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# Ray tracing in factorized anisotropic inhomogeneous media

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**SUMMARY**

A concept of a factorized anisotropic inhomogeneous (FAI) medium is introduced. In the FAI medium, the position dependent density normalized elastic parameters $a_{ijkl}(x_i)$ can be factorized in the following sense: $a_{ijkl}(x_i) = f^2(x_i)A_{ijkl}$, where $A_{ijkl}$ are constants, independent of Cartesian coordinates $x_i$, and $f(x_i)$ is an arbitrary continuous function of $x_i$. Thus, all the density normalized elastic parameters $a_{ijkl}(x_i)$ in the FAI medium depend on the coordinates $x_i$ in the same way, but the generality of anisotropy and of inhomogeneity is not restricted. The factorization of $a_{ijkl}(x_i)$ leads to a factorization of certain important ray theory expressions and equations (eikonal equation, etc.), and to particularly simple ray tracing and dynamic ray tracing. For certain types of FAI media, these equations may even be solved analytically or semianalytically.

**Key words:** anisotropic medium, ray tracing, dynamic ray tracing

## 1 INTRODUCTION

A specification and parameterization of a general 3-D anisotropic inhomogeneous medium is not simple. In a general case, all 21 density normalized elastic parameters $a_{ijkl}$ may depend on Cartesian coordinates in a different way. Thus, we have to specify 21 medium parameters as functions of three coordinates. The cumbersome parameterization of such a general model causes great complications in numerical modelling of seismic wave fields in anisotropic inhomogeneous media (travel time computations, synthetic seismograms computations, etc.). In the solution of inverse problems, any attempt to determine all 21 parameters and their spatial distribution would be hopeless and could hardly lead to reliable results. Moreover, all the ray computations, such as ray tracing and dynamic ray tracing, are computationally very cumbersome and time consuming in the described general model.

For this reason, usually only simpler types of anisotropic inhomogeneous media are considered in computations, particularly in the solution of inverse problems. The simplification of the medium usually follows one of the two following approaches (or both of them). In the first approach, the type of anisotropy is simplified. This reduces the number of elastic parameters. Examples are the transversely isotropic media, the ellipsoidal anisotropic media, etc. In the other approach, the spatial distributions of individual elastic parameters are approximated by some simple functions. For example, the elastic parameters or their square roots are assumed to vary linearly with Cartesian coordinates. Most often, both these approaches are combined: the number of elastic parameters is reduced and only simple spatial distributions of these parameters are considered.

Recently, Shearer & Chapman (1988) proposed a model in which the number of parameters describing an anisotropic inhomogeneous medium is reduced even more. They consider a vertically inhomogeneous medium in which all the density normalized elastic parameters depend quadratically on depth. Thus, the spatial variations of all $a_{ijkl}$ are the same. As well known, the quadratic variations of $a_{ijkl}$ with depth correspond to linear variations of phase and group velocities with depth. The medium is then fully described by parameters $a_{ijkl}(z_0)$, where $z_0$ is an arbitrarily selected depth (corresponding, for example, to the surface of the Earth), and by one parameter specifying the relevant gradients. This approach not only reduces the number of parameters required to describe properly the model, but also leads to simpler ray computations.

In this paper, a more general specification of the anisotropic inhomogeneous medium is proposed. We assume that the spatial variation of all density normalized elastic parameters $a_{ijkl}$ is the same. The types of the anisotropy and of the inhomogeneity are not restricted. See Section 2 for more details. The only natural restriction is that the spatial variations of the parameters are sufficiently smooth to satisfy the validity conditions of the ray method. The proposed medium is called here the factorized anisotropic inhomogeneous medium, or briefly the FAI medium. The consideration of FAI media simplifies considerably the parameterization of the model. For a general anisotropy with 21 independent elastic parameters and a general inhomogeneity, the full description of the FAI medium requires the specification of a spatial variation of one scalar function only and the specification of 21 constants. The number of parameters is, of course, decreased if we consider some simpler types of anisotropy or inhomogeneity. It should be, however, pointed out that the

FAI medium is in some ways a more restrictive model than a general inhomogeneous isotropic model. This is because the $P$- and $S$-wave velocities may vary quite separately in an inhomogeneous isotropic model, whereas the ratios of phase (or group) velocities of $qP$-, $qS1$- and $qS2$-waves propagating in a FAI medium in a fixed direction remain constant throughout the model. Thus, the number of parameters needed to specify the FAI medium is not much larger or is even lower than the number of parameters needed to specify a general isotropic inhomogeneous medium.

The FAI medium has also another important advantage. The ray tracing and the dynamic ray tracing systems in the FAI medium are considerably simpler than in the general anisotropic inhomogeneous media even if we consider the FAI medium described by 21 independent elastic parameters with an arbitrary inhomogeneity.

The ray tracing and dynamic ray tracing in the FAI medium may be still more simplified if we consider some simpler anisotropies or simpler spatial variations. In certain cases, the relevant systems of ordinary differential equations may be even solved analytically or semianalytically. It greatly enhances the numerical efficiency of the proposed approach. Very efficient systems are obtained for FAI media in which the gradient of the 'square of slowness' is constant. (For the meaning of the square of slowness in the FAI medium, refer to Section 6.) A similar result is well known for isotropic inhomogeneous media, see Červený (1985, 1987a) and Virieux, Farra & Madariaga (1988).

As the application of the FAI medium substantially reduces the number of the parameters needed to describe the medium and enhances substantially the numerical efficiency of the ray computations, it is particularly suitable for the solution of inverse problems in anisotropic inhomogeneous media. It will be shown elsewhere that even the ray perturbation equations are much simpler in the FAI media than in the general anisotropic inhomogeneous media.

The FAI medium has also some limitations. It does not offer a possibility to describe properly such models of anisotropic inhomogeneous media in which the type of anisotropy varies smoothly with Cartesian coordinates. Roughly speaking, the type of anisotropy in the FAI medium (layer, block) represents some average type of anisotropy in the region under consideration. However, even if we consider models of the anisotropic inhomogeneous medium in which the type of anisotropy varies spatially, the FAI media may be very useful in the combination with perturbation procedures.

## 2 FACTORIZED ANISOTROPIC INHOMOGENEOUS (FAI) MEDIUM

We shall consider a general anisotropic inhomogeneous medium and denote the density normalized elastic parameters by $a_{ijkl}$,

$$a_{ijkl} = c_{ijkl}/\rho, \tag{1}$$

where $c_{ijkl}$ are elastic parameters and $\rho$ is the density. As we know, the elastic parameters $c_{ijkl}$ and the density normalized elastic parameters satisfy the following symmetry relations,

$$c_{ijkl} = c_{jikl} = c_{ijlk} = c_{klij}, \qquad a_{ijkl} = a_{jikl} = a_{ijlk} = a_{klij}. \tag{2}$$

These symmetry relations reduce the number of independent components of $c_{ijkl}$ and $a_{ijkl}$ from 81 to 21. For details, see Fedorov (1968), Musgrave (1970), Petrashen (1980) and Crampin (1981).

We assume that $c_{ijkl}$ and $a_{ijkl}$ are continuous functions of Cartesian coordinates $x_i$, together with their first derivatives, and with piece-wise continuous second derivatives. The structural interfaces of the first and second order will be briefly discussed later.

We introduce now an important concept of a special type of an anisotropic inhomogeneous medium, and call it the *factorized anisotropic inhomogeneous* (*FAI*) *medium*. In the FAI medium, the density normalized elastic parameters $a_{ijkl}$ are defined by the following relations,

$$a_{ijkl}(x_i) = f^2(x_i)A_{ijkl}, \tag{3}$$

where $A_{ijkl}$ are constants, independent of Cartesian coordinates, and $f(x_i)$ is an arbitrary positive continuous function of Cartesian coordinates. We also assume that the first partial derivatives of $f$ with respect to Cartesian coordinates, $\partial f/\partial x_i$, are continuous, and that the second partial derivatives, $\partial^2 f/\partial x_i\,\partial x_j$, are piece-wise continuous functions of Cartesian coordinates.

Thus, all the density normalized elastic parameters $a_{ijkl}(x_i)$ in the FAI medium depend on Cartesian coordinates in the same way; the relative spatial variations of all $a_{ijkl}$ are equal. The constants $A_{ijkl}$ again satisfy the symmetry relations,

$$A_{ijkl} = A_{jikl} = A_{ijlk} = A_{klij}. \tag{4}$$

As we can see, relation (3) in a way separates the anisotropy from the inhomogeneity; $a_{ijkl}$ are factorized. This factorization of density normalized elastic parameters yields automatically a factorization of many other important expressions and relations, e.g. the factorization of the eikonal equation, etc. For this reason, we call the medium described by relation (3) the factorized anisotropic inhomogeneous medium.

Let us emphasize that the concept of the FAI medium does not restrict the type of anisotropy (all 21 elastic parameters may be non-vanishing and mutually different) and the type of the inhomogeneity [the function $f(x_i)$ may be an arbitrary smoothly varying function of all the three coordinates]. The only requirement is that the relative spatial variations of all density normalized anisotropic parameters $a_{ijkl}$ be the same.

A more general inhomogeneous anisotropic model may be, of course, constructed from layers and blocks, separated by structural interfaces of the first or the second order, in which the density normalized elastic parameters are specified by (3). The functions $f(x_i)$ and constants $A_{ijkl}$ may be different in different layers and blocks. For simplicity, we shall call even such a model the FAI model (or the FAI layered and block model).

## 3 ELASTODYNAMIC EQUATION AND ITS HIGH-FREQUENCY SOLUTIONS. MATRIX $\Gamma$

For the main principles and equations of the seismic ray method for anisotropic inhomogeneous media see Babich

(1961), Vlaar (1968), Červený (1972, 1987b), Červený, Molotkov & Pšenčík (1977), Petrashen (1980), Crampin (1981), Červený & Firbas (1984), Petrashen & Kashtan (1984), Gajewski & Pšenčík (1987), Norris (1987), and Kendall & Thompson (1989). In this paper, we shall follow the general approach and notations used in Červený (1972, 1987b).

The elastodynamic equation for inhomogeneous anisotropic medium can be written in the following form,

$$\frac{\partial}{\partial x_j}\left(c_{ijkl}\frac{\partial u_k}{\partial x_l}\right)=\rho\frac{\partial^2 u_i}{\partial t^2}.$$

Here $u_i$ are cartesian components of the displacement vector $\mathbf{u}$, $t$ is the time. We seek the solution of the elastodynamic equation in the form of a formal ray series (ansatz solution),

$$\mathbf{u}(x_i, t)=\sum_{n=0}^{\infty}\mathbf{U}^{(n)}(x_i)F_n[t-\tau(x_i)],$$

where $F_n$ are high-frequency analytic signals satisfying the relation $dF_n(\theta)=F_{n-1}(\theta)$. It is assumed that the complex-valued amplitude coefficients $\mathbf{U}^{(n)}(x_i)$ and the travel time $\tau(x_i)$ do not depend on time.

Inserting the ray series into the elastodynamic equation yields a basic recurrence system of equations which can be used to find successively the equations for $\tau(x_i)$, $\mathbf{U}^{(0)}(x_i)$, $\mathbf{U}^{(1)}(x_i)$, etc. In this paper we are interested in the evaluation of $\tau(x_i)$ only, so that we shall write only the first equation of the system. It reads

$$(\Gamma_{ik}-\delta_{ik})U_k^{(0)}=0, \qquad i=1, 2, 3, \tag{5}$$

where $\delta_{ik}$ is the Kronecker symbol, and $\Gamma_{ik}$ are components of a $3\times 3$ matrix $\boldsymbol{\Gamma}$, given by the relation

$$\Gamma_{ik}(x_i, p_i)=a_{ijkl}p_jp_l, \qquad p_i=\frac{\partial\tau}{\partial x_i}. \tag{6}$$

We assume here that the components of the slowness vector $p_i$ are real-valued functions. For details and other equations of the basic recurrence system see Červený (1972).

We shall now discuss shortly properties of matrix $\boldsymbol{\Gamma}$, which plays a very important role in the ray method for anisotropic media. It has three important properties.

(i) Matrix $\boldsymbol{\Gamma}$ is symmetric, $\Gamma_{ik}=\Gamma_{ki}$.

(ii) The components of the matrix $\boldsymbol{\Gamma}$ are homogeneous functions of the second order in $p_j$,

$$\Gamma_{ik}(x_i, ap_i)=a^2\Gamma_{ik}(x_i, p_i), \tag{7}$$

where $a$ is an arbitrary non-vanishing constant. As a homogeneous function of the second order, $\Gamma_{ik}$ satisfy the Euler's theorem,

$$p_n\frac{\partial\Gamma_{ik}}{\partial p_n}=2\Gamma_{ik}. \tag{8}$$

(iii) The matrix $\boldsymbol{\Gamma}$ is positive definite. This property follows from energy considerations.

The matrix $\boldsymbol{\Gamma}$ has three eigenvalues, $G_m(x_i, p_i)$, $m=1, 2, 3$, and three corresponding eigenvectors, $\mathbf{g}^{(m)}(x_i, p_i)$. The eigenvalues $G_m$ satisfy the relation,

$$\mathrm{Det}\,(\Gamma_{ik}-G_m\delta_{ik})=0, \tag{9}$$

and the eigenvectors the relations,

$$(\Gamma_{ik}-G_m\delta_{ik})g_k^{(m)}=0, \quad i=1, 2, 3, \tag{10}$$

(no summation over $m$), with an additional equation,

$$g_k^{(m)}g_k^{(m)}=1, \tag{11}$$

(no summation over $m$). Thus, the eigenvectors $\mathbf{g}^{(m)}$ are defined as unit vectors.

The eigenvalues $G_m$ and eigenvectors $\mathbf{g}^{(m)}$, $m=1, 2, 3$, have the following properties.

(i) The eigenvalues $G_m$, $m=1, 2, 3$, are real and positive. This follows from the positive definiteness of the matrix $\boldsymbol{\Gamma}$.

(ii) The eigenvectors $\mathbf{g}^{(m)}$, $m=1, 2, 3$, are real-valued and mutually perpendicular. This again follows from the positive definiteness of the matrix $\boldsymbol{\Gamma}$.

(iii) The eigenvalues $G_m$, $m=1, 2, 3$, are homogeneous functions of the second order in $p_j$. Thus, we have

$$G_m(x_i, ap_i)=a^2G_m(x_i, p_i), \tag{12}$$

and the Euler's theorem,

$$p_n\frac{\partial G_m}{\partial p_n}=2G_m. \tag{13}$$

In a degenerate case of two identical eigenvalues, the direction of the two corresponding eigenvectors cannot be determined from (9) and (10), only the plane in which they are situated can be determined. This plane is perpendicular to the remaining third eigenvector.

*In the FAI medium*, the components of the matrix $\boldsymbol{\Gamma}$ are given by the relation,

$$\Gamma_{ik}(x_i, p_i)=f^2(x_i)A_{ijkl}p_jp_l. \tag{14}$$

Thus, we can write

$$\Gamma_{ik}=f^2\Gamma^0_{ik}, \tag{15}$$

with

$$\Gamma^0_{ik}=A_{ijkl}p_jp_l. \tag{16}$$

We shall call $\Gamma^0_{ik}$ the components of the reduced matrix $\boldsymbol{\Gamma}^0$. The reduced matrix $\boldsymbol{\Gamma}^0$ does not depend explicitly on the spatial coordinates, only on the components of the slowness vector $p_i$.

It is not difficult to show that the eigenvalues $G_m$ of the matrix $\boldsymbol{\Gamma}$ in the FAI medium are given by the following relation,

$$G_m(x_i, p_i)=f^2(x_i)G^0_m(p_i). \tag{17}$$

Here $G^0_m(p_i)$ are the three eigenvalues of the reduced matrix $\boldsymbol{\Gamma}^0$. The eigenvalues $G^0_m$ depend explicitly on $p_j$ only, not on the Cartesian coordinates. Note that the eigenvectors of the reduced matrix $\boldsymbol{\Gamma}^0$ are the same as the eigenvectors of the matrix $\boldsymbol{\Gamma}$.

It is simple to show that the eigenvalues $G^0_m$ of the reduced matrix $\boldsymbol{\Gamma}^0$ and the relevant eigenvectors satisfy again the properties listed above.

## 4 EIKONAL EQUATION. HAMILTONIANS

In the high-frequency approximation, the elastodynamic equation for the anisotropic inhomogeneous medium yields the basic equation of the ray method (5). Equation (5) represents a system of three algebraic equations for $U_1^{(0)}$, $U_2^{(0)}$ and $U_3^{(0)}$. From a comparison with (9) and (10) we can easily see that system (5) has a nontrivial solution only if one of the eigenvalues $G_m$ of the matrix $\boldsymbol{\Gamma}$ equals unity. We shall consider a non-degenerate case of three mutually different eigenvalues, $G_1 \neq G_2 \neq G_3$. Then equation (5) is satisfied if

$$G_m(x_i, p_i) = 1, \quad m = 1 \text{ or } 2 \text{ or } 3. \tag{18}$$

Equation (18) represents a non-linear partial differential equation of the first order and describes the propagation of a wavefront $\tau(x_i) = \text{constant}$. Thus, in an inhomogeneous anisotropic medium three independent wave fronts ($m = 1, 2, 3$), corresponding to one quasi-compressional and two quasi-shear waves can propagate. The propagation of each wavefront is fully described by equation (18). For this reason, we call (18) the *eikonal equation*. Comparing (10) with (5) we can also see that $\mathbf{U}^{(0)}$ has a direction of the relevant eigenvector $\mathbf{g}^{(m)}$.

In the FAI medium, eikonal equation (18) has the following form,

$$f^2(x_i) G_m^0(p_i) = 1. \tag{19a}$$

Eikonal equation (19a) may be written in many other forms. We shall present here two other forms that we shall use in the following. The first is

$$G_m^0(p_i) = 1/f^2(x_i). \tag{19b}$$

Another, more general form of the eikonal equation for the FAI medium which includes also (19b) is

$$(G_m^0)^{n/2} = 1/f^n, \tag{19c}$$

where $n$ is an arbitrary non-vanishing integer. As we can see from equations (19), the term depending explicitly on the Cartesian coordinates is fully separated from the term depending on $p_i$ only.

Eikonal equations (18)–(19c) belong to the class of non-linear partial differential equations referred to as the Hamilton–Jacobi equations (see Kravtsov & Orlov 1980),

$$\mathcal{H}(x_i, p_i) = 0. \tag{20}$$

The function $\mathcal{H}$ is called the Hamiltonian. For general anisotropic inhomogeneous media, the Hamiltonian may be written as follows:

$$\mathcal{H}(x_i, p_i) = \tfrac{1}{2}[G_m(x_i, p_i) - 1]. \tag{21}$$

Now we shall write several alternative forms of the Hamiltonian for the FAI medium. If we just insert (17) into (21), we obtain

$$\mathcal{H}(x_i, p_i) = \tfrac{1}{2}[f^2(x_i) G_m^0(p_i) - 1]. \tag{22a}$$

More useful forms of the Hamiltonian are obtained if we separate the anisotropy and inhomogeneity terms. A very general form we shall use later ensues from (19c),

$$\mathcal{H}(x_i, p_i) = \frac{1}{n}\{[G_m^0(p_i)]^{n/2} - 1/f^n(x_i)\}, \tag{22b}$$

where $n$ is an arbitrary integer. Particularly simple ray tracing system will be obtained from (22b) for $n = 2$,

$$\mathcal{H}(x_i, p_i) = \tfrac{1}{2}[G_m^0(p_i) - 1/f^2(x_i)]. \tag{22c}$$

For $n \to 0$, eq. (22b) is indefinite. Taking a limit, we obtain,

$$\mathcal{H}(x_i, p_i) = \tfrac{1}{2}[\ln G_m^0(p_i) - \ln f^{-2}(x_i)]. \tag{22d}$$

Thus, we shall consider (22d) as a special case of (22b) for $n = 0$.

Of course, it would be possible to find many other suitable forms of Hamiltonians. For example, Norris (1987) and Kendall & Thompson (1989) use, instead of (21), the following Hamiltonian,

$$\mathcal{H}(x_i, p_i) = \det(\Gamma_{ik} - \delta_{ik}),$$

which follows from (9) and (18). In this paper, however, we shall use only the above presented forms (21)–(22).

## 5 RAY TRACING SYSTEMS

The most common way of solving the Hamilton–Jacobi equation (20) is to use the method of characteristics. In seismology, the characteristics represent seismic rays. The characteristics are specified by the following system of equations (ray tracing system),

$$\frac{dx_i}{du} = \frac{\partial \mathcal{H}}{\partial p_i}, \qquad \frac{dp_i}{du} = -\frac{\partial \mathcal{H}}{\partial x_i}, \qquad \frac{d\tau}{du} = p_i \frac{\partial \mathcal{H}}{\partial p_i}. \tag{23}$$

Here $u$ is a variable along the ray. The variable may represent the arclength, the travel time, etc.

We shall first consider a general anisotropic inhomogeneous medium, with the Hamiltonian given by relation (21). Using the Euler's theorem (13) and eikonal equation (18), we obtain

$$\frac{d\tau}{du} = \frac{1}{2} p_i \frac{\partial G_m}{\partial p_i} = G_m = 1.$$

Thus, the parameter along the ray is the travel time $\tau$. The ray tracing system (23) can be then rewritten in the following form,

$$\frac{dx_i}{d\tau} = \frac{1}{2}\frac{\partial G_m}{\partial p_i}, \qquad \frac{dp_i}{d\tau} = -\frac{1}{2}\frac{\partial G_m}{\partial x_i}. \tag{24}$$

Even though the analytic expressions for $G_m$ are rather complicated, the analytic expressions for the derivatives of $G_m$ are simpler (see Červený & Firbas 1984; Červený 1987b),

$$\frac{\partial G_m}{\partial p_i} = 2a_{ijkl} p_l g_k^{(m)} g_j^{(m)},$$

$$\frac{\partial G_m}{\partial x_i} = \frac{\partial a_{ljks}}{\partial x_i} p_l p_s g_k^{(m)} g_j^{(m)}. \tag{25}$$

Remember that $g_i^{(m)}$ are the Cartesian components of the eigenvector $\mathbf{g}^{(m)}$ of the matrix $\boldsymbol{\Gamma}$. Alternative relations for the derivatives of $G_m$ are as follows, see Červený (1972),

$$\frac{\partial G_m}{\partial p_i} = 2a_{ijkl} p_l D_{jk}/D,$$
$$\frac{\partial G_m}{\partial x_i} = \frac{\partial a_{ljks}}{\partial x_i} p_l p_s D_{jk}/D, \tag{26}$$

where

$$\begin{aligned}
D_{11} &= (\Gamma_{22}-1)(\Gamma_{33}-1)-\Gamma_{23}^2,\\
D_{12} &= D_{21} = \Gamma_{13}\Gamma_{23}-\Gamma_{12}(\Gamma_{33}-1)\\
D_{22} &= (\Gamma_{11}-1)(\Gamma_{33}-1)-\Gamma_{13}^2,\\
D_{13} &= D_{31} = \Gamma_{12}\Gamma_{23}-\Gamma_{13}(\Gamma_{22}-1),\\
D_{33} &= (\Gamma_{11}-1)(\Gamma_{22}-1)-\Gamma_{12}^2,\\
D_{23} &= D_{32} = \Gamma_{12}\Gamma_{13}-\Gamma_{23}(\Gamma_{11}-1),\\
D &= \operatorname{tr} D_{jk} = D_{11}+D_{22}+D_{33}.
\end{aligned} \tag{27}$$

As we can see from (24) and (25), the ray tracing system (24) for a general anisotropic inhomogeneous medium is very cumbersome. The computation of the right-hand sides of (24) requires the evaluation of 63 first spatial partial derivatives of the density normalized elastic parameters $a_{ijkl}$. In general, the total number of terms on the right-hand sides of (24) equals 324. The number may be reduced to 144 with the use of the symmetry relations (2), but still the system is extremely cumbersome and numerically non-efficient. The system is not simplified by a special choice of the variable $u$ along the ray, different from $\tau$.

Equations (13) and (24) yield a very important equation

$$p_i\frac{dx_i}{d\tau}=1. \tag{28}$$

This equation is obtained by multiplying the first set of equations (24) by $p_i$ and taking into account (13) and (18).

Now we shall consider the FAI medium. As we shall see, the ray tracing systems will be simplified drastically. We shall first use the general form of the Hamiltonian (22b). From (23), we obtain,

$$\frac{dx_i}{du}=\frac{1}{n}\frac{\partial}{\partial p_i}(G_m^0)^{n/2},\qquad \frac{dp_i}{du}=\frac{1}{n}\frac{\partial}{\partial x_i}f^{-n},$$
$$\frac{d\tau}{du}=f^{-n}=(G_m^0)^{n/2}. \tag{29}$$

It is immediately evident from (29) that the ray tracing system for the FAI medium is considerably simpler than the ray tracing system (24) with (25) or (26) for a general anisotropic inhomogeneous medium. The right-hand side of the equations for $x_i$ do not explicitly depend on $x_i$ and the right-hand side of equations for $p_i$ do not explicitly depend on $p_i$. Note that the relations for $\partial G_m^0/\partial p_i$ are as follows,

$$\frac{\partial G_m^0}{\partial p_i}=2A_{ijkl}p_l g_k^{(m)}g_j^{(m)}=2A_{ijkl}p_l D_{jk}^0/D^0. \tag{30}$$

Functions $D_{jk}^0$ and $D^0$ are again given by the same equations as $D_{jk}$ and $D$, see (27), we only replace $\Gamma_{ik}$ by $\Gamma_{ik}^0$, see (16).

We shall now present several special cases of (29). A very simple ray tracing system is obtained for $n=2$,

$$\frac{dx_i}{d\sigma}=\frac{1}{2}\frac{\partial G_m^0}{\partial p_i},\qquad \frac{dp_i}{d\sigma}=\frac{1}{2}\frac{\partial f^{-2}}{\partial x_i},\qquad \frac{d\tau}{d\sigma}=f^{-2}=G_m^0. \tag{31}$$

As we can see, the variable $\sigma$ along the ray is related to the travel time $\tau$ as follows: $d\sigma=f^2\,d\tau$.

For $n=0$, (29) yields $d\tau/du=1$, so that the variable $u$ along the ray corresponds to the traveltime. By a limiting process from (29), or directly from (22d), we obtain

$$\frac{dx_i}{d\tau}=\frac{1}{2}\frac{\partial \ln G_m^0}{\partial p_i},\qquad \frac{dp_i}{d\tau}=\frac{1}{2}\frac{\partial \ln f^{-2}}{\partial x_i}. \tag{32}$$

Now we shall consider two other important special cases: $n=1$ and $n=-1$. For $n=1$, we denote the variable along the ray by $s$. (In isotropic media, $s$ corresponds to the arclength along the ray.) Then we obtain the ray tracing system as follows,

$$\frac{dx_i}{ds}=\frac{\partial}{\partial p_i}(G_m^0)^{1/2},\qquad \frac{dp_i}{ds}=\frac{\partial f^{-1}}{\partial x_i},$$
$$\frac{d\tau}{ds}=f^{-1}=(G_m^0)^{1/2}. \tag{33}$$

Finally, for $n=-1$ we denote $u$ by $\xi$. The ray tracing system then reads,

$$\frac{dx_i}{d\xi}=-\frac{\partial}{\partial p_i}(G_m^0)^{-1/2},\qquad \frac{dp_i}{d\xi}=-\frac{\partial f}{\partial x_i},$$
$$\frac{d\tau}{d\xi}=f=(G_m^0)^{-1/2}. \tag{34}$$

All the above ray tracing systems (24, 29, 31–34) are written in a Hamiltonian form and are very suitable to finding various analytical and semianalytical solutions and to deriving the dynamic ray tracing systems and the ray perturbation equations. If we are interested in ray tracing and traveltime computations only, we can rewrite these ray tracing systems in alternative forms, using also the eikonal equation, $G_m^0=f^{-2}$. The general ray tracing system (29) then reads,

$$\frac{dx_i}{du}=\frac{1}{2}f^{2-n}\frac{\partial G_m^0}{\partial p_i},\qquad \frac{dp_i}{du}=\frac{1}{n}\frac{\partial f^{-n}}{\partial x_i},\qquad \frac{d\tau}{du}=f^{-n}. \tag{35}$$

Alternative relations for ray tracing systems (31)–(34) are as follows,

$$\begin{aligned}
&\frac{dx_i}{d\sigma}=\frac{1}{2}\frac{\partial G_m^0}{\partial p_i}, && \frac{dp_i}{d\sigma}=\frac{1}{2}\frac{\partial f^{-2}}{\partial x_i}, && \frac{d\tau}{d\sigma}=f^{-2},\\
&\frac{dx_i}{d\tau}=\frac{1}{2}f^2\frac{\partial G_m^0}{\partial p_i}, && \frac{\partial p_i}{d\tau}=-\frac{\partial \ln f}{\partial x_i}, && \\
&\frac{dx_i}{ds}=\frac{1}{2}f\frac{\partial G_m^0}{\partial p_i}, && \frac{dp_i}{ds}=\frac{\partial f^{-1}}{\partial x_i}, && \frac{d\tau}{ds}=f^{-1},\\
&\frac{dx_i}{d\xi}=\frac{1}{2}f^3\frac{\partial G_m^0}{\partial p_i}, && \frac{dp_i}{d\xi}=-\frac{\partial f}{\partial x_i}, && \frac{d\tau}{d\xi}=f.
\end{aligned} \tag{36}$$

All these systems are very suitable for numerical ray tracing. Particularly the first two systems seem to be very attractive. The first system leads to simple analytical solutions in some special cases, as will be shown in Section 7. In the second system, the variable along the ray is the traveltime. Thus, the traveltime is obtained automatically and need not be determined by additional integrations. Moreover, if we evaluate the points along the rays with some constant traveltime increment $\Delta\tau$, we can even plot the wavefronts. The wavefronts may be of great interest in anisotropic inhomogeneous media.

Let us now show how equation (28) is modified in the FAI medium if we use the general variable $u$ instead of the traveltime $\tau$. Using (35), we obtain

$$p_i \frac{dx_i}{du} = f^{-n} = (G_m^0)^{n/2}. \quad (37)$$

## 6 PHASE AND GROUP VELOCITIES

We denote the phase velocity vector by $\mathbf{V}^{\mathrm{PH}}$, the group velocity vector by $\mathbf{V}^{\mathrm{G}}$, and likewise the phase velocity by $V^{\mathrm{PH}}$ and the group velocity by $V^{\mathrm{G}}$. In general anisotropic inhomogeneous media, both these velocities are functions of the position and of the direction of the normal to the wavefront of the propagating wave under consideration.

The phase velocity $V^{\mathrm{PH}}$ and the phase velocity vector $\mathbf{V}^{\mathrm{PH}}$ are related to the slowness vector $\mathbf{p}$ by the relations,

$$\mathbf{p} = \frac{\mathbf{N}}{V^{\mathrm{PH}}}, \qquad \mathbf{V}^{\mathrm{PH}} = V^{\mathrm{PH}}\mathbf{N} = (V^{\mathrm{PH}})^2\mathbf{p}. \quad (38)$$

Here $\mathbf{N}$ denotes the unit vector perpendicular to the wavefront. If we use eikonal equation (18) and the Euler's theorem (13), we obtain,

$$G_m(x_i, p_i) = G_m\left(x_i, \frac{N_i}{V^{\mathrm{PH}}}\right) = \left(\frac{1}{V^{\mathrm{PH}}}\right)^2 G_m(x_i, N_i) = 1.$$

This yields

$$V^{\mathrm{PH}} = [G_m(x_i, N_i)]^{1/2}. \quad (39)$$

For the FAI medium, we use (17) and obtain,

$$V^{\mathrm{PH}} = f(x_i)[G_m^0(N_i)]^{1/2}, \quad V_i^{\mathrm{PH}} = f(x_i)[G_m^0(N_i)]^{1/2} N_i. \quad (40)$$

This can be also rewritten in the following form,

$$V^{\mathrm{PH}} = f(x_i)(V^{\mathrm{PH}})_0, \qquad V_i^{\mathrm{PH}} = f(x_i)(V_i^{\mathrm{PH}})_0. \quad (41)$$

Here $(V^{\mathrm{PH}})_0$ and $(V_i^{\mathrm{PH}})_0$ are the phase velocity and the components of the phase velocity vector in the reference homogeneous anisotropic medium with $f(x_i) = 1$, i.e. with $a_{ijkl} = A_{ijkl}$.

The components of the group velocity vector in a general anisotropic inhomogeneous medium are given by the relation,

$$V_i^{\mathrm{G}} = \frac{dx_i}{d\tau} = \frac{1}{2}\frac{\partial G_m}{\partial p_i} = a_{ijkl} p_l g_k^{(m)} g_j^{(m)} = a_{ijkl} p_l D_{jk}/D. \quad (42)$$

For the FAI medium, we obtain,

$$V_i^{\mathrm{G}} = f^2 A_{ijkl} p_l g_k^{(m)} g_j^{(m)} = f A_{ijkl} \frac{N_l}{(V^{\mathrm{PH}})_0} g_k^{(m)} g_j^{(m)}. \quad (43)$$

We again denote by $(V_i^G)_0$ the components of the group velocity vector in the reference homogeneous anisotropic medium with $f(x_i) = 1$, i.e. with $a_{ijkl} = A_{ijkl}$,

$$(V_i^{\mathrm{G}})_0 = A_{ijkl} \frac{N_l}{(V^{\mathrm{PH}})_0} g_k^{(m)} g_j^{(m)}. \quad (44)$$

Then we obtain,

$$V^{\mathrm{G}} = f(x_i)(V^G)_0, \qquad V_i^{\mathrm{G}} = f(x_i)(V_i^{\mathrm{G}})_0. \quad (45)$$

Thus, in the FAI medium, the phase and group velocities are directly proportional to the function $f(x_i)$. The directional variations of $V^{\mathrm{PH}}$ and $V^{\mathrm{G}}$ remain the same in the whole medium, only the magnitudes of $V^{\mathrm{G}}$ and $V^{\mathrm{PH}}$ vary due to changes of $f(x_i)$. If the spatial gradient of $f$ is constant, even the gradients of the phase and group velocities are constant (for $\mathbf{N}$ fixed). Consequently, we shall use the following convention: if the gradient of $f$ is constant, we speak about a constant gradient of velocities. Similarly, for a constant gradient of $1/f$, we shall speak about a constant gradient of slowness, etc.

As we can see from (38) and (42), equation (28) gives a general relation between the phase and group velocity vectors for a given $\mathbf{N}$,

$$p_i \frac{dx_i}{d\tau} = \frac{1}{(V^{\mathrm{PH}})^2} V_i^{\mathrm{PH}} V_i^{\mathrm{G}} = 1. \quad (46)$$

If we denote the angle between the phase and group velocity vectors for a given $\mathbf{N}$ by $\theta$, we obtain,

$$\cos\theta = \frac{V^{\mathrm{PH}}}{V^{\mathrm{G}}}. \quad (47)$$

It follows from (41), (45) and (47) that the angle $\theta$ does not depend on the function $f(x_i)$ in the FAI medium and is quite the same as in the reference medium with $f(x_i) = 1$, for a given $\mathbf{N}$.

The properties of the phase and group velocities in the FAI medium discussed above suggest another advantage of the FAI medium. Let us first consider a general anisotropic inhomogeneous medium, in which the spatial variations of all density normalized elastic parameters are specified independently. Then unsuitable spatial variations of one or more parameters (caused, for example, by oscillations due to spline approximations) may *locally* generate non-realistic, anomalous anisotropic properties. To avoid such situations, we would have to supplement the algorithm for the model specification by some sophisticated inspection procedures. In the FAI media, such anomalous anisotropic properties cannot be locally generated since the directional variations of $V^{\mathrm{PH}}$ and $V^{\mathrm{G}}$ remain the same in the whole medium (layer, block), only their magnitudes vary.

## 7 ANALYTIC AND SEMIANALYTIC SOLUTIONS OF THE RAY TRACING SYSTEMS

In this section, we shall consider several types of FAI media in which the ray tracing systems are particularly simple. We shall mainly discuss certain simple approximations of $f(x_i)$, not of anisotropy, the anisotropy may be quite general. We shall briefly discuss simpler anisotropy symmetries in Section 7.3.

### 7.1 Constant gradient of $f^{-n}$

In this case, we shall use (29). We specify the spatial distribution by the following relation,

$$f^{-n}(x_i) = A_0 + A_1 x_1 + A_2 x_2 + A_3 x_3. \quad (48)$$

Then we obtain a simple analytic solution for the slowness vector along the ray,

$$p_i(u) = p_i(u_0) + \frac{1}{n} A_i (u - u_0), \quad (49)$$

where $u$ is the parameter along the ray related to the travel time $\tau$ in the following way: $d\tau = f^{-n}\, du$. The remaining equations of ray tracing system (29) read,

$$\frac{dx_i}{du} = \frac{1}{n}\frac{\partial}{\partial p_i}(G_m^0)^{n/2}, \qquad \frac{d\tau}{du} = (G_m^0)^{n/2}. \tag{50}$$

As we can see, the right-hand side of (50) do not explicitly depend on $x_i$, they depend only on $p_i$. From (49) we see that the slowness vector components $p_i$ depend only on $u$. Thus, equations (50) can be solved by quadratures,

$$\begin{aligned} x_i(u) &= x_i(u_0) + \frac{1}{n}\int_{u_0}^{u} \frac{\partial}{\partial p_i}(G_m^0)^{n/2}\, du, \\ \tau(u) &= \tau(u_0) + \int_{u_0}^{u} (G_m^0)^{n/2}\, du. \end{aligned} \tag{51}$$

Equations (49) and (51) give the final solution to the problem. They are valid for an arbitrary anisotropic medium, with all $A_{ijkl}$ non-vanishing and mutually independent; only the function $f(x_i)$ is specified by (48).

In the numerical ray tracing, it may be suitable to evaluate $x_i(u)$ first, and only after this $\tau(u)$. In this case, we can use the eikonal equation to obtain,

$$\tau(u) = \tau(u_0) + \int_{u_0}^{u} f^{-n}\, du. \tag{52}$$

### 7.2 Special cases of constant gradients

Here we shall present four special cases of the ray tracing equations derived in Section 7.1.

Very simple equations are obtained for the constant gradient of $f^{-2}$, i.e. for $n = 2$. We speak about the model of the constant gradient of the square of slowness. We obtain,

$$\begin{aligned} f^{-2}(x_i) &= A_0 + A_1x_1 + A_2x_2 + A_3x_3, \\ p_i(\sigma) &= p_i(\sigma_0) + \tfrac{1}{2}A_i(\sigma - \sigma_0), \\ x_i(\sigma) &= x_i(\sigma_0) + \frac{1}{2}\int_{\sigma_0}^{\sigma} \frac{\partial}{\partial p_i} G_m^0\, d\sigma, \\ \tau(\sigma) &= \tau(\sigma_0) + \int_{\sigma_0}^{\sigma} G_m^0\, d\sigma = \tau(\sigma_0) + \int_{\sigma_0}^{\sigma} f^{-2}\, d\sigma. \end{aligned} \tag{53}$$

The next case we shall consider is the constant gradient of $f$, i.e. $n = -1$. We speak about the model of the constant gradient of velocity. We obtain,

$$\begin{aligned} f(x_i) &= A_0 + A_1x_1 + A_2x_2 + A_3x_3, \\ p_i(\xi) &= p_i(\xi_0) - A_i(\xi - \xi_0), \\ x_i(\xi) &= x_i(\xi_0) - \int_{\xi_0}^{\xi} \frac{\partial}{\partial p_i}(G_m^0)^{-1/2}\, d\xi, \\ \tau(\xi) &= \tau(\xi_0) + \int_{\xi_0}^{\xi} (G_m^0)^{-1/2}\, d\xi = \tau(\xi_0) + \int_{\xi_0}^{\xi} f\, d\xi. \end{aligned} \tag{54}$$

For this case, we shall also briefly discuss an alternative approach to the ray tracing at the end of this section.

Now we shall discuss the model of the constant gradient of slowness, i.e. the constant gradient of $f^{-1}$. We choose $n = 1$ and obtain

$$\begin{aligned} f^{-1}(x_i) &= A_0 + A_1x_1 + A_2x_2 + A_3x_3, \\ p_i(s) &= p_i(s_0) + A_i(s - s_0), \\ x_i(s) &= x_i(s_0) + \int_{s_0}^{s} \frac{\partial}{\partial p_i}(G_m^0)^{1/2}\, ds, \\ \tau(s) &= \tau(s_0) + \int_{s_0}^{s} (G_m^0)^{1/2}\, ds = \tau(s_0) + \int_{s_0}^{s} f^{-1}\, ds. \end{aligned} \tag{55}$$

The final special case we shall consider is $n = 0$. It corresponds to the model of the constant gradient of logarithmic velocity ($\ln f$). It reads

$$\begin{aligned} \ln f &= A_0 + A_1x_1 + A_2x_2 + A_3x_3, \\ p_i(\tau) &= p_i(\tau_0) - A_i(\tau - \tau_0), \\ x_i(\tau) &= x_i(\tau_0) + \frac{1}{2}\int_{\tau_0}^{\tau} \frac{\partial}{\partial p_i} \ln G_m^0\, d\tau. \end{aligned} \tag{56}$$

The variable along the ray corresponds to the traveltime $\tau$ so that no equation for $\tau$ is necessary.

Remember that $\partial G_m^0/\partial p_i$ is given by (30) in all the relations given above.

We shall now derive one interesting equation which may be used alternatively to compute rays for $n = -1$ (constant gradient of velocity). From (49), we easily obtain $x_i\, dp_i/du = n^{-1}A_ix_i$. Combining this with (37) and (48), we obtain,

$$\frac{d}{du}(x_ip_i) = A_0 + A_ix_i\frac{n+1}{n}. \tag{57}$$

If we take $A_0 = 0$ and $n = -1$ (constant gradient of $f$) we obtain

$$\frac{d}{d\xi}(x_ip_i) = 0, \quad \text{i.e. } x_ip_i = \text{constant}. \tag{58}$$

A similar relation was first derived and used to find suitable analytical ray tracing in an anisotropic medium with a constant gradient of velocity by Shearer & Chapman (1988). Let us rotate and shift the Cartesian coordinate system in such a way that $f = A_3x_3$. Thus, we consider the case of a linear increase of velocity with $x_3$. We also choose the axis $x_2$ in such a way that $p_2(\xi_0) = 0$. Then we have, see (54),

$$\begin{aligned} p_1(\xi) &= p_1(\xi_0), \qquad p_2(\xi) = 0, \\ p_3(\xi) &= p_3(\xi_0) - A_3(\xi - \xi_0). \end{aligned} \tag{59}$$

The eikonal equation (19c) for $n = -1$ yields

$$f(x_i)[G_m^0(p_1, p_3)]^{1/2} = A_3x_3[G_m^0(p_1, p_3)]^{1/2} = 1. \tag{60}$$

As $p_1$ is constant and $p_2 = 0$, the only variable in the reduced eigenvalue $G_m^0$ is $p_3$. Thus, if we express $p_3$ in terms of $p_1$, $x_1$ and $x_3$ in (60), equation (60) will represent an equation $x_3 = x_3(x_1)$ for the projection of the ray onto the plane $x_2 = 0$.

Such an equation for $p_3$ is given by (58). If we choose the initial point $\xi_0$ in such a way that $x_i(\xi_0)p_i(\xi_0) = 0$, we obtain,

$$x_1p_1 + x_3p_3 = 0, \quad \text{i.e.} \quad p_3 = -x_1p_1/x_3. \tag{61}$$

Inserting (61) into (60) and using the Euler's theorem (13), we finally obtain,

$$A_3[G_m^0(p_1x_3, -p_1x_1)]^{1/2}=1. \tag{62}$$

This is the final equation for the projection of the ray onto the plane $x_2=0$. The quantities $x_2$ (which are in general non-vanishing even for $p_2=0$) and $\tau$ may be then evaluated by quadratures along the known ray. For more details and numerical examples refer to Shearer & Chapman (1988).

### 7.3 Simpler anisotropies

As mentioned above, simple solutions of ray tracing systems are obtained for the model of the constant gradient of the square of slowness ($n=2$), see (53). Using (30), we can rewrite relations for $x_i(\sigma)$ as follows,

$$\begin{aligned} x_i(\sigma) &= x_i(\sigma_0)+A_{ijkl}\int_{\sigma_0}^{\sigma} p_l g_k^{(m)} g_j^{(m)}\, d\sigma \\ &= x_i(\sigma_0)+A_{ijkl}\int_{\sigma_0}^{\sigma} p_l D_{jk}^0/D^0\, d\sigma. \end{aligned} \tag{63}$$

For a general anisotropy, the integrand of the integrals (63) can be written as $P(\sigma)/Q(\sigma)$, where $P(\sigma)$ is a polynomial of the fifth order and $Q(\sigma)$ polynomial of the fourth order in $\sigma$. It is perhaps simpler to evaluate the integrals numerically than to seek analytical solutions in this case.

For various simpler anisotropies, these integrals can be solved analytically in a closed form. Let us consider a 2-D medium, with $A_2=0$. For $p_2(\sigma_0)=0$, we obtain $p_2(\sigma)=0$ along the whole ray. If we consider a transversely isotropic medium, the polynomial $P(\sigma)$ is of the third order and $Q(\sigma)$ of the second order in $\sigma$. The integrals can be simply evaluated analytically. We shall not present here relevant relations, but they are straightforward.

It is interesting to note that the simple solutions (49)–(56) of the ray tracing system have not been known even for isotropic media, with the exception of the case $n=2$ (see Červený 1987). In an isotropic medium, we have $G_m^0=p_ip_i$, $f=v$, and

$$\frac{1}{n}\frac{\partial}{\partial p_i}(G_m^0)^{n/2}=(G_m^0)^{(n/2)-1}p_i.$$

For $G_m^0$, we can write

$$G_m^0=p_ip_i=p_i(u_0)p_i(u_0)+\frac{2}{n}A_ip_i(u_0)(u-u_0)+\frac{1}{n^2}A_iA_i(u-u_0)^2.$$

Thus, $G_m^0$ is a polynomial of the second order in $u$. Then integrals (51) can be solved analytically for any $n$. For $n=2$, integrals (51) yield a polynomial of the second order for $x_i(\sigma)$ and a polynomial of the third order for $\tau(\sigma)$.

## 8 DYNAMIC RAY TRACING. RAY PROPAGATOR MATRICES

In this section, we shall derive the dynamic ray tracing equations for general anisotropic inhomogeneous media and for FAI media. For derivation, we shall consider the ray tracing systems written in a Hamiltonian form.

In the derivation, we shall use the Cartesian coordinates. In isotropic inhomogeneous media, particularly simple dynamic ray tracing equations are obtained if we use the ray-centred coordinate system. In anisotropic inhomogeneous media, the Cartesian coordinates seem to lead to simpler results.

We consider a central ray $\Omega$ specified by initial point $x_{i0}=x_i(u_0)$ and by initial slowness vector components $p_{i0}=p_i(u_0)$. The trajectory of the ray $\Omega$ is described by equations $x_i(u, x_{i0}, p_{i0})$ and the slowness vector at the point $x_i$ by $p_i(u, x_{i0}, p_{i0})$. Here $u$ is a variable along the ray, $x_i$ and $p_i$ are evaluated by ray tracing. Consider now a neighbouring ray, specified by initial conditions $x_{i0}+\delta x_{i0}$ and $p_{i0}+\delta p_{i0}$, where $\delta x_{i0}$ and $\delta p_{i0}$ are small. For $u$ fixed, the position and the slowness vector component on the neighbouring ray is $x_i+\delta x_i$ and $p_i+\delta p_i$. To determine $\delta x_i$ and $\delta p_i$, we can use the dynamic ray tracing system.

### 8.1 General anisotropic inhomogeneous medium

The ray tracing system for a general anisotropic inhomogeneous medium, written in a Hamiltonian form, is given by (23). It leads simply to the dynamic ray tracing system in the following form,

$$\begin{aligned} \frac{d\delta x_i}{\delta u} &= \frac{\partial^2\mathcal{H}}{\partial p_i\,\partial x_j}\delta x_j+\frac{\partial^2\mathcal{H}}{\partial p_i\,\partial p_j}\delta p_j, \\ \frac{d\delta p_i}{du} &= -\frac{\partial^2\mathcal{H}}{\partial x_i\,\partial x_j}\delta x_j-\frac{\partial^2\mathcal{H}}{\partial x_i\,\partial p_j}\delta p_j. \end{aligned} \tag{64}$$

The same dynamic ray tracing system is obtained even for $\partial x_i/\partial\gamma_J$ and $\partial p_i/\partial\gamma_J$, where $\gamma_J$ ($J=1, 2$) are ray parameters (see Červený 1972).

In a matrix form, Eq. (64) may be rewritten as follows,

$$\frac{d\mathbf{X}}{du}=\mathbf{SX}, \tag{65}$$

where

$$\mathbf{S}=\begin{pmatrix} \dfrac{\partial^2\mathcal{H}}{\partial p_i\,\partial x_j} & \dfrac{\partial^2\mathcal{H}}{\partial p_i\,\partial p_j} \\ -\dfrac{\partial^2\mathcal{H}}{\partial x_i\,\partial x_j} & -\dfrac{\partial^2\mathcal{H}}{\partial x_i\,\partial p_j} \end{pmatrix}, \qquad \mathbf{X}=\begin{pmatrix}\delta x_i\\ \delta p_i\end{pmatrix}. \tag{66}$$

We shall now introduce the $6\times 6$ propagator matrix of (65) which equals the $6\times 6$ identity matrix at the point $u=u_0$. We denote it $\boldsymbol{\pi}(u, u_0)$ and call it the ray propagator matrix. The ray propagator matrix $\boldsymbol{\pi}(u, u_0)$ is composed of six linearly independent solutions of (65). It is not difficult to show that the ray propagator matrix $\boldsymbol{\pi}(u, u_0)$ satisfies the following properties.

(i) It is symplectic, i.e.

$$\boldsymbol{\pi}^{\mathrm{T}}(u, u_0)\mathbf{J}\boldsymbol{\pi}(u, u_0)=\mathbf{J}, \qquad \mathbf{J}=\begin{pmatrix}\mathbf{0} & \mathbf{I}\\ -\mathbf{I} & \mathbf{0}\end{pmatrix}. \tag{67}$$

Here $\mathbf{0}$ and $\mathbf{I}$ are $3\times 3$ zero and identity matrices, respectively.

(ii) The determinant of the ray propagator matrix equals unity along the whole ray. This follows from (67).

(iii) The ray propagator matrix satisfies the chain property,

$$\boldsymbol{\pi}(u, u_0) = \boldsymbol{\pi}(u, u_1)\boldsymbol{\pi}(u_1, u_0), \tag{68}$$

where $u_1$ is an arbitrary point on the ray (situated even outside $u$, $u_0$).

(iv) The inverse of the ray propagator matrix may be simply computed from the known ray propagator matrix,

$$\boldsymbol{\pi}^{-1}(u, u_0) = \boldsymbol{\pi}(u_0, u) = \begin{pmatrix} \mathbf{P}_2^{\mathrm{T}}(u, u_0) & -\mathbf{Q}_2^{\mathrm{T}}(u, u_0) \\ -\mathbf{P}_1^{\mathrm{T}}(u, u_0) & \mathbf{Q}_1^{T}(u, u_0) \end{pmatrix}. \tag{69}$$

Here we have used a notation,

$$\boldsymbol{\pi}(u, u_0) = \begin{pmatrix} \mathbf{Q}_1(u, u_0) & \mathbf{Q}_2(u, u_0) \\ \mathbf{P}_1(u, u_0) & \mathbf{P}_2(u, u_0) \end{pmatrix}, \tag{70}$$

with $3 \times 3$ matrices $\mathbf{Q}_1$, $\mathbf{Q}_2$, $\mathbf{P}_1$, $\mathbf{P}_2$.

If we know the ray propagator matrix $\boldsymbol{\pi}(u, u_0)$, we can find the solution of the dynamic ray tracing system at $u$ for any initial conditions $\mathbf{X}(u_0)$ using the relation,

$$\mathbf{X}(u) = \boldsymbol{\pi}(u, u_0)\mathbf{X}(u_0). \tag{71}$$

Unfortunately, the solution of the dynamic ray tracing system in a general anisotropic inhomogeneous medium is very time-consuming since the matrix $\mathbf{S}$ in (65), given by (66), is very cumbersome and we must solve 36 equations to compute $\boldsymbol{\pi}(u, u_0)$. Analytical relations for the elements of $\mathbf{S}$ for the dynamic ray tracing system corresponding to the ray tracing system (24) with (26) were found by Červený (1972). Only recently were they used for computations by Gajewski & Pšenčík (1988) and by Kendall & Thompson (1989). The computations are straightforward, but very cumbersome. In a general case, matrix $\mathbf{S}$ contains 189 first and second spatial derivatives of density normalized elastic parameters. All these derivatives must be evaluated at each integration step. Not only the number of derivatives, but also the manipulations with them are very time-consuming and numerically inefficient. See also a discussion in Hanyga (1982a).

### 8.2 Dynamic ray tracing in the FAI media

Dynamic ray tracing system (65) with (66) simplifies drastically for the FAI medium. Let us first consider the general ray tracing system (29) (with $n$ arbitrary). Then the matrix $\mathbf{S}$ reads,

$$\mathbf{S} = \frac{1}{n}\begin{pmatrix} \mathbf{0} & \dfrac{\partial^2}{\partial p_i\, \partial p_j}(G_m^0)^{n/2} \\ \dfrac{\partial^2 f^{-n}}{\partial x_i\, \partial x_j} & \mathbf{0} \end{pmatrix}. \tag{72}$$

Even in this very general case, matrix $\mathbf{S}$ is really much simpler than (66). To evaluate $\mathbf{S}$, we need to determine only six second spatial derivatives of $f^{-n}$. The expression $\partial^2(G_m^0)^{n/2}/\partial p_i\, \partial p_j$ does not explicitly depend on $x_i$ at all, it only depends on $p_i$.

It may be useful to write the matrix $\mathbf{S}$ even for $n = 0$. Then the variable $u$ along the ray in (65) equals the travel time $\tau$ and matrix $\mathbf{S}$ reads, see (22d),

$$\mathbf{S} = \begin{pmatrix} \mathbf{0} & \dfrac{1}{2}\dfrac{\partial^2 \ln G_m^0}{\partial p_i\, \partial p_j} \\ -\dfrac{\partial^2 \ln f}{\partial x_i\, \partial x_j} & \mathbf{0} \end{pmatrix}. \tag{73}$$

Equation (72) offers even other simplifications. For a medium with a constant gradient of $f^{-n}$, we obtain

$$\mathbf{S} = \begin{pmatrix} \mathbf{0} & \dfrac{1}{n}\dfrac{\partial^2 (G_m^0)^{n/2}}{\partial p_i\, \partial p_j} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}. \tag{74}$$

The simplest matrix $\mathbf{S}$ is obtained for the FAI media with a constant gradient of the square of slowness, $f^{-2}$, when $n = 2$,

$$\mathbf{S} = \begin{pmatrix} \mathbf{0} & \dfrac{1}{2}\dfrac{\partial^2 G_m^0}{\partial p_i\, \partial p_j} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}. \tag{75}$$

For $\mathbf{S}$ given by (74) and (75), we can write analytical relations for the ray propagator matrices.

For a general $n$, we obtain from (74),

$$\boldsymbol{\pi}(u, u_0) = \begin{pmatrix} \mathbf{I} & \dfrac{1}{n}\displaystyle\int_{u_0}^{u} \frac{\partial^2}{\partial p_i\, \partial p_j}(G_m^0)^{n/2}\, du \\ \mathbf{0} & \mathbf{I} \end{pmatrix}, \tag{76}$$

and for $n = 2$,

$$\boldsymbol{\pi}(\sigma, \sigma_0) = \begin{pmatrix} \mathbf{I} & \dfrac{1}{2}\displaystyle\int_{\sigma_0}^{\sigma} \frac{\partial^2 G_m^0}{\partial p_i\, \partial p_j}\, d\sigma \\ \mathbf{0} & \mathbf{I} \end{pmatrix}. \tag{77}$$

All the properties of the ray propagator matrix listed in Section 8.1 remain valid even in FAI media.

## 9 CONCLUDING REMARKS

Even though the concept of the FAI medium has certain limitations, it can find many important applications both in the numerical modelling of seismic wave fields in complex structures and in the solution of inverse problems for anisotropic inhomogeneous media. In a way, it allows the anisotropy effects to be separated from the inhomogeneity effects and to be treated independently. It does not restrict the type of anisotropy and the type of inhomogeneity; both of them may be quite arbitrary. The concept of the FAI medium offers an effective and numerically efficient ray tracing, dynamic ray tracing and the travel time computations. All necessary equations are derived and discussed in this paper. Thus, FAI media can find useful applications particularly in the solution of inverse problems, as they decrease the number of the medium parameters and considerably increase the efficiency of computations.

The treatment of the FAI media presented in this paper is not complete; certain problems have not been treated here and will be discussed elsewhere. We shall list several of them.

We have discussed here only smooth FAI media, without interfaces. In the numerical modelling of seismic wavefields in realistic media, we must consider block and layered structures. To do this, we must solve the problems of (i) ray tracing across interfaces, (ii) dynamic ray tracing across interfaces, (iii) ray amplitudes across interfaces. The solution of these problems is known for general anisotropic inhomogeneous media and can be rewritten also for FAI media.

As well known, the dynamic ray tracing and the ray propagator matrices have found many important applications in the solution of direct and inverse problems in isotropic inhomogeneous layered and block structures, see Červený, Klimeš & Pšenčík (1988). It would be useful to solve corresponding problems also for general anisotropic inhomogeneous layered and block structures, particularly for the case of FAI media in individual layers and blocks.

In the solution of inverse problems for isotropic inhomogeneous layered and block structures, a very important role is played by ray perturbation equations (see, e.g. Farra & Madariaga 1987; Nowack & Lutter 1988; Nowack & Lyslo 1989). Equations for the travel time perturbations due to perturbations of the medium parameters are also known for general anisotropic inhomogeneous media (Červený 1982; Hanyga 1982b; Červený & Jech 1982, Červený & Firbas 1984). Recently, the travel time perturbation equations have been derived even for regions in which two eigenvalues are close to each other (shear wave singularities, etc.), see Jech & Pšenčík (1989). It would be very useful to write all perturbation equations (perturbations of travel times, rays, propagator matrices, ect.) for general FAI layered and block structures. The concept of FAI media will again greatly simplify these equations, in comparison with general anisotropic inhomogeneous media.

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