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First-order perturbation method for anisotropic media

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SUMMARY

The first-order perturbation method is used to evaluate approximate phase velocities and polarization vectors in elastic anisotropic media. Formulae are given which make possible computations of perturbations of these parameters for quasi-compressional as well as quasi-shear waves, no matter whether the unperturbed medium is isotropic or anisotropic. Approximate results for an extremely anisotropic material and relatively large deviations of parameters of unperturbed and perturbed media closely resemble the results computed exactly. It is, therefore, expected that the application of the perturbation method to realistic media with generally weaker anisotropy and for smaller deviations between unperturbed and perturbed medium parameters should give satisfactory results. The method will find the most important applications in the investigation of high-frequency wave propagation in inhomogeneous anisotropic media and in solving inverse problems for anisotropic structures. Several possible applications are listed and briefly discussed.

Key words: anisotropic structures, first-order perturbations, polarization vectors, ray method, wave surfaces

1 INTRODUCTION

Computations of synthetic seismic wavefields in laterally varying layered anisotropic structures, based on the ray method, are considerably more time consuming and complicated than similar computations for isotropic structures. This is mainly due to the following facts. As a rule, the computations for anisotropic structures must be performed in 3-D models, since 2-D rays are an exception in general inhomogeneous anisotropic structures. Instead of compressional and shear waves propagating in isotropic media, in anisotropic media we generally deal with three independent waves propagating along different paths, namely with two quasi-shear and one quasi-compressional wave. The two quasi-shear waves must always be considered together, otherwise the resulting synthetic wavefield would be distorted. This substantially increases the number of elementary waves to be considered in the computations of synthetic wavefields. Another cause of complexity of ray computations in anisotropic media is the fact that ray tracing and dynamic ray tracing equations for anisotropic media have much more complicated structures than those for isotropic media. Special treatment is required for quasi-shear waves in regions or directions in which they propagate with nearly the same phase velocity, which also complicates the computations.

It is, therefore, understandable that attempts have been made to obtain faster, but approximate ways of computation of ray synthetic wavefields in anisotropic media or, at least, of some of their parameters.

A possible way of solving wave propagation problems in anisotropic media by approximate methods is to use the perturbation theory. In its first approximation, it corresponds to linearization. Important linearized formulae for phase velocities in weakly anisotropic homogeneous media were derived by Backus (1965), see also Thomsen (1986). Červený (1982) and Hanyga (1982), following Romanov (1972), derived linearized formulae for travel times of seismic body waves propagating in slightly perturbed inhomogeneous anisotropic media. The perturbation of the travel time was obtained by integrating perturbations of the medium parameters along the ray in the unperturbed medium. The application of these formulae was connected with some complications in case the unperturbed medium was isotropic and the ray considered corresponded to the shear wave. Therefore, a linearized procedure starting from the isotropic medium was applied only to computing travel times of quasi-compressional waves or the sum of travel times of quasi-shear waves, see Červený & Jech (1982), Červený & Firbas (1984) and Firbas (1984).

Studying asymptotic electromagnetic wave propagation in inhomogeneous anisotropic media, Kravtsov & Orlov (1980) derived similar formulae for the computation of travel time (phase) perturbations which are applicable even in the case of an isotropic unperturbed medium.

In this paper, we use first order perturbations to solve approximately the Christoffel equation, controlling propagation of waves in anisotropic media. It is equivalent to expanding eigenvalues and eigenvectors into perturbation series due to the perturbation of the corresponding eigenvalue problem and keeping the leading terms of these series. This procedure is well-known in quantum mechanics (see, e.g. Landau & Lifschitz 1974). Recently a similar procedure was applied by Garmany (1988) to seismic wave propagation in stratified anisotropic media.

Application of perturbation methods to wave propagation problems in isotropic inhomogeneous media has a long history. First-order perturbations of travel times are routinely used in linearized kinematic inversions. Significant progress in the application of perturbation methods to the computation of complete ray wavefields in isotropic media was achieved by Farra & Madariaga (1987), see also Nowack & Lutter (1988).

As an unperturbed medium, any anisotropic medium can be used, including an isotropic medium. The use of an isotropic unperturbed medium is especially attractive, since efficient and reliable procedures for computations of ray synthetic wavefields in rather general 2-D or 3-D laterally varying layered isotropic structures have been developed in recent years.

The perturbation procedure is different for the case of an unperturbed medium in which the two quasi-shear waves propagate independently with different phase velocities (such a case is referred to here as non-degenerate) and for the case of the medium in which both quasi-shear phase velocities coincide (e.g. in singular directions in anisotropic media or everywhere in isotropic media—such a case is referred to here as degenerate).

In Section 2 we present formulae for the first-order perturbations of phase velocities and polarization vectors for both non-degenerate and degenerate cases. The effectivity of the perturbation procedure is demonstrated on examples, presented in Section 3. The section contains numerical examples of approximate computations of wave surfaces and polarization vectors. The quality of approximate results, obtained by the perturbation procedure, is checked by comparing them with the results of exact computations. In Section 4, a list of possible applications of the perturbation method in anisotropic media is given.

2 FIRST ORDER PERTURBATIONS OF THE CHRISTOFFEL EQUATION

The kinematics of wave propagation in inhomogeneous anisotropic media is controlled by the Christoffel equation (see, e.g. Červený 1972):

$$(\hat{\Gamma}_{jk} - G_m \delta_{jk}) g_j^{(m)} = 0, \tag{1}$$

where

$$\hat{\Gamma}_{jk} = a_{ijkl} p_i p_l, \qquad a_{ijkl} = c_{ijkl}/\rho, \qquad p_i = \partial\tau/\partial x_i. \tag{1'}$$

The Einstein summation convention applies over all repeated lower indices with the exception of m and n (no summation is performed over m and n!). The indices take values 1, 2 and 3. In (1) $c_{ijkl} = c_{ijkl}(x_n)$ are elastic parameters describing the properties of the medium, $\rho = \rho(x_n)$ is the density. The partial derivatives of the phase function $\tau = \tau(x_n)$ with respect to x_i are the components of the slowness vector p_i, a vector perpendicular to the wavefront. Symbol G_m in (1) ($m = 1, 2, 3$) denotes an eigenvalue (non-dimensional quantity) of the matrix $\hat{\Gamma}_{jk}$ which is subject to the condition $G_m = 1$, see Červený (1972). Symbol $g_j^{(m)}$ denotes the corresponding eigenvector, also called a polarization vector.

In the following, we shall consider (1) in a form which is more convenient for the purposes of this paper, namely

$$(\Gamma_{jk} - V_m^2 \delta_{jk}) g_j^{(m)} = 0, \tag{2}$$

with

$$\Gamma_{jk} = a_{ijkl} n_i n_l, \tag{2'}$$

n_i being a unit vector perpendicular to the wavefront, also called phase normal. The eigenvalues of Γ_{jk} are squares of phase velocity.

Let us denote the perturbed quantities by a tilde over the corresponding symbol. The perturbed matrix $\tilde{\Gamma}_{jk}$ with perturbed eigenvalues $\tilde{V}_m^2$ and eigenvectors $\tilde{g}_j^{(m)}$ can be expressed as a sum of the unperturbed matrix Γ_{jk} and first-order perturbation $\Delta\Gamma_{jk}$,

$$\tilde{\Gamma}_{jk} = \Gamma_{jk} + \Delta\Gamma_{jk}. \tag{3}$$

From (2′), we can see that the perturbation of the matrix Γ_{jk} can be caused either by perturbation Δa_{ijkl} of the medium parameters or by perturbation Δn_i of the phase normal. In the following, we mostly concentrate on the perturbation caused by Δa_{ijkl}. Therefore, we often refer to $\Delta\Gamma_{jk}$ as to the perturbation of the medium.

Our task is to find formulae expressing the perturbations ΔV_m and $\Delta g_j^{(m)}$ of the phase velocities and polarization vectors in terms of V_m and $g_j^{(m)}$ (corresponding to the unperturbed medium) and perturbations $\Delta\Gamma_{jk}$ of the medium. To do so, we proceed in the way used by Garmany (1988).

From the requirement that $\tilde{g}_j^{(m)}$ are unit vectors, we get

$$g_j^{(m)} \Delta g_j^{(m)} = 0. \tag{4}$$

For $\Delta g_j^{(1)}$, say, we can therefore write

$$\Delta g_j^{(1)} = \alpha g_j^{(2)} + \beta g_j^{(3)}. \tag{5}$$

In the perturbed medium, equation (2) can be rewritten as follows;

$$[\Gamma_{jk} + \Delta\Gamma_{jk} - (V_m + \Delta V_m)^2 \delta_{jk}](g_j^{(m)} + \Delta g_j^{(m)}) = 0, \tag{6}$$

which yields

$$(\Gamma_{jk} - V_m^2 \delta_{jk}) \Delta g_j^{(m)} + (\Delta\Gamma_{jk} - 2V_m \Delta V_m \delta_{jk}) g_j^{(m)} + (\Delta\Gamma_{jk} - 2V_m \Delta V_m \delta_{jk}) \Delta g_j^{(m)} = 0. \tag{7}$$

In (7), we omitted the term $(\Gamma_{jk} - V_m^2 \delta_{jk}) g_j^{(m)}$ which is, due to (2), equal to zero.

Multiplying (7) by $g_k^{(m)}$ and neglecting second order terms, we arrive at (no summation over m)

$$\Delta\Gamma_{jk} g_j^{(m)} g_k^{(m)} - 2V_m \Delta V_m = 0,$$

which yields

$$\Delta V_m = \tfrac{1}{2} V_m^{-1} B_{mm}. \tag{8}$$

In (8),

$$B_{mn} = \Delta\Gamma_{jk} g_j^{(m)} g_k^{(n)} \tag{8'}$$

is a symmetric 3×3 matrix.

Equations (8) give a first order perturbation of the phase velocity due to perturbation $\Delta\Gamma_{jk}$ of the non-degenerate matrix Γ_{jk} (which has all three eigenvalues different).

To find the perturbations of eigenvectors, let us specify

(7) for, say, $m=1$ and multiply it by $g_k^{(2)}$. Taking into account (5) and neglecting second order terms, we obtain

$$\alpha\Gamma_{jk}g_j^{(2)}g_k^{(2)} + \Delta\Gamma_{jk}g_j^{(1)}g_k^{(2)} - \alpha V_1^2\delta_{jk}g_j^{(2)}g_k^{(2)} = 0.$$

From this we simply get, assuming $V_1 \neq V_2$,

$$\alpha = (V_1^2 - V_1^2)^{-1}B_{12} \tag{9}$$

and similarly for β, assuming $V_1 \neq V_3$,

$$\beta = (V_1^2 - V_3^2)^{-1}B_{13}. \tag{10}$$

Generally, we can write

$$\Delta g_j^{(m)} = \sum_{n=1}^{3} c_{mn}g_j^{(n)} \tag{11}$$

with

$$\begin{aligned} c_{mn} &= (V_m^2 - V_n^2)^{-1}B_{mn} \quad \text{for } m \neq n \\ c_{mn} &= 0 \quad \text{for } m = n. \end{aligned} \tag{12}$$

It follows immediately from (11) and (12) that the condition of applicability of these formulae is

$$B_{mn} \ll |V_m^2 - V_n^2|. \tag{13}$$

Thus formulae (11) and (12) can be used for the determination of the first-order perturbations of polarization vectors for non-degenerate unperturbed media for which $V_m \neq V_n$ for all $m \neq n$. For unperturbed media in which $V_m = V_n$ for $m \neq n$, different formulae must be used.

In the case of, say, $V_1 = V_2$ the only specification of the corresponding polarization vectors $g_i^{(1)}$, $g_i^{(2)}$ is that they are situated in the plane perpendicular to $g_i^{(3)}$ and are mutually orthogonal. Therefore, we consider these vectors in the following form,

$$g_i^{(1)} = a_{11}e_i^{(1)} + a_{12}e_i^{(2)}, \qquad g_i^{(2)} = a_{21}e_i^{(1)} + a_{22}e_i^{(2)}. \tag{14}$$

The mutually perpendicular unit vectors $e_i^{(1)}$, $e_i^{(2)}$ are arbitrarily chosen in the plane perpendicular to $g_i^{(3)}$. The coefficients a_{ij} are to be determined. Since a_{ij} are directional cosines, they satisfy the following relations

$$a_{12} = -a_{21}, \qquad a_{11} = a_{22}. \tag{15}$$

Let us insert (14) into (7), specified for $m = 1$, and multiply the resulting equation successively by $e_k^{(1)}$ and $e_k^{(2)}$. We arrive at a system of two equations for a_{ij},

$$\begin{aligned} \bar{B}_{11}a_{11} + \bar{B}_{12}a_{12} - 2V_1\Delta V_1 a_{11} &= 0, \\ \bar{B}_{12}a_{11} + \bar{B}_{22}a_{12} - 2V_1\Delta V_1 a_{12} &= 0, \end{aligned} \tag{16}$$

in which

$$\bar{B}_{mn} = \Delta\Gamma_{jk}e_j^{(m)}e_k^{(n)}.$$

From the condition of solvability of (16),

$$\begin{bmatrix} \bar{B}_{11} - 2V_1\Delta V_1 & \bar{B}_{12} \\ \bar{B}_{12} & \bar{B}_{22} - 2V_1\Delta V_1 \end{bmatrix} = 0$$

we get

$$\Delta V_I = \frac{1}{4V_1}\{(\bar{B}_{11} + \bar{B}_{22}) \pm [(\bar{B}_{11} - \bar{B}_{22})^2 + 4\bar{B}_{12}^2]^{1/2}\}, \tag{17}$$

$I = 1, 2$. One of the two signs in (17) corresponds to ΔV_1, the other to ΔV_2 [we would arrive at the same condition leading to (17) if we inserted (14) into (7), specified for $m = 2$]. Equation (17) gives a first order perturbation of phase velocity due to perturbation $\Delta\Gamma_{jk}$ of the degenerate matrix Γ_{jk}. For $\bar{B}_{12} = 0$, we get

$$\Delta V_1 = \frac{\bar{B}_{11}}{2V_1}, \qquad \Delta V_2 = \frac{\bar{B}_{22}}{2V_1}. \tag{18}$$

For $\bar{B}_{12} = 0$ we also immediately get from (16) and (14)

$$g_i^{(1)} = e_i^{(1)}, \qquad g_i^{(2)} = e_i^{(2)}. \tag{19}$$

For $\bar{B}_{12} \neq 0$, we have from (16)

$$\begin{aligned} a_{11} &= \frac{\bar{B}_{12}}{|\bar{B}_{12}|}\left[\frac{1}{2}\left(1 + \frac{\bar{B}_{11} - \bar{B}_{22}}{D}\right)\right]^{1/2}, \\ a_{12} &= -\frac{\bar{B}_{12}}{|\bar{B}_{12}|}\left[\frac{1}{2}\left(1 - \frac{\bar{B}_{11} - \bar{B}_{22}}{D}\right)\right]^{1/2}, \end{aligned} \tag{20}$$

with

$$D = [(\bar{B}_{11} - \bar{B}_{22})^2 + 4\bar{B}_{12}^2]^{1/2}.$$

Inserting (20) and (15) into (14), we get the exact specification of polarization vectors in the plane perpendicular to $g_i^{(3)}$ in the unperturbed medium! Although the unperturbed medium is degenerate, the specification of the polarization vectors in it is unique. It is controlled by the perturbations of the matrix Γ_{jk}. Generally, for different $\Delta\Gamma_{jk}$, we get different $g_i^{(1)}$ and $g_i^{(2)}$. Landau & Lifschitz (1974) refer to this effect as that 'the perturbation removes degeneration'.

In order to determine first order perturbations of polarization vectors let us assume that the polarization vectors $g_i^{(m)}$, specified by (14) and (20), have been found and let us again seek perturbations in the form (5). Inserting (5) into (7) specified for $m = 1$ and multiplying the resulting equation by $g_k^{(3)}$, we get

$$\beta(V_3^2 - V_1^2) + B_{13} = 0,$$

from which

$$\beta = \frac{B_{13}}{V_1^2 - V_3^2}. \tag{21}$$

Multiplying the same equation as above but by $g_k^{(2)}$, we get

$$\alpha(B_{22} - 2V_1\Delta V_1) + \beta B_{23} = 0,$$

from which taking into account (21) we get

$$\alpha = \frac{B_{13}B_{23}}{2V_1(\Delta V_1 - \Delta V_2)(V_1^2 - V_3^2)}. \tag{22}$$

Deriving (22), we took into account that $B_{22} = 2V_1\Delta V_2$, see (8).

Generally, we can write

$$\Delta g_j^{(m)} = \sum_{n=1}^{3} c_{mn}g_j^{(n)} \tag{23}$$

with

$$\begin{aligned} c_{m3} &= \frac{B_{m3}}{V_m^2 - V_3^2} \quad \text{for } m \neq 3, \\ c_{mn} &= \frac{B_{m3}B_{n3}}{2V_m(\Delta V_m - \Delta V_n)(V_m^2 - V_3^2)} \quad \text{for } m, n \neq 3 \text{ and } m \neq n, \\ c_{mn} &= -c_{nm}, \qquad c_{mm} = 0. \end{aligned} \tag{24}$$

For B_{mn} see (8').

Let us note that it would be possible to derive, in a similar way as above, higher order perturbation terms for the situations discussed. They would have, of course, a more complex form than the first order terms presented above.

3 APPROXIMATE EVALUATION OF WAVE SURFACES AND POLARIZATION VECTORS

To test the perturbation formulae, given in Section 2, we use a homogeneous anisotropic material used by Shearer & Chapman (1989) to represent elastic properties at the surface of their earth's crust model. The effective anisotropy of the model is caused by aligned, vertical, water-filled cracks. The model is anisotropic with hexagonal symmetry, the axis of symmetry being horizontal. The density normalized elastic parameters for the model are: $a_{1111} = 19.63$, $a_{2222} = 20.16$, $a_{1212} = 3.48$, $a_{2323} = 6.38$, $a_{1122} = 7.26\ \mathrm{km^2\,s^{-2}}$. The model is extremely anisotropic with quasi-compressional anisotropy of about 9 per cent and quasi-shear anisotropy of about 30 per cent. We use such extremely anisotropic material on purpose to show the possibilities and limitations of the perturbation method. For detailed description of the model see Shearer & Chapman (1989).

For comparison, exactly determined phase velocities and polarization vectors were used. Analytic expressions for these quantities in the symmetry plane of an anisotropic medium with hexagonal symmetry were derived from formulae by Červený, Molotkov & Pšenčík (1977).

In Fig. 1, as a first example, we show intersections of phase velocity surfaces of quasi-compressional and two quasi-shear waves with the symmetry plane. An isotropic medium with P-wave velocity of $4.4\ \mathrm{km\,s^{-1}}$ and S-wave velocity of $2.2\ \mathrm{km\,s^{-1}}$ was considered as the unperturbed medium. It is shown in Fig. 1 by dashed lines. Perturbations which are denoted by dots were obtained from formula (17) for a degenerate unperturbed medium. Exactly determined phase velocity surfaces for the anisotropic material are shown by full lines. The fit of approximate and exact surfaces is very good. An interesting phenomenon is 'the split' of the single shear surface in the isotropic medium into two quasi-shear surfaces in the anisotropic medium. As can be seen from Fig. 1, the unperturbed model was chosen very close to the final anisotropic model. Tests have been made, which have shown that the parameters of the perturbed medium may be substantially different from the unperturbed ones, and still the performance of perturbation formulae is very good, if squares of the phase velocity are perturbed. If the phase velocity itself is perturbed, as in this paper, the results begin to depend more strongly on the proximity of the parameters of the perturbed and unperturbed media. Let us note that in applications of the perturbation method

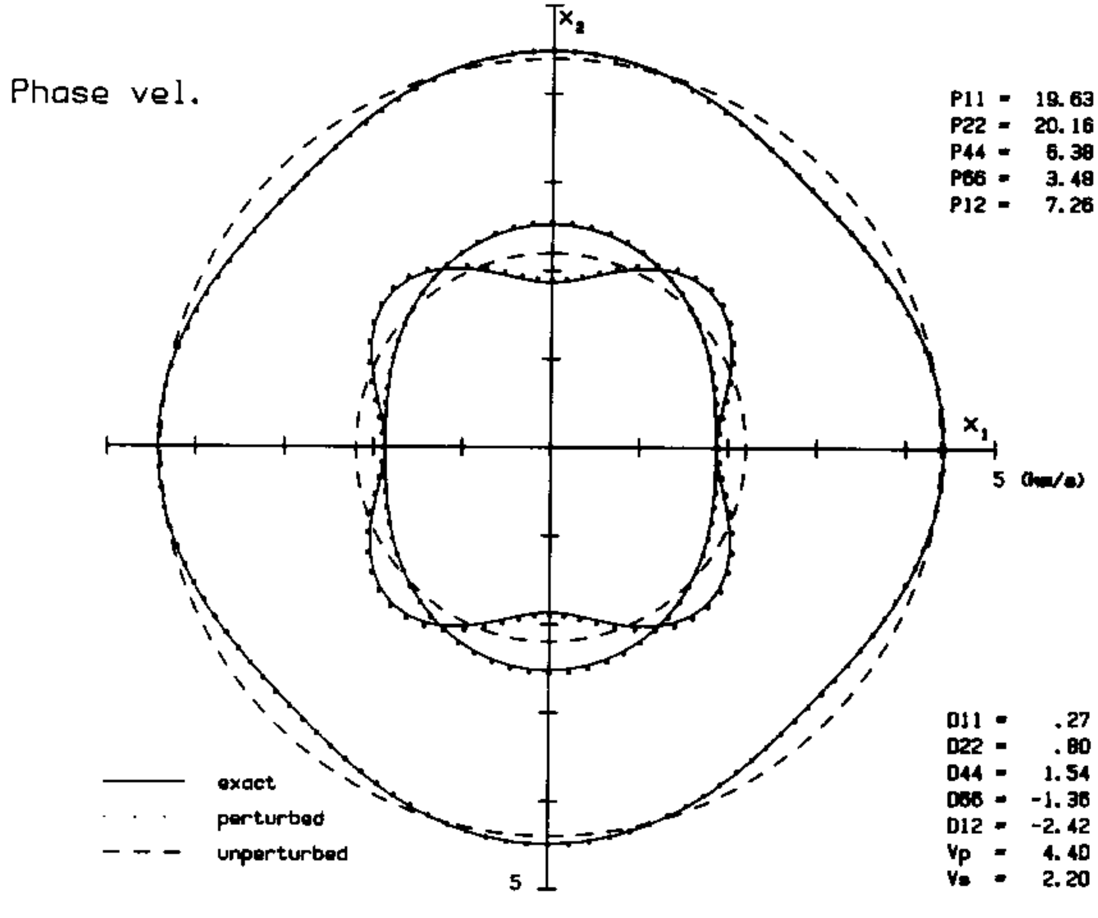

Figure 1. Intersections of phase velocity surfaces with the symmetry plane. Perturbation formula (17) is used. The parameters of the perturbed medium are expressed in compressed form: $P11 = a_{1111}$, $P22 = a_{2222}$, $P66 = a_{1212}$, $P44 = a_{2323}$, $P12 = a_{1122}$. The isotropic unperturbed medium is specified by V_P, V_S, D11, D22, . . . , denote the differences between parameters of the perturbed and the unperturbed medium.

to the approximate computation of rays and ray amplitudes, the perturbed quantity is the quantity G_m, see (1), which corresponds to the square of the phase velocity. Thus, in such applications, a good performance of the perturbation method should be expected.

In Fig. 2 we show how important it is to use equation (17) instead of (8) for a degenerate unperturbed medium. For the same situation as in Fig. 1, perturbation formulae (8) for a non-degenerate unperturbed medium were used with polarization vectors ill-specified in such a way that $g_i^{(1)}$ is 35° out of the symmetry plane. This is a completely different orientation of vectors $g_i^{(m)}$ than that which would be obtained if formulae (14) with (20) were used (which would yield $g_i^{(1)}$ in the symmetry plane and $g_i^{(2)}$ perpendicular to it). As expected, the picture is the same for the quasi-compressional wave for which the perturbation formulae hold universally (the quasi-compressional sheet does not coincide with another sheet). For quasi-shear waves, however, the approximate results significantly deviate from the exact ones.

Another experiment, the examples of which are not presented here, was to compute perturbations of phase velocity due to perturbations Δn_i of the phase normal for a given anisotropic medium. No perturbations of medium parameters were considered in this case. The results were not as satisfactory as the previous ones. Probably, higher order perturbations would be required, since the problem is strongly non-linear.

In Figs 3 and 4, the distribution of polarization vectors at intersections of the phase velocity surfaces of quasi-compressional and one quasi-shear wave (q*SP* in the notation of Shearer & Chapman 1989) with the symmetry plane is shown. In this case, an anisotropic medium (also of hexagonal symmetry) was used as an unperturbed medium. It was, therefore, possible to use perturbation formulae for a non-degenerate case, see (11) and (12). The perturbed medium has again the same parameters as in Fig. 1. For the identification of the parameters of the unperturbed medium matrix A, and for the perturbed medium matrix P are used in the figures. D again denotes the difference between both models. The origins of symbols for polarization vectors are plotted on phase velocity surfaces.

In Fig. 3, the parameters of the unperturbed medium were chosen quite 'far' from those of the perturbed medium, especially for quasi-shear waves. The differences are largest in horizontal and vertical directions, they decrease in between them. Differences in horizontal and vertical quasi-compressional velocities make 10.76 and 6.38 per cent, respectively. For quasi-shear velocities, they are the same for both directions and make 23.84 per cent. Note that Nowack & Lutter (1988) consider that maximum differences in perturbed parameters of an isotropic medium should not

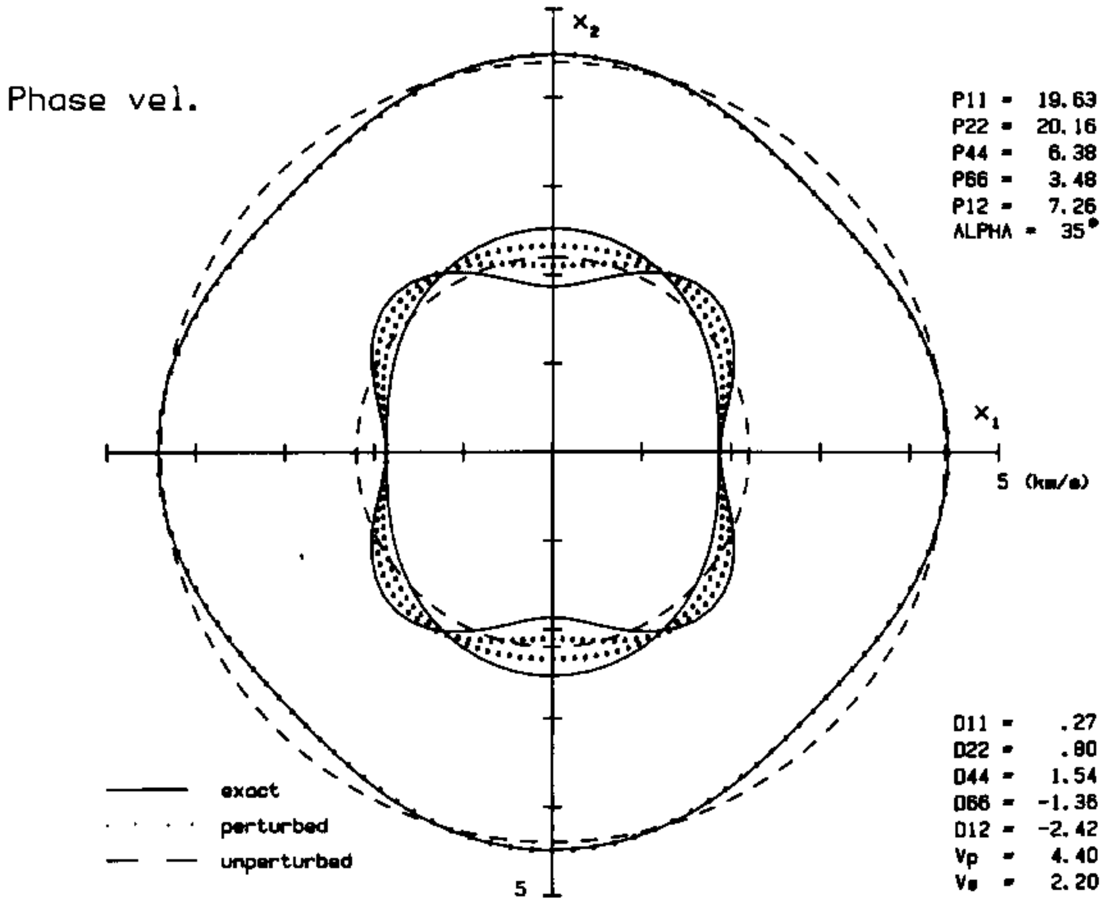

Figure 2. Intersections of phase velocity surfaces with the symmetry plane. Perturbation formulae (8) and (8') are used with a 'wrong' choice of polarization vectors $g_i^{(1)}$, $g_i^{(2)} - g_i^{(1)}$ is 35° out of the symmetry plane. Other parameters are the same as in Fig. 1.

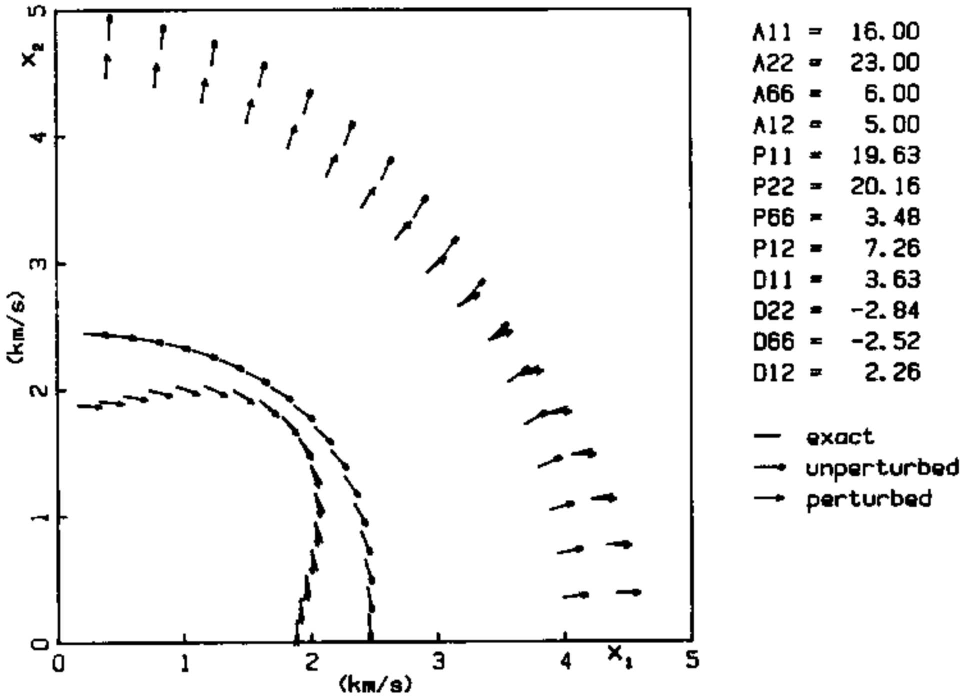

Figure 3. Polarization vectors at intersections of phase velocity surfaces of quasi-compressional wave and one quasi-shear (q*SP*) wave with the symmetry plane. The perturbation formulae (11) and (12) are used. The unperturbed anisotropic medium (of hexagonal symmetry) is specified by parameters A11, A22, . . . Other notations are the same as in Fig. 1.

exceed 8 per cent. Although the differences in Fig. 3 are large, the resemblance of perturbed and exact results is surprisingly good. Observable discrepancies are around the direction of about 35° from the horizontal axis. They are caused by different curvatures of the wave surfaces, corresponding to unperturbed and perturbed media.

In Fig. 4, the differences between the parameters of the unperturbed and perturbed medium are reduced, but the maximum difference for quasi-shear waves still amounts to 17.4 per cent. The perturbed polarization vectors of both waves, however, resemble the exactly computed polarization vectors very well.

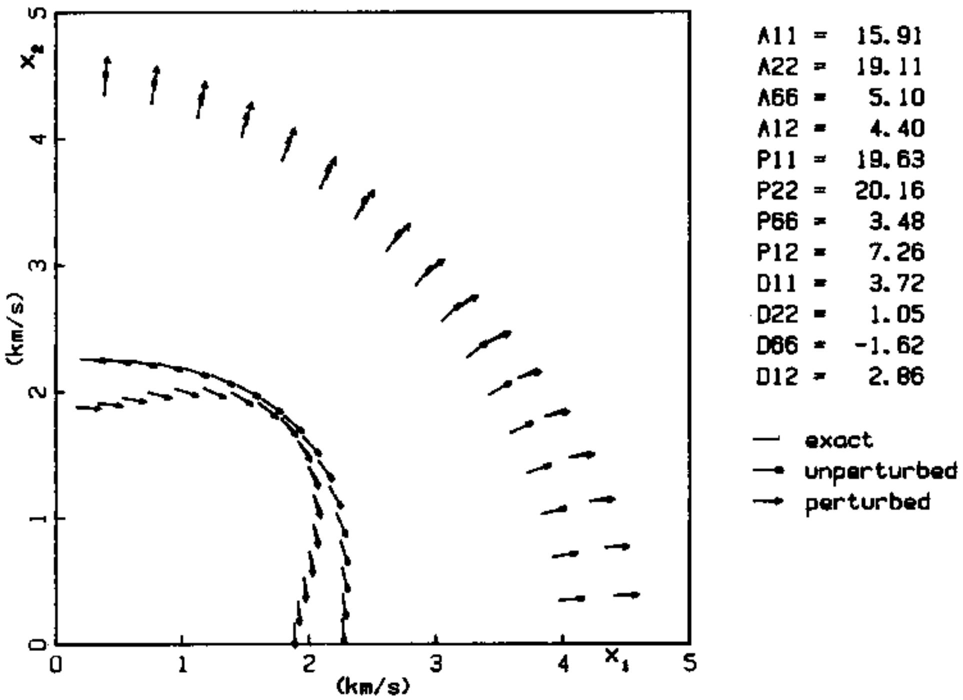

Figure 4. The same as Fig. 3, but for a different unperturbed medium.

4 POSSIBLE APPLICATIONS OF PERTURBATION FORMULAE

The numerical examples in the preceding section were chosen only as an illustration of the effectivity and reliability of perturbation formulae for anisotropic media. The method will find more important applications in the approximate evaluation of kinematic and dynamic parameters of seismic waves or, possibly, of complete synthetic wavefields, propagating in inhomogeneous anisotropic media. Let us mention some of the possible applications.

4.1 Approximate computation of travel times

For the non-degenerate case ($G_m \neq G_n$ for all $m \neq n$) the formula

$$\Delta\tau = -\frac{1}{2}\int_{\tau_0}^{\tau} \hat{B}_{mn}\, d\tau = -\frac{1}{2}\int_{\tau_0}^{\tau} \Delta a_{ijkl} p_i p_l g_j^{(m)} g_k^{(m)}\, d\tau \tag{25}$$

derived by Červený (1982) and Hanyga (1982) can be used. The quantities $\hat{B}_{mn}$ in (25) differ from B_{mn} in Section 2 by the substitution of n_i by p_i, *cf.* (2) and (1). In (25), τ denotes the travel time along the ray in the unperturbed medium. Integration of (25) along this ray path yields the correction $\Delta\tau$ of the travel time, caused by the perturbation of the medium Δa_{ijkl}. Applications of equation (25) can be found, for example, in Červený & Jech (1982) or Guiziou (1988). Equation (25) is a generalization of a popular formula for the perturbation of the travel time due to a perturbation of slowness in an isotropic medium. For layered media, equation (25) was generalized by Firbas (1984).

For the degenerate case, e.g. for the computation of travel times of quasi-shear waves using an isotropic unperturbed medium, the alternative formula to (25) has the form

$$\Delta\tau_I = -\frac{1}{4}\int_{\tau_0}^{\tau} \{(\hat{B}_{11} + \hat{B}_{22}) \pm [(\hat{B}_{11} - \hat{B}_{22})^2 + 4\hat{B}_{12}^2]^{1/2}\}\, d\tau, \tag{26}$$

$I = 1, 2,$ see the analogous formula for electromagnetic waves in Kravtsov & Orlov (1980). Here the quantities $\hat{B}_{mn}$ differ from $\bar{B}_{mn}$ in Section 2 by the substitution of n_i by p_i. From (26), we simply get expressions for other useful quantities:

$$|\tau_1 - \tau_2| = \frac{1}{2}\int_{\tau_0}^{\tau} [(\hat{B}_{11} - \hat{B}_{22})^2 + 4\hat{B}_{12}^2]^{1/2}\, d\tau, \tag{27}$$

$$\tau_1 + \tau_2 = -\frac{1}{2}\int_{\tau_0}^{\tau} (\hat{B}_{11} + \hat{B}_{22})\, d\tau. \tag{28}$$

Equation (28) is a well-known relation derived by Červený (1982), see also Červený & Jech (1982) and Červený & Firbas (1984).

As already mentioned by all the above referred authors, the formulae above are of considerable importance in solving inverse kinematic problems for anisotropic media. Examples of such applications can be found in Hirahara & Ishikawa (1984) and Jech (1988).

4.2 Effects of slight absorption

The ray perturbation method offers a simple way of evaluating absorbing effects of slightly dissipative media. This may be done by allowing a_{ijkl} to become complex,

$$\tilde{a}_{ijkl} = a_{ijkl} + \Delta a_{ijkl}, \tag{29}$$

with small imaginary part $\Delta a_{ijkl} = -iq_{ijkl}$. The global absorption factor τ^* is then given as follows,

$$\tau^* = \int_{\tau_0}^{\tau} q_{ijkl} p_i p_l g_j^{(m)} g_k^{(m)}\, d\tau \tag{30}$$

for any of the waves propagating in an anisotropic, slightly dissipative medium.

4.3 Approximate determination of the slowness vector of a single wave generated at an interface

The transformation of the slowness vector across an interface between two anisotropic media generally requires the solution of a sixth-order polynomial equation (see, e.g. Gajewski & Pšenčík 1987). As a result, we obtain slowness vectors of all generated waves on one side of the interface. With the use of perturbation formulae, this procedure can be substituted by a simple approximate formula for a single generated wave (as it is common in isotropic media). The slowness vector of the generated wave in the unperturbed medium can be expressed in the form

$$p_i = b_i + \xi N_i. \tag{31}$$

In (31), b_i is the vectorial component of the slowness vector in the plane tangential to the interface at the point of incidence. Symbol ξ denotes the component of the slowness vector on the normal N_i to the interface at the same point. The correction $\Delta\xi$ of the component ξ due to the perturbation of the medium Δa_{ijkl} can then be determined as follows:

$$\Delta\xi = \frac{-\hat{B}_{mm}}{2a_{ijkl} p_i N_l g_j^{(m)} g_k^{(m)}}. \tag{32}$$

To increase the accuracy of the determination of the slowness vector of the generated wave in a perturbed medium, formula (32) can be used iteratively. More details on this procedure will be given elsewhere.

4.4 Approximate determination of rays and ray amplitudes

The ray perturbation theory by Farra & Madariaga (1987) can easily be extended to anisotropic media. As an example, we present here ray tracing equations for a perturbed medium (in fact, they have the same form as for isotropic media, only the function G_m is different)

$$\begin{aligned} \frac{d\Delta x_i}{d\tau} &= \frac{1}{2}\left[\frac{\partial^2 G_m}{\partial p_i\, \partial x_k}\Delta x_k + \frac{\partial^2 G_m}{\partial p_i\, \partial p_k}\Delta p_k + \frac{\partial \Delta G_m}{\partial p_i}\right], \\ \frac{d\Delta p_i}{d\tau} &= -\frac{1}{2}\left[\frac{\partial^2 G_m}{\partial x_i\, \partial x_k}\Delta x_k + \frac{\partial^2 G_m}{\partial x_i\, \partial p_k}\Delta p_k + \frac{\partial \Delta G_m}{\partial x_i}\right]. \end{aligned} \tag{33}$$

Equations (33) are linear differential equations for the perturbations Δx_i, Δp_i of the ray path and slowness vectors due to the perturbation of parameters of the medium Δa_{ijkl}. Symbol ΔG_m denotes the perturbation of the eigenvalue G_m (see equation 1) due to Δa_{ijkl}. The integration of (33) is performed along the ray in the unperturbed medium. Note that omitting the terms with ΔG_m in each equation leads to standard dynamic ray tracing equations for the unperturbed medium. For more details see Farra & Madariaga (1987) and Nowack & Lutter (1988). System (33) can be used for an approximate, but fast determination of rays in the perturbed medium. For the evaluation of amplitudes, standard dynamic ray tracing equations can be solved in the perturbed medium along the approximate ray trajectories, as suggested by Nowack & Lyslo (1989). An alternative way is to write the dynamic ray tracing equations for perturbed quantities, as was done by Farra & Madariaga, and then to compute amplitudes approximately.

The perturbation formulae will doubtlessly find also important applications in modifications of the ray method for singular regions with coupled quasi-shear waves.

5 CONCLUSIONS

The numerical examples presented in Section 3 show the limitations and possibilities of ray perturbation formulae. It seems that for realistic anisotropic models in which anisotropy is expected to be weaker than in the examples presented, say up to 10 per cent, and in which the deviations of the parameters of the perturbed medium from the parameters of the unperturbed medium do not exceed 10 per cent, the perturbation method can give satisfactory results. It is necessary to keep in mind that the results for quasi-shear waves will be of lower quality in singular regions in which phase velocities of quasi-shear waves are close to each other.

The list of possible applications of perturbation formulae to computing synthetic seismic wavefields in inhomogeneous anisotropic media suggests that the perturbation method may become a useful tool for solving both direct and inverse problems of seismic wave propagation in inhomogeneous anisotropic media.

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