation equations are applicable to both real–valued and complex–valued paraxial matrices. The equations are expressed in terms of a general Hamiltonian function and are applicable to the transformation of paraxial matrices in both isotropic and anisotropic media. The interface is specified by an implicit equation. No local coordinates are needed for the transformation.

Keywords

Ray theory, Hamilton–Jacobi equation, Hamilton's equations, geodesic deviation, paraxial rays, paraxial matrices, reflection or refraction at curved interfaces, anisotropy, heterogeneous media, paraxial approximation, Gaussian beams, wave propagation.

1. Introduction

In smooth media, rays (geodesics) can be calculated by solving the non–linear ordinary differential equations for rays derived by Hamilton (1837) and called Hamilton's equations. Hamilton's equations represent extremely convenient tools for solving a general partial differential equation of the first order, called the Hamilton–Jacobi equation. Hamilton's equations are applicable to arbitrary spatial coordinates, including curvilinear coordinates. At smooth curved interfaces, rays can be transformed using Snell's law in the form derived by Hamilton (1837, eq. C^7).

Paraxial matrices are the derivatives of the phase–space coordinates of rays with respect to the initial conditions for Hamilton's equations. In smooth media, the paraxial matrices can be calculated along rays by solving the linear ordinary differential equations derived by Červený (1972). We shall refer to these linear differential equations as the Hamiltonian equations of geodesic deviation, because they are equivalent to the covariant equation of geodesic deviation used in the Finsler geometry. They are also often called the paraxial ray tracing equations or the dynamic ray tracing equations. The paraxial matrices found many applications in the theory of wave propagation (Červený, 2001).

In this paper, we derive the explicit equations for transforming paraxial matrices at a general smooth curved interface between two arbitrary media. The transformation

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equations are applicable to both real–valued and complex–valued paraxial matrices. The equations are expressed in terms of a general Hamiltonian function and are applicable to the transformation of paraxial matrices in generally heterogeneous media, both isotropic and generally anisotropic. The interface represents the surface at which the Hamiltonian function or its phase–space derivatives of an arbitrary order may be discontinuous. The interface is specified by an implicit equation. No local coordinates are needed for the transformation of paraxial matrices. The derived transformation equations are applicable to all paraxial matrices including non–eikonal ones.

The derived transformation equations are identical to the equations of Farra & Le Bégat (1995, eq. A12). Analogously derived transformation equations of Gajewski & Pšenčík (1990, eqs. 37a–37b) for transforming the paraxial matrices of orthonomic systems of rays are not applicable to non–eikonal paraxial matrices. Equivalent but less explicit transformation equations were derived for a parametrically specified interface (Moser, 2005, eqs. 31, 38; Moser & Červený, 2007, eq. 28).

We use the componental notation for vectors and matrices. For example, p_i stands for the covariant vector with components p_i. The Einstein summation over repetitive indices is used throughout the paper.

2. Paraxial matrices in smooth media

2.1. Hamiltonian function and the Hamilton-Jacobi equation

We consider Hamiltonian function

$$H(x^m, p_n) \tag{1}$$

of spatial coordinates x^m in a manifold and of slowness–vector components p_i of covariant vectors from the cotangent spaces. In a D–dimensional manifold, x^m and p_i form $2D$ coordinates in the *phase space*.

The corresponding Hamilton–Jacobi equation for complex–valued travel time $\tau = \tau(x^k)$ reads

$$H\big(x^m, \tau_{,n}(x^k)\big) = C \quad , \tag{2}$$

where

$$\tau_{,i}(x^k) = \frac{\partial}{\partial x^i}\tau(x^k) \quad . \tag{3}$$

The value of constant C is determined by the form and meaning of the Hamiltonian function.

Hamiltonian function $H = H(x^i, p_j)$ is a function of phase–space coordinates x^i and p_i. We use notation

$$H^{;a...f}_{,i...n} = \frac{\partial}{\partial p^a}\cdots\frac{\partial}{\partial p^f}\frac{\partial}{\partial x^i}\cdots\frac{\partial}{\partial x^n}H \tag{4}$$

for the phase–space partial derivatives of the Hamiltonian function, and the analogous notation for the spatial partial derivatives of other functions, see (3).

2.2. Hamilton's equations of rays

Rays are determined by Hamilton's equations

$$\frac{\mathrm{d}x^i}{\mathrm{d}\gamma} = H^{,i}(x^m, p_n) \quad , \tag{5}$$

$$\frac{\mathrm{d}p_i}{\mathrm{d}\gamma} = -H_{,i}(x^m, p_n) \quad , \tag{6}$$

where the meaning of parameter γ along rays is determined by the form of the Hamiltonian function. The slowness vector calculated along rays represents the gradient of travel time,

$$p_n = \tau_{,i} \quad . \tag{7}$$

Travel time τ satisfying Hamilton–Jacobi equation (2) is determined along rays by equation

$$\frac{\mathrm{d}\tau}{\mathrm{d}\gamma} = p_i\, H^{,i}(x^m, p_n) \quad . \tag{8}$$

The initial conditions for p_n in equations (5) and (6) should satisfy condition

$$H(x^m, p_n) = C \quad , \tag{9}$$

where constant C is defined by Hamilton–Jacobi equation (2).

2.3. Initial parameters and "non-eikonal" solutions

We denote parameters, which parametrize the initial conditions for Hamilton's equations (5) and (6), by $\gamma^{\underline{a}}$. The coordinates of points of rays and the corresponding components of slowness vectors are the functions of these *initial parameters* $\gamma^{\underline{a}}$ and of parameter γ along rays:

$$x^i = x^i(\gamma^{\underline{a}}, \gamma) \quad , \tag{10}$$

$$p_i = p_i(\gamma^{\underline{a}}, \gamma) \quad . \tag{11}$$

An *orthonomic system of rays* corresponds to a particular solution of Hamilton–Jacobi equation (2) for given initial conditions. The initial conditions for an orthonomic system of rays are determined by the initial travel time specified along an initial surface and by condition (9). In D–dimensional space, the initial conditions for an orthonomic system of rays may be parametrized by $D-1$ mutually independent initial parameters $\gamma^{\underline{a}}$, called ray parameters in this case. These $D-1$ ray parameters $\gamma^{\underline{a}}$ together with parameter γ along rays form D ray coordinates.

If constant C in Hamilton–Jacobi equation (2) and in condition (9) is a priori given and independent of initial parameters $\gamma^{\underline{a}}$, we speak of the *eikonal solutions* of Hamilton's equations. In D–dimensional space, the initial conditions for all eikonal rays, i.e., rays satisfying condition (9), may be parametrized by $2D-1$ mutually independent initial parameters $\gamma^{\underline{a}}$.

If constant C in condition (9) is allowed to depend on initial parameters $\gamma^{\underline{a}}$, i.e., if $C = C(\gamma^{\underline{a}})$, we speak of the *non–eikonal solutions* of Hamilton's equations. Note that even if C depends on initial parameters $\gamma^{\underline{a}}$, it is constant along each ray. The non–eikonal solutions of Hamilton's equations (5) and (6) may be sorted into eikonal solutions corresponding to various constants C. The initial conditions for non–eikonal rays in D–dimensional space, may be parametrized by $2D$ mutually independent initial parameters $\gamma^{\underline{a}}$.

2.4. Paraxial matrices and Hamiltonian equations of geodesic deviation

We define the matrices of the partial derivatives of coordinates (10) and of corresponding slowness–vector components (11) with respect to the initial parameters,

$$Q^i{}_{\underline{a}} = \frac{\partial x^i}{\partial \gamma^{\underline{a}}}(\gamma^{\underline{c}}, \gamma) \quad , \tag{12}$$

$$P_{i\underline{a}} = \frac{\partial p_i}{\partial \gamma^{\underline{a}}}(\gamma^{\underline{c}}, \gamma) \quad . \tag{13}$$

Since the partial derivatives in (12) and (13) are calculated for fixed γ, the definition of paraxial matrices $Q^i{}_{\underline{a}}$ and $P_{i\underline{a}}$ depends on the kind of parameter γ along rays, which is in turn determined by the form of the Hamiltonian.

We shall refer to matrix $Q^i{}_{\underline{a}}$ as the *matrix of geodesic deviation*, to matrix $P_{i\underline{a}}$ as the *matrix of paraxial slowness vectors*, and to both matrices as the *paraxial matrices*. Matrix $Q^i{}_{\underline{a}}$ is often called the *matrix of geometric spreading*.

System

$$\frac{\mathrm{d}}{\mathrm{d}\gamma} Q^i{}_{\underline{a}} = \quad H^{,i}_{,j}(x^q, p_r)\; Q^j{}_{\underline{a}} + H^{,ij}(x^q, p_r)\; P_{j\underline{a}} \quad , \tag{14}$$

$$\frac{\mathrm{d}}{\mathrm{d}\gamma} P_{i\underline{a}} = -H_{,ij}(x^q, p_r)\; Q^j{}_{\underline{a}} - H^{,j}_{,i}(x^q, p_r)\; P_{j\underline{a}} \tag{15}$$

of the *Hamiltonian equations of geodesic deviation* (also called the *paraxial ray tracing equations* or the *dynamic ray tracing equations*) can be obtained by differentiating Hamilton's equations (5)–(6) with respect to $\gamma^{\underline{a}}$ (Červený, 1972). Note that matrices $Q^i = \partial x^i / \partial \gamma = H^{,i}$ and $P_i = \partial p_i / \partial \gamma = -H_{,i}$ also satisfy Hamiltonian equations (14)–(15) of geodesic deviation.

2.5 Propagator matrix of geodesic deviation

We may choose six initial parameters $\gamma^{\underline{a}}$ equal to the six phase–space coordinates x^i_0 and p^0_i of the initial point of a ray. The *propagator matrix of geodesic deviation* from point x^n_0 to point x^m is defined by equation

$$\mathbf{\Pi}(x^m, x^n_0) = \begin{pmatrix} \frac{\partial x^i}{\partial x^j_0} & \frac{\partial x^i}{\partial p^0_j} \\ \frac{\partial p_i}{\partial x^j_0} & \frac{\partial p_i}{\partial p^0_j} \end{pmatrix} \quad , \tag{16}$$

where the derivatives are taken at fixed parameter γ along rays. Different forms of the Hamiltonian function may yield different propagator matrices of geodesic deviation.

The propagator matrix of geodesic deviation describes all $2D$ linearly independent non–eikonal solutions of the Hamiltonian equations of geodesic deviation, and may thus be used for calculating paraxilal matrices for arbitrary initial conditions.

The Hamiltonian equations (14)–(15) of geodesic deviation for the propagator matrix read

$$\frac{\mathrm{d}}{\mathrm{d}\gamma} \mathbf{\Pi}(x^m, x^n_0) = \begin{pmatrix} H^{,i}_{,j} & H^{,ij} \\ -H_{,ij} & -H^{,j}_{,i} \end{pmatrix} \mathbf{\Pi}(x^m, x^n_0) \tag{17}$$

with unit initial conditions,

$$\mathbf{\Pi}(x^m_0, x^n_0) = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix} \quad . \tag{18}$$

The propagator matrix of geodesic deviation obeys the chain rule,

$$\mathbf{\Pi}(x^m, x_0^n) = \mathbf{\Pi}(x^m, x_1^j)\, \mathbf{\Pi}(x_1^k, x_0^n) \quad , \tag{19}$$

and is symplectic,

$$\mathbf{\Pi}^{\mathrm{T}} \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{pmatrix} \mathbf{\Pi} = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{pmatrix} \quad , \tag{20}$$

where $\mathbf{0}$ and $\mathbf{1}$ are the 3×3 zero and identity matrices.

3. Smooth interface

We consider a smooth interface implicitly described by equation

$$F(x^i) = 0 \tag{21}$$

(Hamilton, 1837, eq. B^7). The interface represents the surface at which the Hamiltonian function or its phase–space derivatives of an arbitrary order may be discontinuous.

3.1. Hamilton-Jacobi equations for incident travel time and reflected or refracted travel time

We denote the incident travel time by $\tilde{\tau}(x^i)$. It satisfies Hamilton–Jacobi equation

$$\widetilde{H}\big(x^i, \tilde{\tau}_{,j}(x^m)\big) = \widetilde{C} \quad , \tag{22}$$

where $\widetilde{H} = \widetilde{H}(x^m, p_n)$ is given Hamiltonian function (1) corresponding to the incident travel time. Reflected or refracted travel time $\tau(x^i)$ satisfies Hamilton–Jacobi equation

$$H\big(x^i, \tau_{,j}(x^m)\big) = C \tag{23}$$

(Hamilton, 1837, eq. F^7), where $H = H(x^m, p_n)$ is given Hamiltonian function (1) corresponding to the reflected or refracted travel time.

The domains for solving equations (22) and (23) are situated at the same side of the interface for reflected travel time $\tau(x^i)$, and at the opposite sides for refracted travel time $\tau(x^i)$.

For eikonal solutions, constant C in Hamilton–Jacobi equation (23) is independent of $\gamma^{\underline{a}}$ and is a priori given. In this case, The form of Hamiltonian function $H(x^i, p_j)$ may be different from the form of $\widetilde{H}(x^i, p_j)$, and constant C may differ from $\widetilde{C}$.

For non–eikonal solutions, $C = C(\gamma^{\underline{a}})$ depends on $\gamma^{\underline{a}}$. In this case, we assume that

$$C(\gamma^{\underline{a}}) = \widetilde{C}(\gamma^{\underline{a}}) \quad . \tag{24}$$

The reflected or refracted travel time must be equal to the incident travel time along the interface:

$$\tau(x^i) = \tilde{\tau}(x^i) \tag{25}$$

(Hamilton, 1837, eq. A^7) for x^i satisfying equation (21). Equation (25) thus represents the initial conditions for Hamilton–Jacobi equation (23).

3.2. Incident rays and reflected or refracted rays

We denote the phase–space coordinates of the incident ray by $\tilde{x}^i = \tilde{x}^i(\gamma^{\underline{a}}, \gamma)$, $\tilde{p}_i = \tilde{p}_i(\gamma^{\underline{a}}, \gamma)$, and the phase–space coordinates of the reflected or refracted ray by $x^i = x^i(\gamma^{\underline{a}}, \gamma)$, $p_i = p_i(\gamma^{\underline{a}}, \gamma)$.

In this notation, Hamilton's equations (5)–(6) for the incident rays read

$$\frac{\mathrm{d}\tilde{x}^i}{\mathrm{d}\gamma} = \widetilde{H}^{,i}(\tilde{x}^m, \tilde{p}_n) \quad , \tag{26}$$

$$\frac{\mathrm{d}\tilde{p}_i}{\mathrm{d}\gamma} = -\widetilde{H}_{,i}(\tilde{x}^m, \tilde{p}_n) \quad , \tag{27}$$

and for the reflected or refracted rays read

$$\frac{\mathrm{d}x^i}{\mathrm{d}\gamma} = H^{,i}(x^m, p_n) \quad , \tag{28}$$

$$\frac{\mathrm{d}p_i}{\mathrm{d}\gamma} = -H_{,i}(x^m, p_n) \quad . \tag{29}$$

Definitions (12)–(13) of the paraxial matrices for the incident rays read

$$\widetilde{Q}^i_{\underline{a}} = \frac{\partial \tilde{x}^i}{\partial \gamma^{\underline{a}}}(\gamma^{\underline{c}}, \gamma) \quad , \tag{30}$$

$$\widetilde{P}_{i\underline{a}} = \frac{\partial \tilde{p}_i}{\partial \gamma^{\underline{a}}}(\gamma^{\underline{c}}, \gamma) \quad , \tag{31}$$

and for the reflected or refracted rays read

$$Q^i_{\ \underline{a}} = \frac{\partial x^i}{\partial \gamma^{\underline{a}}}(\gamma^{\underline{c}}, \gamma) \quad , \tag{32}$$

$$P_{i\underline{a}} = \frac{\partial p_i}{\partial \gamma^{\underline{a}}}(\gamma^{\underline{c}}, \gamma) \quad . \tag{33}$$

3.3. Intersections of rays with an interface

The value of independent parameter γ along the ray at the point of intersection with interface (21) depends on initial parameters $\gamma^{\underline{a}}$:

$$\gamma = \gamma|_F(\gamma^{\underline{a}}) \quad . \tag{34}$$

We denote by

$$\tilde{x}^i|_F(\gamma^{\underline{a}}) = \tilde{x}^i\big(\gamma^{\underline{a}}, \gamma|_F(\gamma^{\underline{c}})\big) \tag{35}$$

and

$$\tilde{p}_i|_F(\gamma^{\underline{a}}) = \tilde{p}_i\big(\gamma^{\underline{a}}, \gamma|_F(\gamma^{\underline{c}})\big) \tag{36}$$

the functions describing the dependence of the phase–space coordinates $\tilde{x}^i$ and $\tilde{p}_i$ of the point of intersection of the incident ray with the interface on initial parameters $\gamma^{\underline{a}}$.

We analogously denote by

$$x^i|_F(\gamma^{\underline{a}}) = x^i\big(\gamma^{\underline{a}}, \gamma|_F(\gamma^{\underline{c}})\big) \tag{37}$$

and

$$p_i|_F(\gamma^{\underline{a}}) = p_i\big(\gamma^{\underline{a}}, \gamma|_F(\gamma^{\underline{c}})\big) \tag{38}$$

the functions describing the dependence of the phase–space coordinates x^i and p_i of the point of intersection of the reflected or refracted ray with the interface on initial parameters $\gamma^{\underline{a}}$.

Spatial coordinates (35) of the point of intersection of the incident ray with the interface obviously satisfy relation

$$F\big(\tilde{x}^i|_F(\gamma^{\underline{a}})\big) = 0 \quad . \tag{39}$$

The spatial coordinates of the point of intersection of the reflected or refracted ray with the interface are equal to the spatial coordinates of the point of intersection of the incident ray with the interface:

$$x^i|_F(\gamma^{\underline{a}}) = \tilde{x}^i|_F(\gamma^{\underline{a}}) \quad . \tag{40}$$

3.4. Snell's law

Slowness vector p_i should satisfy Snell's law

$$p_i = \tilde{p}_i + F_{,i}\,\lambda \tag{41}$$

(Hamilton, 1837, eq. C^7; Klimeš, 2010, eq. 7).

For eikonal solutions satisfying Hamilton–Jacobi equation (23) with a priori given constant C, Lagrange multiplier λ can be calculated from non–linear algebraic equation

$$H\big(\tilde{x}^i, \tilde{p}_j + F_{,j}\,\lambda\big) = C \tag{42}$$

(Hamilton, 1837, eq. F^7; Klimeš, 2010, eq. 9), where $\tilde{x}^i = \tilde{x}^i|_F(\gamma^{\underline{a}})$ is the point of incidence of the ray at the interface and $\tilde{p}_i = \tilde{p}_i|_F(\gamma^{\underline{a}})$ is the incident slowness vector.

For non–eikonal solutions with $C = C(\gamma^{\underline{a}})$, Lagrange multiplier λ can be calculated from non–linear algebraic equation

$$H\big(\tilde{x}^i, \tilde{p}_j + F_{,j}\,\lambda\big) = \widetilde{H}(\tilde{x}^i, \tilde{p}_j) \quad , \tag{43}$$

where $\tilde{x}^i = \tilde{x}^i|_F(\gamma^{\underline{a}})$ is the point of incidence of the ray at the interface and $\tilde{p}_i = \tilde{p}_i|_F(\gamma^{\underline{a}})$ is the incident slowness vector, see (24).

4. Transformation of the matrix of geodesic deviation at an interface

We differentiate relation (40) with respect to initial parameters $\gamma^{\underline{a}}$ and obtain equation

$$\frac{\partial x^i|_F}{\partial \gamma^{\underline{a}}} = \frac{\partial \tilde{x}^i|_F}{\partial \gamma^{\underline{a}}} \quad . \tag{44}$$

We now differentiate definitions (35) and (37) with respect to initial parameters $\gamma^{\underline{a}}$ and obtain relations

$$\frac{\partial \tilde{x}^i|_F}{\partial \gamma^{\underline{a}}} = \frac{\partial \tilde{x}^i}{\partial \gamma^{\underline{a}}} + \frac{\partial \tilde{x}^i}{\partial \gamma}\frac{\partial \gamma|_F}{\partial \gamma^{\underline{a}}} \quad , \tag{45}$$

$$\frac{\partial x^i|_F}{\partial \gamma^{\underline{a}}} = \frac{\partial x^i}{\partial \gamma^{\underline{a}}} + \frac{\partial x^i}{\partial \gamma}\frac{\partial \gamma|_F}{\partial \gamma^{\underline{a}}} \quad . \tag{46}$$

We insert the first Hamilton's equation (26) or (28) and definition (30) or (32) into (45) and (46):

$$\frac{\partial \tilde{x}^i|_F}{\partial \gamma^{\underline{a}}} = \widetilde{Q}^i{}_{\underline{a}} + \widetilde{H}^{,i}\,\frac{\partial \gamma|_F}{\partial \gamma^{\underline{a}}} \tag{47}$$

$$\frac{\partial x^i|_F}{\partial \gamma^{\underline{a}}} = Q^i{}_{\underline{a}} + H^{,i}\,\frac{\partial \gamma|_F}{\partial \gamma^{\underline{a}}} \quad . \tag{48}$$

Inserting (47) and (48) into (44), we obtain transformation relation

$$Q^i{}_{\underline{a}} = \widetilde{Q}^i{}_{\underline{a}} + \left(\widetilde{H}^{,i} - H^{,i}\right)\frac{\partial \gamma|_F}{\partial \gamma^{\underline{a}}} \quad . \tag{49}$$

To calculate $\partial\gamma|_F/\partial\gamma^{\underline{a}}$, we differentiate relation (39) with respect to initial parameters $\gamma^{\underline{a}}$:

$$F_{,i}\,\frac{\partial \tilde{x}^i|_F}{\partial \gamma^{\underline{a}}} = 0 \quad . \tag{50}$$

Inserting (47) into (50), we arrive at

$$F_{,i}\widetilde{Q}^i{}_{\underline{a}} + F_{,i}\widetilde{H}^{,i}\,\frac{\partial \gamma|_F}{\partial \gamma^{\underline{a}}} = 0 \quad . \tag{51}$$

We normalize gradient $F_{,i}$ to $\widetilde{N}_i$ and N_i so that scalar products $\widetilde{H}^{,i}\widetilde{N}_i$ and $H^{,i}N_i$ are unit:

$$\widetilde{N}_i = \left(\widetilde{H}^{,q}F_{,q}\right)^{-1}F_{,i} \quad , \tag{52}$$

$$N_i = \left(H^{,q}F_{,q}\right)^{-1}F_{,i} \quad . \tag{53}$$

Equation (51) with (52) then yields

$$\frac{\partial \gamma|_F}{\partial \gamma^{\underline{a}}} = -\widetilde{N}_i\widetilde{Q}^i{}_{\underline{a}} \quad . \tag{54}$$

We insert (54) into (49) and obtain transformation relation

$$Q^i{}_{\underline{a}} = \widetilde{Q}^i{}_{\underline{a}} + \left(H^{,i} - \widetilde{H}^{,i}\right)\widetilde{N}_j\,\widetilde{Q}^j{}_{\underline{a}} \quad . \tag{55}$$

We finally express this transformation relation in form

$$Q^i{}_{\underline{a}} = C^i{}_j\,\widetilde{Q}^j{}_{\underline{a}} \quad , \tag{56}$$

where

$$C^i{}_j = \delta^i_j + \left(H^{,i} - \widetilde{H}^{,i}\right)\widetilde{N}_j \quad . \tag{57}$$

5. Transformation of the matrix of paraxial slowness vectors at an interface

In Snell's law (41), we express the dependence on initial parameters $\gamma^{\underline{a}}$ explicitly:

$$p_i|_F(\gamma^{\underline{a}}) = \tilde{p}_i|_F(\gamma^{\underline{a}}) + F_{,i}\big(\tilde{x}^m|_F(\gamma^{\underline{a}})\big)\,\lambda(\gamma^{\underline{c}}) \quad . \tag{58}$$

We differentiate this Snell's law with respect to initial parameters $\gamma^{\underline{a}}$ and obtain equation

$$\frac{\partial p_i|_F}{\partial\gamma^{\underline{a}}} = \frac{\partial\tilde{p}_i|_F}{\partial\gamma^{\underline{a}}} + \lambda F_{,ij}\frac{\partial\tilde{x}^j|_F}{\partial\gamma^{\underline{a}}} + F_{,i}\frac{\partial\lambda}{\partial\gamma^{\underline{a}}} \quad . \tag{59}$$

We now differentiate definitions (36) and (38) with respect to initial parameters $\gamma^{\underline{a}}$ and obtain relations

$$\frac{\partial\tilde{p}_i|_F}{\partial\gamma^{\underline{a}}} = \frac{\partial\tilde{p}_i}{\partial\gamma^{\underline{a}}} + \frac{\partial\tilde{p}_i}{\partial\gamma}\frac{\partial\gamma|_F}{\partial\gamma^{\underline{a}}} \quad , \tag{60}$$

$$\frac{\partial p_i|_F}{\partial\gamma^{\underline{a}}} = \frac{\partial p_i}{\partial\gamma^{\underline{a}}} + \frac{\partial p_i}{\partial\gamma}\frac{\partial\gamma|_F}{\partial\gamma^{\underline{a}}} \quad . \tag{61}$$

We insert the second Hamilton's equation (27) or (29) and definition (31) or (33) into (60) and (61):

$$\frac{\partial\tilde{p}_i|_F}{\partial\gamma^{\underline{a}}} = \widetilde{P}_{i\underline{a}} - \widetilde{H}_{,i}\frac{\partial\gamma|_F}{\partial\gamma^{\underline{a}}} \tag{62}$$

$$\frac{\partial p_i|_F}{\partial\gamma^{\underline{a}}} = P_{i\underline{a}} - H_{,i}\frac{\partial\gamma|_F}{\partial\gamma^{\underline{a}}} \quad . \tag{63}$$

Inserting (47), (62) and (63) into (59), we obtain transformation relation

$$P_{i\underline{a}} = \widetilde{P}_{i\underline{a}} - \big(\widetilde{H}_{,i} - H_{,i}\big)\frac{\partial\gamma}{\partial\gamma^{\underline{a}}} + \lambda F_{,ij}\left(\widetilde{Q}^j{}_{\underline{a}} + \widetilde{H}^{,j}\frac{\partial\gamma}{\partial\gamma^{\underline{a}}}\right) + F_{,i}\frac{\partial\lambda}{\partial\gamma^{\underline{a}}} \quad . \tag{64}$$

We insert (54) into (64),

$$P_{i\underline{a}} = \widetilde{P}_{i\underline{a}} + \big(\widetilde{H}_{,i} - H_{,i}\big)\,\widetilde{N}_j\widetilde{Q}^j{}_{\underline{a}} + \lambda F_{,il}\left(\widetilde{Q}^l{}_{\underline{a}} - \widetilde{H}^{,l}\widetilde{N}_j\widetilde{Q}^j{}_{\underline{a}}\right) + F_{,i}\frac{\partial\lambda}{\partial\gamma^{\underline{a}}} \quad , \tag{65}$$

and express this transformation equation in form

$$P_{i\underline{a}} = \widetilde{P}_{i\underline{a}} + \big(\widetilde{H}_{,i} - H_{,i}\big)\widetilde{N}_j\widetilde{Q}^j{}_{\underline{a}} + \lambda F_{,il}\left(\delta^l_j - \widetilde{H}^{,l}\widetilde{N}_j\right)\widetilde{Q}^j{}_{\underline{a}} + F_{,i}\frac{\partial\lambda}{\partial\gamma^{\underline{a}}} \quad . \tag{66}$$

To calculate $\partial\lambda/\partial\gamma^{\underline{a}}$, we multiply equation (66) by $H^{,i}$,

$$H^{,i}P_{i\underline{a}} = H^{,i}\widetilde{P}_{i\underline{a}} + H^{,i}\big(\widetilde{H}_{,i} - H_{,i}\big)\widetilde{N}_j\widetilde{Q}^j{}_{\underline{a}} + \lambda H^{,i}F_{,il}\left(\delta^l_j - \widetilde{H}^{,l}\widetilde{N}_j\right)\widetilde{Q}^j{}_{\underline{a}} + H^{,i}F_{,i}\frac{\partial\lambda}{\partial\gamma^{\underline{a}}} \ , \tag{67}$$

and obtain expression

$$\frac{\partial\lambda}{\partial\gamma^{\underline{a}}} = \big(H^{,r}F_{,r}\big)^{-1}\big[H^{,k}\big(P_{k\underline{a}} - \widetilde{P}_{k\underline{a}}\big) - H^{,k}\big(\widetilde{H}_{,k} - H_{,k}\big)\widetilde{N}_j\widetilde{Q}^j{}_{\underline{a}} - \lambda H^{,k}F_{,kl}\big(\delta^l_j - \widetilde{H}^{,l}\widetilde{N}_j\big)\widetilde{Q}^j{}_{\underline{a}}\big] \, . \tag{68}$$

Since $\partial\lambda/\partial\gamma^{\underline{a}}$ stands on the right–hand side of transformation equation (66), we need to remove $H^{,k}P_{k\underline{a}}$ from the right–hand side of equation (68).

To calculate $H^{,k}P_{k\underline{a}}$, we differentiate equation

$$H\big(\tilde{x}^i|_F(\gamma^{\underline{a}}), p_j|_F(\gamma^{\underline{a}})\big) = \widetilde{H}\big(\tilde{x}^i|_F(\gamma^{\underline{a}}), \tilde{p}_j|_F(\gamma^{\underline{a}})\big) \quad , \tag{69}$$

following from (22)–(24), with respect to initial parameters $\gamma^{\underline{a}}$:

$$H^{,k}\frac{\partial p_k|_F}{\partial\gamma^{\underline{a}}}+H_{,k}\frac{\partial\tilde{x}^k|_F}{\partial\gamma^{\underline{a}}}=\widetilde{H}^{,k}\frac{\partial\tilde{p}_k|_F}{\partial\gamma^{\underline{a}}}+\widetilde{H}_{,k}\frac{\partial\tilde{x}^k|_F}{\partial\gamma^{\underline{a}}}\quad . \tag{70}$$

Equation (69) is applicable to both eikonal and non–eikonal solutions, but under constraint $C=\widetilde{C}$, which need not be required for eikonal solutions. For eikonal solutions with $C\neq\widetilde{C}$, we obtain equation (70) by differentiating equations

$$H\big(\tilde{x}^i|_F(\gamma^{\underline{a}}),p_j|_F(\gamma^{\underline{a}})\big)=C \tag{71}$$

and

$$\widetilde{H}\big(\tilde{x}^i|_F(\gamma^{\underline{a}}),\tilde{p}_j|_F(\gamma^{\underline{a}})\big)=\widetilde{C}\quad , \tag{72}$$

following from (22) and (23), with respect to initial parameters $\gamma^{\underline{a}}$.

Note that if we differentiated equation (71) only, we would obtain the equation similar to (70) but with zero right–hand side. This equation would be applicable to eikonal solutions, but not to non–eikonal solutions. If we proceeded in this way, the final transformation equations would not be applicable to the propagator matrix of geodesic deviation which contains also non–eikonal solutions. The transformation equations for paraxial matrices obtained in this way would be identical to the transformation equations of Gajewski & Pšenčík (1990, eqs. 37a–37b).

Inserting (47), (62) and (63) into (70), we arrive at

$$H^{,k}P_{k\underline{a}}=\widetilde{H}^{,k}\widetilde{P}_{k\underline{a}}-\big(\widetilde{H}^{,k}\widetilde{H}_{,k}-H^{,k}H_{,k}\big)\frac{\partial\gamma}{\partial\gamma^{\underline{a}}}+\big(\widetilde{H}_{,k}-H_{,k}\big)\Big(\widetilde{Q}^k_{\ \underline{a}}+\widetilde{H}^{,k}\frac{\partial\gamma}{\partial\gamma^{\underline{a}}}\Big)\quad . \tag{73}$$

We insert (54) into (73),

$$H^{,k}P_{k\underline{a}}=\widetilde{H}^{,k}\widetilde{P}_{k\underline{a}}+\big(\widetilde{H}^{,k}\widetilde{H}_{,k}-H^{,k}H_{,k}\big)\widetilde{N}_j\widetilde{Q}^j_{\ \underline{a}}+\big(\widetilde{H}_{,k}-H_{,k}\big)\big(\widetilde{Q}^k_{\ \underline{a}}-\widetilde{H}^{,k}\widetilde{N}_j\widetilde{Q}^j_{\ \underline{a}}\big)\ , \tag{74}$$

and express this equation in form

$$H^{,k}P_{k\underline{a}}=\widetilde{H}^{,k}\widetilde{P}_{k\underline{a}}+\big(\widetilde{H}^{,k}\widetilde{H}_{,k}-H^{,k}H_{,k}\big)\widetilde{N}_j\widetilde{Q}^j_{\ \underline{a}}+\big(\widetilde{H}_{,k}-H_{,k}\big)\big(\delta^k_j-\widetilde{H}^{,k}\widetilde{N}_j\big)\,\widetilde{Q}^j_{\ \underline{a}}\ . \tag{75}$$

Inserting (75) into (68), we obtain relation

$$\begin{aligned}\frac{\partial\lambda}{\partial\gamma^{\underline{a}}}=\big(H^{,r}F_{,r}\big)^{-1}\big[&\big(\widetilde{H}^{,k}-H^{,k}\big)\widetilde{P}_{k\underline{a}}+\big(\widetilde{H}^{,k}\widetilde{H}_{,k}-H^{,k}H_{,k}\big)\widetilde{N}_j\widetilde{Q}^j_{\ \underline{a}}\\&+\big(\widetilde{H}_{,k}-H_{,k}\big)\big(\delta^k_j-\widetilde{H}^{,k}\widetilde{N}_j\big)\,\widetilde{Q}^j_{\ \underline{a}}-H^{,k}\big(\widetilde{H}_{,k}-H_{,k}\big)\widetilde{N}_j\widetilde{Q}^j_{\ \underline{a}}\\&-\lambda H^{,k}F_{,kl}\,\big(\delta^l_j-\widetilde{H}^{,l}\widetilde{N}_j\big)\,\widetilde{Q}^j_{\ \underline{a}}\big]\quad .\end{aligned} \tag{76}$$

We insert (76) into (66) and arrive at transformation relation

$$\begin{aligned}P_{i\underline{a}}=\widetilde{P}_{i\underline{a}}&+N_i\,\big(\widetilde{H}^{,k}-H^{,k}\big)\,\widetilde{P}_{k\underline{a}}+N_i\,\big(\widetilde{H}^{,k}\widetilde{H}_{,k}-H^{,k}H_{,k}\big)\,\widetilde{N}_j\,\widetilde{Q}^j_{\ \underline{a}}\\&+N_i\,\big(\widetilde{H}_{,k}-H_{,k}\big)\big(\delta^k_j-\widetilde{H}^{,k}\widetilde{N}_j\big)\,\widetilde{Q}^j_{\ \underline{a}}+\big(\delta^k_i-N_iH^{,k}\big)\big(\widetilde{H}_{,k}-H_{,k}\big)\,\widetilde{N}_j\,\widetilde{Q}^j_{\ \underline{a}}\\&+\lambda\,\big(\delta^k_i-N_iH^{,k}\big)F_{,kl}\,\big(\delta^l_j-\widetilde{H}^{,l}\widetilde{N}_j\big)\,\widetilde{Q}^j_{\ \underline{a}}\quad .\end{aligned} \tag{77}$$

We finally express this transformation relation in form

$$\boxed{P_{i\underline{a}}=E_i^{\ j}\,\widetilde{P}_{j\underline{a}}+D_{ij}\,\widetilde{Q}^j_{\ \underline{a}}\quad ,} \tag{78}$$

where

$$\boxed{E_i^{\ j}=\delta^j_i+N_i\,\big(\widetilde{H}^{,j}-H^{,j}\big)} \tag{79}$$

and

$$\boxed{\begin{aligned} D_{ij} = {} & \lambda\left(\delta^k_i - N_i H^{,k}\right) F_{,kl} \left(\delta^l_j - \widetilde{H}^{,l} \widetilde{N}_j\right) \\ & + N_i\left(\widetilde{H}_{,k} - H_{,k}\right)\left(\delta^k_j - \widetilde{H}^{,k}\widetilde{N}_j\right) + \left(\delta^k_i - N_i H^{,k}\right)\left(\widetilde{H}_{,k} - H_{,k}\right)\widetilde{N}_j \\ & + N_i\left(\widetilde{H}^{,r}\widetilde{H}_{,r} - H^{,r}H_{,r}\right)\widetilde{N}_j \quad . \end{aligned}} \tag{80}$$

Equation (80) may also be expressed in form

$$\begin{aligned} D_{ij} = {} & \lambda\left(\delta^k_i - N_i H^{,k}\right) F_{,kl} \left(\delta^l_j - \widetilde{H}^{,l} \widetilde{N}_j\right) \\ & + N_i\left(\widetilde{H}_{,j} - H_{,j}\right) + \left(\widetilde{H}_{,i} - H_{,i}\right)\widetilde{N}_j + N_i\left(\widetilde{H}^{,r}H_{,r} - H^{,r}\widetilde{H}_{,r}\right)\widetilde{N}_j \quad . \end{aligned} \tag{81}$$

The right–hand side of equation (80) represents the decomposition of the right–hand side of equation (81) into projection matrices $\delta^k_i - N_i H^{,k}$ and $N_i H^{,k}$ from the left, and into projection matrices $\delta^l_j - \widetilde{H}^{,l}\widetilde{N}_j$ and $\widetilde{H}^{,l}\widetilde{N}_j$ from the right.

Equations (57), (79) and (81) are identical to the transformation equations of Farra & Le Bégat (1995, eq. A12). Equations (57), (79) and (81) differ from the transformation equations of Gajewski & Pšenčík (1990, eqs. 37a–37b) by terms $+N_i\widetilde{H}^{,j}$ in (79) and $+N_i\widetilde{H}_{,j}$ in (81).

6. Transformation of both paraxial matrices at an interface

Transformation of both paraxial matrices at an interface reads

$$\begin{pmatrix} Q^i{}_{\underline{a}} \\ P_{i\underline{a}} \end{pmatrix} = \begin{pmatrix} C^i{}_j & 0^{ij} \\ D_{ij} & E_i{}^j \end{pmatrix} \begin{pmatrix} \widetilde{Q}^j{}_{\underline{a}} \\ \widetilde{P}_{j\underline{a}} \end{pmatrix} \quad , \tag{82}$$

where 0^{ij} are the components of the zero matrix. Matrices $C^i{}_j$, D_{ij} and $E_i{}^j$ are given by equations (57), (80) and (79).

The analogous transformation equation for the propagator matrix (16) of geodesic deviation reads

$$\mathbf{\Pi} = \begin{pmatrix} \mathbf{C} & \mathbf{0} \\ \mathbf{D} & \mathbf{E} \end{pmatrix} \tilde{\mathbf{\Pi}} \quad . \tag{83}$$

Matrix $\left(\begin{smallmatrix} \mathbf{C} & \mathbf{0} \\ \mathbf{D} & \mathbf{E} \end{smallmatrix}\right)$ is often called the interface propagator matrix (Červený, 2001, eq. 4.7.12; Moser, 2005; Moser & Červený, 2007).

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