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Arrival-time Residuals and Hypocentre Mislocation

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Abstract—It is shown that the arrival-time residuals and hypocentre mislocation are two mutually independent consequences of the inaccurate seismic model and inaccurately measured arrival times. The minimum residuals resulting from the kinematic hypocentre determination contain no information on the accuracy in determining the hypocentre position.

Key words: Hypocentre determination, seismic model, accuracy, arrival-time residuals.

1. Introduction

Iterative linearized procedures of kinematic hypocentre determination are, as a rule, based on minimization of residuals of measured arrival times, with respect to unknown hypocentral time plus calculated travel times. This minimization is, of course, correct (if the linearization leads to the correct minimum of the objective function), but tempts seismologists to erroneously consider the error of the final hypocentral position to be proportional to the resulting minimum arrival-time residuals.

The aim of this paper is to demonstrate that the resulting minimum arrival-time residuals carry information pertinent to the accuracy of the model, but no information on the accuracy of the hypocentre determination. As a consequence, if the residuals are decreased by adjusting the model (without additional information, using the arrival times not only for hypocentre determination but also to update the model), the accuracy of the corresponding hypocentral position cannot be improved, but is often made worse.

Assume, for the kinematic hypocentre determination, that the arrival times of some elementary waves at each of several receivers are measured. Let us denote the total number of measured arrival times N. For the sake of simplicity, we shall refer to the corresponding N arrivals as individual "waves", although some of the arrivals may correspond to the same elementary wave recorded at different receivers. For the

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sake of simplicity, we shall consider only a Gaussian error distribution, in agreement with the conventional linearized inversion.

2. Exact Hypocentral Position and Time

Let us assume, in this section, that the hypocentre is located at its exact position and at exact hypocentral time. Let us emphasize that the exact hypocentral position and time are unknown and cannot be determined.

Denote by $T^{(i)}_{\text{mod}}$ the difference between synthetic and exact travel times caused by the inaccurate seismic model of the medium, and by $T^{(i)}_{\text{wave}}$ the difference between inaccurately measured and exact arrival times of the ith "wave" used to determine the hypocentre. Consequently the difference between measured and calculated arrival times of the ith "wave" is

$$T^{(i)} = T^{(i)}_{\text{wave}} - T^{(i)}_{\text{mod}} . \tag{1}$$

These differences form an N-dimensional vector

$$\mathbf{T} = (T^{(1)}, T^{(2)}, \ldots, T^{(N)})^{\mathrm{T}}, \tag{2}$$

where N is the number of arrival times used for hypocentre determination. Superscript $^{\mathrm{T}}$ denotes transpose.

3. Approximate Hypocentral Position and Time

The dependence of arrival-time residuals

$$\mathbf{R} = (R^{(1)}, R^{(2)}, \ldots, R^{(N)})^{\mathrm{T}} \tag{5}$$

on hypocentre mislocation

$$\mathbf{X} = (\delta x^{(1)}, \delta x^{(2)}, \delta x^{(3)}, \delta x^{(4)} = \delta t)^{\mathrm{T}} \tag{6}$$

is, in a linear approximation,

$$\mathbf{R}(\mathbf{X}) = \mathbf{T} - \mathbf{P}\mathbf{X}, \tag{7}$$

where

$$\mathbf{P} = \begin{bmatrix} p_1^{(1)} & p_2^{(1)} & p_3^{(1)} & p_4^{(1)} = 1 \\ p_1^{(2)} & p_2^{(2)} & p_3^{(2)} & p_4^{(2)} = 1 \\ \vdots & \vdots & \vdots & \vdots \\ p_1^{(N)} & p_2^{(N)} & p_3^{(N)} & p_4^{(N)} = 1 \end{bmatrix}, \tag{8}$$

$p_\alpha^{(i)}$, $\alpha = 1, 2, 3, 4$ being the derivatives of the ith synthetic arrival time with respect to the hypocentral coordinates and hypocentral time (i.e., the ith space-time slowness vector at the hypocentre). The use of linear approximation (7) is correct here, because expansion (7) is centred at the exact hypocentral position, $\mathbf{X}$ representing the error. The possible quadratic term in Taylor expansion (7) would then be proportional to the square of the error and should thus be negligible for reasonable solutions.

Since the exact hypocentral position is unknown, an approximate hypocentral position is determined by minimizing the objective function

$$y(\mathbf{X}) = \tfrac{1}{2}\mathbf{R}(\mathbf{X})^{\mathrm{T}}\mathbf{C}^{-1}\mathbf{R}(\mathbf{X}). \tag{9}$$

Here

$$\mathbf{C} = \mathbf{C}_{\text{wave}} + \mathbf{C}_{\text{mod}} \tag{10}$$

is the data covariance matrix describing the standard deviations of both the measured arrival times and synthetic travel times.

Objective function (9) achieves its minimum for

$$\mathbf{X} = (\mathbf{P}^{\mathrm{T}}\mathbf{C}^{-1}\mathbf{P})^{-1}\mathbf{P}^{\mathrm{T}}\mathbf{C}^{-1}\mathbf{T} \tag{11}$$

and the resulting residuals are

$$\mathbf{R} = \mathbf{T} - \mathbf{P}\ (\mathbf{P}^{\mathrm{T}}\mathbf{C}^{-1}\mathbf{P})^{-1}\mathbf{P}^{\mathrm{T}}\mathbf{C}^{-1}\mathbf{T}. \tag{12}$$

Let us now remind the reader that both final mislocation (11) and resulting minimum residuals (12) are due to the unknown error $\mathbf{T}$ in measured arrival times and synthetic travel times.

4. Independence between Arrival-time Residuals and Hypocentre Mislocation

Operator $\mathbf{E}$ is a *projection operator* if $\mathbf{EE} = \mathbf{E}$. As a consequence, if $\mathbf{E}$ is a projection operator, $\mathbf{1} - \mathbf{E}$ is a projection operator, too. Here $\mathbf{1}$ is an identity operator.

Introducing two complementary projection operators

$$\mathbf{P}_S = \mathbf{P}\ (\mathbf{P}^{\mathrm{T}}\mathbf{C}^{-1}\mathbf{P})^{-1}\mathbf{P}^{\mathrm{T}}\mathbf{C}^{-1} \tag{13}$$

and

$$\mathbf{P}_R = \mathbf{1} - \mathbf{P}_S \tag{14}$$

[see (12)], the unknown error vector $\mathbf{T}$ may be decomposed as

$$\mathbf{T} = \mathbf{S} + \mathbf{R}, \tag{15}$$

with the two components

$$\mathbf{S} = \mathbf{P}_S\mathbf{T} \quad \text{and} \quad \mathbf{R} = \mathbf{P}_R\mathbf{T}. \tag{16}$$

Vector $\mathbf{S}$ contains those parts of arrival-time errors (2) which were eliminated by misplacing the hypocentre in space and time.

Inserting (15), (16), (14), and (13) into (11), we see that final mislocation (11) is fully caused by vector $\mathbf{S}$,

$$\mathbf{X} = (\mathbf{P}^{\mathrm{T}}\mathbf{C}^{-1}\mathbf{P})^{-1}\mathbf{P}^{\mathrm{T}}\mathbf{C}^{-1}\mathbf{S}, \tag{17}$$

and is independent of the resulting minimum arrival-time residuals $\mathbf{R}$. Similarly, the arrival-time residuals $\mathbf{R}$ are independent of vector $\mathbf{S}$.

Covariance matrix $\mathbf{C}_S = \mathbf{P}_S\mathbf{C}\ \mathbf{P}_S^{\mathrm{T}}$ corresponding to $\mathbf{S}$ is

$$\mathbf{C}_S = \mathbf{P}\ (\mathbf{P}^{\mathrm{T}}\mathbf{C}^{-1}\mathbf{P})^{-1}\mathbf{P}^{\mathrm{T}}, \tag{18}$$

see (13), and covariance matrix $\mathbf{C}_R = \mathbf{P}_R\mathbf{C}\ \mathbf{P}_R^{\mathrm{T}}$ corresponding to $\mathbf{R}$ is

$$\mathbf{C}_R = \mathbf{C} - \mathbf{C}_S. \tag{19}$$

The cross-variance $\mathbf{C}_{SR} = \mathbf{P}_S\mathbf{C}\ \mathbf{P}_R^{\mathrm{T}}$ between $\mathbf{S}$ and $\mathbf{R}$ is zero,

$$\mathbf{C}_{SR} = 0, \tag{20}$$

if the data covariance matrix $\mathbf{C}$ in (9) is chosen correctly. The cross-variance between $\mathbf{X}$ and $\mathbf{R}$ is then zero too.

Only if an incorrect matrix $\mathbf{C}$ in (9) were chosen, would there be a statistical relation between the residuals and mislocation: the statistical expectation of both of them would be larger than for the correct value of $\mathbf{C}$ in (9). The incorrect choice of data covariance matrix $\mathbf{C}$ is discussed in more detail in the next section which serves as an example and may be bypassed by readers not detail-oriented.

5. Consequences of Incorrect Data Covariance Matrix

Assume that the correct data covariance matrix $\mathbf{C}$ in (9) is not known and is replaced by incorrect estimate $\tilde{\mathbf{C}}$ during the hypocentre determination procedure, and thus equations (11) to (17) are also affected in this way. For instance, projection matrices (13) and (14) are replaced by

$$\tilde{\mathbf{P}}_S = \mathbf{P}\ (\mathbf{P}^{\mathrm{T}}\tilde{\mathbf{C}}^{-1}\mathbf{P})^{-1}\mathbf{P}^{\mathrm{T}}\tilde{\mathbf{C}}^{-1} \tag{21}$$

and

$$\tilde{\mathbf{P}}_R = \mathbf{1} - \tilde{\mathbf{P}}_S. \tag{22}$$

Variances (18) to (20) are then replaced by

$$\tilde{\mathbf{C}}_S = \tilde{\mathbf{P}}_S\mathbf{C}\ \tilde{\mathbf{P}}_S^{\mathrm{T}}, \tag{23}$$

$$\tilde{\mathbf{C}}_R = \tilde{\mathbf{P}}_R\mathbf{C}\ \tilde{\mathbf{P}}_R^{\mathrm{T}}, \tag{24}$$

$$\tilde{\mathbf{C}}_{SR} = \tilde{\mathbf{P}}_S\mathbf{C}\ \tilde{\mathbf{P}}_R^{\mathrm{T}}. \tag{25}$$

Let us emphasize that if data covariance matrix $\mathbf{C}$ is known, variances $\mathbf{C}_S$, $\mathbf{C}_R$ and $\mathbf{C}_{SR}$ may be determined using (18) to (20), whereas if the correct data covariance matrix $\mathbf{C}$ is not known and is replaced by $\tilde{\mathbf{C}}$, variances (23) to (25) cannot be determined.

Using identity

$$(\tilde{\mathbf{P}}_S - \mathbf{P}_S)\mathbf{C}\ \mathbf{P}_S^{\mathrm{T}} = \mathbf{0}, \tag{26}$$

the differences of variances (23) to (25) with respect to variances (18) to (20) may be expressed as

$$\tilde{\mathbf{C}}_S - \mathbf{C}_S = (\tilde{\mathbf{P}}_S - \mathbf{P}_S)\mathbf{C}\ (\tilde{\mathbf{P}}_S - \mathbf{P}_S)^{\mathrm{T}}, \tag{27}$$

$$\tilde{\mathbf{C}}_R - \mathbf{C}_R = (\tilde{\mathbf{P}}_S - \mathbf{P}_S)\mathbf{C}\ (\tilde{\mathbf{P}}_S - \mathbf{P}_S)^{\mathrm{T}} - (\tilde{\mathbf{P}}_S - \mathbf{P}_S)\mathbf{C} - \mathbf{C}\ (\tilde{\mathbf{P}}_S - \mathbf{P}_S)^{\mathrm{T}}, \tag{28}$$

$$\tilde{\mathbf{C}}_{SR} - \mathbf{C}_{SR} = -(\tilde{\mathbf{P}}_S - \mathbf{P}_S)\mathbf{C}\ (\tilde{\mathbf{P}}_S - \mathbf{P}_S)^{\mathrm{T}} + (\tilde{\mathbf{P}}_S - \mathbf{P}_S)\mathbf{C}. \tag{29}$$

Thus, if the correct data covariance matrix $\mathbf{C}$ is replaced by its incorrect approximation $\tilde{\mathbf{C}}$, the results of the kinematic hypocentre determination are changed in the following way:

(a) Operator $\tilde{\mathbf{P}}_S$ projects on the same subspace as $\mathbf{P}_S$, generated by columns of matrix $\mathbf{P}$ defined by (8). Vector $\mathbf{S}$ thus still belongs to the same 4-D subspace. Vector $\mathbf{S}$ may be decreased or increased with the same probability, depending on difference $\tilde{\mathbf{P}}_S - \mathbf{P}_S$. For reasonable approximations $\tilde{\mathbf{C}}$ of $\mathbf{C}$, i.e., for small $\tilde{\mathbf{P}}_S - \mathbf{P}_S$, covariance matrix $\tilde{\mathbf{C}}_S$ remains quite similar to $\mathbf{C}_S$, whereas for large $\tilde{\mathbf{P}}_S - \mathbf{P}_S$, covariance matrix $\tilde{\mathbf{C}}_S$ increases, see (27). The same applies to mislocation vector $\mathbf{X}$ because mapping (17) of $\mathbf{S}$ on $\mathbf{X}$ is independent of the choice of $\mathbf{C}$. In particular: The resulting hypocentral position and time are shifted in an unknown direction and with the same probability of both orientations, i.e., the mislocation may be decreased or increased with the same probability. The mislocation thus remains nearly the same from the statistical point of view for small difference $\tilde{\mathbf{P}}_S - \mathbf{P}_S$ and increases for large $\tilde{\mathbf{P}}_S - \mathbf{P}_S$, see (27) and (17). The accuracy of the hypocentre determination can no longer be estimated because of the unknown correct value of $\mathbf{C}$.

(b) The projection subspace of operator $\tilde{\mathbf{P}}_R$ is rotated with respect to the projection subspace of $\mathbf{P}_R$. Residuals $\mathbf{R}$ thus may have to be rotated from the unknown projection subspace of $\mathbf{P}_R$ to the unknown projection subspace of $\tilde{\mathbf{P}}_R$, see terms $(\tilde{\mathbf{P}}_S - \mathbf{P}_S)\mathbf{C}$ and $\mathbf{C}\ (\tilde{\mathbf{P}}_S - \mathbf{P}_S)^{\mathrm{T}}$ in (28) and (29), otherwise they are changed with a random orientation. For small $\tilde{\mathbf{P}}_S - \mathbf{P}_S$, the size of the residuals remains approximately the same from the statistical point of view, whereas for large $\tilde{\mathbf{P}}_S - \mathbf{P}_S$, the arrival-time residuals become larger from the statistical point of view, see term $(\tilde{\mathbf{P}}_S - \mathbf{P}_S)\mathbf{C}\ (\tilde{\mathbf{P}}_S - \mathbf{P}_S)^{\mathrm{T}}$ in (28).

(c) A statistical dependence arises between the residuals and hypocentre mislocation, see (29), however it is unknown.

6. *Conclusions*

The arrival-time residuals and hypocentre mislocation are two mutually independent consequences of the inaccurate seismic model and inaccurately measured arrival times. The resulting minimum residuals contain no information regarding the accuracy in determining the hypocentre position.

This conclusion has been derived for reasonably small mislocation vectors **X**, but is even more valid in cases of larger mislocations or non-unique solutions.

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