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Applications of dynamic ray tracing

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Dynamic ray tracing has recently found broad application in the numerical modelling of high-frequency seismic wave fields in complex 2-D and 3-D layered structures. It involves solving a system of ordinary differential equations (dynamic ray tracing system) along a ray. This paper briefly explains the main principles of dynamic ray tracing in ray centred coordinates, introduces the ray propagator matrix and describes its properties, summarizes the most important applications of dynamic ray tracing and of the ray propagator matrix. Together with known applications, several new applications are discussed in greater detail. Among these, a method to determine the geometrical spreading from the travel-time measurements is proposed.

1. Introduction

In the seismological investigation of the structure of the lithosphere and the asthenosphere, particularly in the investigation of their lateral inhomogeneities, an increasingly important role is played by the interpretational methods based on high-frequency approximations. Traditionally, the ray method has been broadly applied in these interpretations, both in the numerical modelling of seismic wave fields in complex laterally varying layered structures and in the solution of inverse problems. Recently, several extensions of the ray method have been proposed. In all of these extensions dynamic ray tracing is usually required as well as ray tracing. The procedure of dynamic ray tracing is relatively new and is not so well known as the standard kinematic ray tracing. This article briefly explains the main principles and applications of dynamic ray tracing. Although it has partially, but not fully, a review character, several applications are proposed here for the first time (see, e.g., section 3.17).

In standard ray tracing, the slowness vector $\mathbf{p}$ is determined at each point of the ray. It is well known that the covariant components of the slowness vector are given by the relationship $p_i = \partial\tau/\partial x^i$ ($i = 1, 2, 3$), where τ is the travel time and x^i are general coordinates. Thus, standard ray tracing yields the first partial derivatives of the travel time field with respect to general coordinates along the whole ray. This offers an opportunity to determine approximately the travel-time field not only along the ray, but also in its vicinity. The information obtained in this way corresponds to a plane wave front approximation.

In practical seismological applications, however, the wavefronts are curved as a rule, so that

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the possibility to extrapolate the travel times assuming a plane wavefront is limited to within a very close vicinity of the ray only. The extrapolation corresponds to the Taylor expansion of the travel-time field in the vicinity of the ray up to first-order terms.

If the second spatial derivatives of the travel-time field were known along the ray, in addition to the first derivatives, the extrapolation of the travel-time field from the ray to its vicinity would be considerably more accurate. It would correspond to the Taylor expansion of the travel-time field up to second-order terms.

Dynamic ray tracing is a procedure which can be used to evaluate the second derivatives of the travel-time field along the ray. It has, however, found many other important applications in seismic ray theory.

Dynamic ray tracing involves solving a system of ordinary differential equations along a ray and can be written in many forms and in various coordinate systems. It was first used by Belonosova et al. (1967) to calculate geometrical spreading in a 2-D laterally varying structure. A similar procedure to calculate the geometrical spreading in a 3-D laterally varying layered structure was proposed by Červený et al. (1974). An especially simple form of the dynamic ray tracing system is obtained in the ray centred coordinate system connected with the ray under consideration. The ray centred coordinate system was used by Popov and Pšenčík (1978a,b) in connection with the computation of geometrical spreading. These authors also proposed a very simple dynamic ray tracing system which is broadly used even at present. Červený and Hron (1980) suggested calling the above system of ordinary differential equations the dynamic ray tracing system and to call the procedure dynamic ray tracing. Many papers have been devoted to various aspects of dynamic ray tracing; for a long list of references see Červený (1985a). In this article we shall only give the more recent references: exceptionally, we shall include for completeness some of the important older references listed in Červený (1985a)

Although the first applications of the dynamic ray tracing system were devoted especially to the evaluation of geometrical spreading, it was soon found that dynamic ray tracing offers many other applications. The well-known applications include the computation of the second derivatives of the travel-time field, curvature of the wavefront, paraxial travel times, paraxial rays, etc. These and other applications are listed in the papers by Červený et al. (1984), Červený (1985a, b, 1987b,c). The number of possible applications is, however, increasing rapidly. In this paper we discuss known applications, together with several new ones.

2. Dynamic ray tracing in ray centred coordinates

2.1. The ray centred coordinate system: polarization vectors

In seismological terminology, dynamic ray tracing may be also interpreted as an approximate ray tracing of paraxial rays (rays situated close to a central ray). In dynamic ray tracing, it is useful to use a specific curvilinear orthogonal coordinate system connected with the ray, called the ray centred coordinate system. Let us consider an arbitrarily selected ray, Ω, specified by ray parameters, γ_1, γ_2. The ray centred coordinates q_1, q_2, q_3 connected with the ray Ω are defined in the following way (see Fig. 1). One coordinate, say q_3, corresponds to the arc length s along the ray Ω, measured from an arbitrary reference point $s = s_0$. (Instead of s, of course, we can also use other parameters along the ray.) The coordinates q_1, q_2 form a 2-D Cartesian coordinate system in the

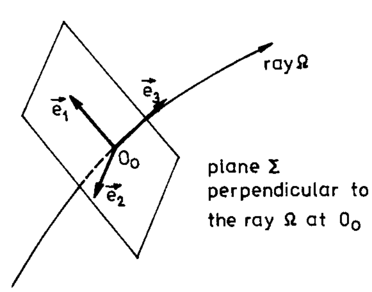

Fig. 1. Basis vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ of the ray centred coordinate system connected with the ray Ω. $\mathbf{e}_3$ is the unit vector tangent to the ray, $\mathbf{e}_1$ and $\mathbf{e}_2$ are perpendicular to the ray.

plane perpendicular to Ω at q_3, with its origin at Ω. The coordinates q_1, q_2 may be chosen in many ways; we choose them in such a way as to make the ray centred coordinate system orthogonal. This condition determines the ray centred coordinate system uniquely along the whole ray Ω, once it has been specified at any reference point of the ray. Thus, the ray Ω is the axis of the ray centred coordinate system. Paraxial means close to the axis, that is, close to the central ray Ω. Similarly, the paraxial ray approximation is an approximation in the vicinity of Ω, and the paraxial ray is a ray situated close to Ω. Note the difference between the ray centred coordinates q_1, q_2, q_3 and the ray coordinates γ_1, γ_2, γ_3. The ray centred coordinates are connected with a selected ray Ω while the ray coordinates are connected with the whole two-parametric system of rays. Each ray in the system is specified by different ray coordinates γ_1, γ_2 (also called ray parameters). The third ray coordinate, γ_3, is an arbitrary monotonic parameter along the ray, e.g. the arc length s.

The basis unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ of the ray centred coordinate system have a very interesting property. In smoothly varying inhomogeneous media, the polarization of the relevant seismic body wave propagating along the ray Ω remains fixed with respect to the basis vectors. The P wave is polarized linearly in the direction of the unit vector $\mathbf{e}_3 = \mathbf{t}$ tangent to the ray, one of the basis vectors. Similarly, the displacement vector of the S wave remains fixed with respect to the vectors $\mathbf{e}_1$, $\mathbf{e}_2$. The polarization of the S wave, however, may be more complicated; the S wave may be polarized elliptically. Even the polarization ellipse remains fixed with respect to $\mathbf{e}_1$, $\mathbf{e}_2$ in this case. Therefore, the basis vectors of the ray centred coordinate system $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3 = \mathbf{t}$ are often called the polarization vectors.

If the ray Ω crosses an interface, Σ, knowledge of the polarization vectors $\mathbf{e}_1$, $\mathbf{e}_2$ is not sufficient to determine the polarization of the reflected/transmitted S wave. The actual polarization of the S wave is influenced by the reflection/transmission coefficients of SV and SH wave types at the point of incidence on Σ. It is necessary to supplement the computation of $\mathbf{e}_1$, $\mathbf{e}_2$ by the computation of the vectorial complex-valued amplitudes to be able to study the polarization of the reflected/transmitted S wave. The concept of the basis vectors $\mathbf{e}_1$, $\mathbf{e}_2$ is useful, however, even in this case: they form a frame into which the vectorial complex-valued amplitude of the S wave is decomposed.

The polarization vectors $\mathbf{e}_1$, $\mathbf{e}_2$ can be determined by a solution of simple ordinary differential equations of the first order along the ray Ω. These equations in Cartesian coordinates can be found in Popov and Pšenčík (1978a,b). For general coordinates, see Červený et al. (1987a). The vector $\mathbf{e}_3$ can be determined from the ray tracing, $\mathbf{e}_3 = v\mathbf{p}$.

2.2. Dynamic ray tracing in ray centred coordinates

The dynamic ray tracing system in ray centred coordinates consists of four ordinary differential equations of the first order. These equations can be solved either along a known ray Ω, or together with the ray tracing of Ω. The dynamic ray tracing system is as follows

$$\frac{d\mathbf{W}}{ds} = v\,\mathbf{S}\mathbf{W}, \qquad \mathbf{S} = \begin{pmatrix} \mathbf{O} & \mathbf{I} \\ -v^{-3}\mathbf{V} & \mathbf{O} \end{pmatrix} \tag{1}$$

Here s is the arc length along the ray Ω, v is the relevant velocity and $\mathbf{V}$ is a 2×2 matrix of the second derivatives of the velocity with respect to the ray centred coordinates q_1, q_2

$$V_{IJ} = \left(\frac{\partial^2 v}{\partial q_I \partial q_J} \right)_{q_1 = q_2 = 0}, \qquad I, J = 1, 2 \tag{2}$$

$\mathbf{O}$ is a 2×2 null matrix and $\mathbf{I}$ the 2×2 identity matrix. Finally, $\mathbf{W}$ is a 4×1 column matrix. The elements of $\mathbf{W}$ are the ray centred coordinates q_1, q_2 of a paraxial ray, and the corresponding ray centred components of the slowness vector $p_I^{(q)} = \partial\tau/\partial q_I, I = 1,2$

$$\mathbf{W} = \left(q_1, q_2, p_1^{(q)}, p_2^{(q)}\right)^T \tag{3}$$

Alternatively, the dynamic ray tracing system can also be written in the following form

$$\frac{d\mathbf{X}}{ds} = v\mathbf{S}\mathbf{X}, \qquad \mathbf{X} = \begin{pmatrix} \mathbf{Q} \\ \mathbf{P} \end{pmatrix} \tag{4}$$

Here $\mathbf{Q}$ and $\mathbf{P}$ are 2×2 matrices defined by relations

$$Q_{IJ} = \left(\partial q_I / \partial \gamma_J\right)_{q_1 = q_2 = 0},$$
$$P_{IJ} = \left(\partial p_I^{(q)} / \partial \gamma_J\right)_{q_1 = q_2 = 0} = \left(\partial^2 \tau / \partial q_I \partial \gamma_J\right)_{q_1 = q_2 = 0} \tag{5}$$

Matrix **Q** measures the deviations of paraxial rays from the ray Ω and is also referred to as the matrix of geometrical spreading. The matrices **Q** and **P** can also be interpreted as transformation matrices: **Q** is a transformation matrix from the ray coordinates γ_1, γ_2 to the ray centred coordinates q_1, q_2, **P** is a transformation matrix from the ray coordinates γ_1, γ_2 to the phase space coordinates $p_1^{(q)}$, $p_2^{(q)}$. We can define the 2×2 matrix **M** of second derivatives of the travel-time field with respect to the ray centred coordinates q_1, q_2 by the relationship

$$M_{IJ} = \left(\partial^2 \tau / \partial q_I \partial q_J\right)_{q_1 = q_2 = 0} \tag{6}$$

Then we can write

$$\mathbf{M} = \mathbf{P}\mathbf{Q}^{-1} \tag{7}$$

2.3. Ray propagator matrix

We denote by $\boldsymbol{\pi}(s, s_0)$ the fundamental 4×4 matrix of linearly independent solutions of eq. 1, with the initial conditions

$$\boldsymbol{\pi}(s_0, s_0) = \mathbf{I} \tag{8}$$

where **I** is the 4×4 identity matrix. Such a fundamental matrix is called the ray propagator matrix.

Any solution of eq. 1 or eq. 4 can be then expressed in the following form

$$\mathbf{W}(s) = \boldsymbol{\pi}(s, s_0)\mathbf{W}(s_0), \quad \mathbf{X}(s) = \boldsymbol{\pi}(s, s_0)\mathbf{X}(s_0) \tag{9}$$

Here $\mathbf{W}(s_0)$ and $\mathbf{X}(s_0)$ specify relevant initial conditions.

The ray propagator matrix $\boldsymbol{\pi}(s, s_0)$ can also be written in the following form

$$\boldsymbol{\pi}(s, s_0) = [\mathbf{X}_1(s, s_0)\ \mathbf{X}_2(s, s_0)] = \begin{bmatrix} \mathbf{Q}_1(s, s_0) & \mathbf{Q}_2(s, s_0) \\ \mathbf{P}_1(s, s_0) & \mathbf{P}_2(s, s_0) \end{bmatrix} \tag{10}$$

Here $\mathbf{X}_1$, $\mathbf{X}_2$ are 4×2 matrices, $\mathbf{Q}_1$, $\mathbf{Q}_2$, $\mathbf{P}_1$, $\mathbf{P}_2$ are 2×2 matrices. It is not difficult to see that the first matrix solution, $\mathbf{X}_1 = \begin{pmatrix} \mathbf{Q}_1 \\ \mathbf{P}_1 \end{pmatrix}$, corresponds to the initially parallel ray field at $s = s_0$ (see eq. 8). It is also called the telescopic, or plane wavefront solution. The second solution, $\mathbf{X}_2 = \begin{pmatrix} \mathbf{Q}_2 \\ \mathbf{P}_2 \end{pmatrix}$ corresponds to the initially central ray field at $s = s_0$, and is also called the point source solution. Thus, any solution $\mathbf{X}(s) = \begin{pmatrix} \mathbf{Q}(s) \\ \mathbf{P}(s) \end{pmatrix}$ of the dynamic ray tracing system can be obtained as a linear combination of the telescopic and point source solutions.

The ray propagator matrix $\boldsymbol{\pi}(s, s_0)$ satisfies several important relationships along the ray Ω.

(1) The chain rule of the ray propagator matrix: if s' is an arbitrary point on the ray Ω, not necessarily between s_0 and s, we can write

$$\boldsymbol{\pi}(s, s_0) = \boldsymbol{\pi}(s, s')\boldsymbol{\pi}(s', s_0) \tag{11}$$

(2) The symplectic property of the ray propagator matrix $\boldsymbol{\pi}(s, s_0)$

$$\boldsymbol{\pi}^T \mathbf{J} \boldsymbol{\pi} = \mathbf{J}, \text{ with } \mathbf{J} = \begin{pmatrix} \mathbf{O} & \mathbf{I} \\ -\mathbf{I} & \mathbf{O} \end{pmatrix} \tag{12}$$

Here **I** is a 2×2 identity matrix, and **O** is a 2×2 null matrix.

(3) The determinant of the ray propagator matrix $\boldsymbol{\pi}(s, s_0)$ is constant along the ray Ω

$$\det \boldsymbol{\pi}(s, s_0) = \det \boldsymbol{\pi}(s_0, s_0) = 1 \tag{13}$$

(4) The inverse of the ray propagator matrix is given by the relation

$$\boldsymbol{\pi}(s_0, s) = \boldsymbol{\pi}^{-1}(s, s_0) = \begin{pmatrix} \mathbf{P}_2^T(s, s_0) & -\mathbf{Q}_2^T(s, s_0) \\ -\mathbf{P}_1^T(s, s_0) & \mathbf{Q}_1^T(s, s_0) \end{pmatrix} \tag{14}$$

Note that the properties (3) and (4) follow from eqs. 8 and 12.

Using the propagator matrix, we can also define the relative geometrical spreading at s, corresponding to a point source at $s = s_0$, by the relationship

$$J(s, s_0) = |\det \mathbf{Q}_2(s, s_0)| \tag{15}$$

The relative geometrical spreading is of great importance in the calculation of ray amplitudes (see

Section 3.2). It follows from eq. 14 that the relative geometrical spreading is reciprocal

$$J(s, s_0) = J(s_0, s) \qquad (16)$$

as $\det[-\mathbf{Q}_2^T(s, s_0)] = \det \mathbf{Q}_2(s, s_0)$.

The propagator matrix is discontinuous across an interface. All necessary relationships relevant to this case can be found in Červený (1985b), where most of the other relationships presented in this section are also derived. The properties of the propagator matrix listed above remain valid even for multiply reflected, possibly converted waves.

In addition to the references devoted to dynamic ray tracing and propagator matrices given in Červený (1985a,b), we present here several other references (Hanyga et al., 1984; Chapman, 1985; Thompson and Chapman, 1985; Cormier, 1986; Červený, 1987b, c; Farra and Madariaga, 1987).

3. Application of the dynamic ray tracing and ray propagator matrices

The dynamic ray tracing and the ray propagator matrices offer many important seismological applications. We describe several of them here. Some other applications of the ray propagator matrices and of the dynamic ray tracing are sure to be found in the future; they may be even more important than those dealt with below.

Let us consider a ray, Ω, from the point O_0 to the point O_s and assume that the slowness vectors at O_0 and O_s and the propagator matrix from O_0 to O_s are known. We denote the propagator matrix from O_0 to O_s by $\boldsymbol{\pi}(O_s, O_0)$. Then the following seismological applications are possible.

3.1. Transformation matrices Q and P

Matrix $\mathbf{Q}(O_s)$, also called the matrix of geometrical spreading, fully describes the geometry of the elementary ray tube at O_s. Matrix $\mathbf{P}(O_s)$ describes the behaviour of the slowness vectors in the close vicinity of the central ray Ω. Matrices $\mathbf{Q}(O_s)$ and $\mathbf{P}(O_s)$ are transformation matrices along the whole Ω if they are transformation matrices at any reference point of the ray, say O_0. The ray propagator matrix $\boldsymbol{\pi}(O_s, O_0)$ can then be used to determine $\mathbf{Q}(O_s)$, and $\mathbf{P}(O_s)$, see eqs. 9 and 10. More specifically, we obtain

$$\mathbf{Q}(O_s) = \mathbf{Q}_1(O_s,O_0)\mathbf{Q}(O_0) + \mathbf{Q}_2(O_s,O_0)\mathbf{P}(O_0)$$

$$\mathbf{P}(O_s) = \mathbf{P}_1(O_s,O_0)\mathbf{Q}(O_0) + \mathbf{P}_2(O_s,O_0)\mathbf{P}(O_0) \qquad (17)$$

3.2. Geometrical spreading

The geometrical spreading at O_s is usually defined by the relationship

$$J(O_s) = |\det \mathbf{Q}(O_s)|$$

see Červený (1985a). It measures the expansion and contraction of the ray tube at O_s and is of great importance in the computation of ray amplitudes. The ray amplitudes are inversely proportional to $J(O_s)^{1/2}$.

The determination of $J(O_s)$ from the ray propagator matrix and from $\mathbf{Q}(O_0)$ and $\mathbf{P}(O_0)$ at a given reference point, O_0, of the ray Ω is straightforward, see eq. 17. For a point source at O_0, we have $\mathbf{Q}(O_0) = \mathbf{O}$, so that

$$J(O_s) = |\det \mathbf{Q}_2(O_s,O_0) \det \mathbf{P}(O_0)|$$
$$= J(O_s,O_0)\,|\det \mathbf{P}(O_0)|$$

where $J(O_s,O_0)$ is the relative geometrical spreading.

It is not difficult to show that the relative geometrical spreading $J(O_s,O_0)$ does not depend on the local orientation of the polarization vectors $\mathbf{e}_1$, $\mathbf{e}_2$ in the plane perpendicular to Ω at O_0 and O_s.

3.3. Matrix of the second derivatives of the travel-time field

The second derivatives of the travel-time field are needed in many seismological applications. If we know $\mathbf{M}(O_s)$, the orientation of the polarization vectors $\mathbf{e}_1$, $\mathbf{e}_2$ and the gradient of velocity at O_s, we can also easily determine the 3×3 matrix of the second derivatives of the travel-time field with respect to general coordinates at O_s, with elements $\partial^2\tau/\partial x_i \partial x_j$. The continuation formula for $\mathbf{M}(O_s)$ along the ray Ω can be readily obtained from eqs. 7 and 17

$$\mathbf{M}(O_s) = [\mathbf{P}_1(O_s,O_0) + \mathbf{P}_2(O_s,O_0)\mathbf{M}(O_0)] \times [\mathbf{Q}_1(O_s,O_0) + \mathbf{Q}_2(O_s,O_0)\mathbf{M}(O_0)]^{-1} \quad (18)$$

If $\mathbf{Q}(O_0)$ and $\mathbf{P}(O_0)$ are known, eq. 18 can be rewritten as follows

$$\mathbf{M}(O_s) = [\mathbf{P}_1(O_s,O_0)Q(O_0) + \mathbf{P}_2(O_s,O_0)\mathbf{P}(O_0)] \times [Q_1(O_s,O_0)\mathbf{Q}(O_0) + \mathbf{Q}_2(O_s,O_0)\mathbf{P}(O_0)]^{-1} \quad (19)$$

Note that matrix $\mathbf{M}(O_s)$ can be determined from the travel-time measurements if the distribution of the travel time is known in the plane perpendicular to the ray Ω at O_s. To specify the plane, the slowness vector $\mathbf{p}(O_s)$ must be known.

Alternatively, the second derivatives of the travel-time field may be determined along another surface, Σ, passing through O_s, not necessarily along the plane perpendicular to Ω. To transform the second derivatives from the plane Σ to the plane perpendicular to Ω, the velocity, gradient of velocity and the curvature of Σ at O_s must be known. The relevant transformation equation can be found in Klimeš (1984), Červený (1985b), and Červený (1987c).

3.4. Curvature matrix of the wavefront

The 2×2 curvature matrix $\mathbf{K}(O_s)$ of the wavefront at O_s is given by the relation

$$\mathbf{K}(O_s) = v(O_s)\mathbf{M}(O_s) \quad (20)$$

The matrix $\mathbf{K}(O_s)$ yields a full geometry of the wavefront at O_s. The continuation of the curvature matrix $\mathbf{K}(O_s)$ along the ray Ω is given by the equation

$$\mathbf{K}(O_s) = v(O_s)[\mathbf{P}_1(O_s,O_0) + \mathbf{P}_2(O_s,O_0)v^{-1}(O_0)\mathbf{K}(O_0)] \times [\mathbf{Q}_1(O_s,O_0) + \mathbf{Q}_2(O_s,O_0)v^{-1}(O_0)\mathbf{K}(O_0)]^{-1} \quad (21)$$

Thus, if we know the geometry of the wavefront at O_0, we can determine the geometry of the wavefront at O_s, and vice versa.

3.5. Paraxial travel times

Let us consider a ray Ω, a point O_s situated on Ω and another point S situated close to O_s, possibly outside the ray Ω. If the distance between S and O_s is small, the travel-time field at S can be obtained by the Taylor expansion at O_s, up to second order terms. Remember that the first derivatives of the travel-time field at O_s are known from standard ray tracing, and the second derivatives can be easily determined from the ray propagator matrix $\pi(O_s, O_0)$.

The simplest relationships for the paraxial travel times are obtained in ray centred coordinates of points O_s and S. They can be simply written, however, even if the points O_s and S are specified in a general coordinate system.

3.6. Fresnel volumes and Fresnel zones

We consider here a central ray Ω and two points O_0 and O_s situated on the ray Ω. Let us assume that the point O_0 represents a point source and O_s the position of the receiver. Then the Fresnel volume corresponding to the two points O_0 and O_s represents a vicinity of the central ray Ω which actually contributes to the wave field at

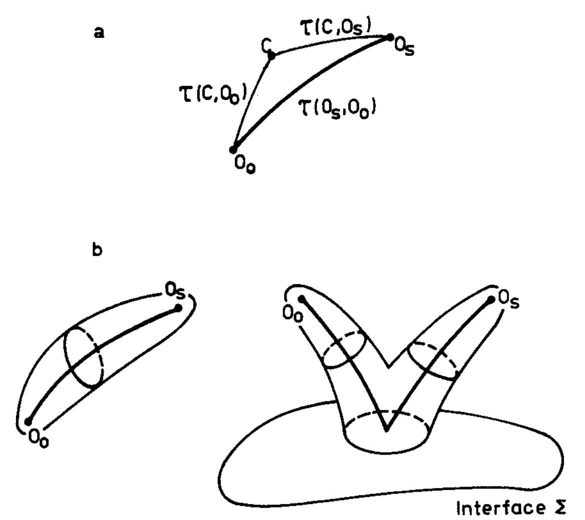

Fig. 2. Fresnel volumes. (a) Point C satisfying the condition of eq. 22 form the Fresnel volume corresponding to points O_0 and O_s. (b) Fresnel volumes for a direct wave and for the wave reflected from an interface Σ.

O_s. The Fresnel volume is sometimes called the physical ray, see Kravtsov and Orlov (1980).

Let us denote the travel time at a point C due to a point source at A by $\tau(C,A)$. Then the Fresnel volume corresponding to the two points O_0, O_s is composed of point C, for which the following condition is valid

$$|\tau(C, O_0) + \tau(C, O_s) - \tau(O_s, O_0)| < \tfrac{1}{2}T \qquad (22)$$

where T is the period of the wave under consideration. See Fig. 2a. As $\tau(C, A) = \tau(A, C)$, the same Fresnel volume corresponds to the receiver position at O_0 and to the point source at O_s.

Two examples of Fresnel volumes, corresponding to points O_0 and O_s, for a direct wave and for a wave reflected from an interface Σ, are shown in Fig. 2b.

The Fresnel volumes can be easily evaluated by some algebraic manipulations with the ray propagator matrices, for any type of elementary wave (including multiply reflected and converted waves) propagating in a 3-D laterally varying medium. For details see Červený (1987c).

3.7. *Boundary-value ray tracing problems for paraxial rays*

Any boundary-value ray tracing problem for paraxial rays can be solved analytically if the ray propagator matrix is known along the ray Ω. This includes the two-point ray tracing, the ray tracing of normal rays (ray from O_0 perpendicular to a given surface Σ), etc. The solution is only approximate, but it can be repeated iteratively. For details and relevant equations see Červený et al. (1984), and Červený (1987c).

3.8. *Finite source computations*

In the numerical modelling of high-frequency radiation from earthquake sources of a finite extent, it is usually necessary to evaluate the travel times from any point on the rupture surface Σ to the receiver O_0. For a not too large surface Σ, it is often sufficient to shoot just one ray Ω from O_0 to O_s, where O_s is situated on the rupture surface Σ or close to it. If the ray propagator matrix $\pi(O_s, O_0)$ is determined along Ω, we can evaluate the travel time from O_0 to any point S of the surface Σ analytically, without additional ray tracing. If the rupture surface Σ is larger, it is, of course, necessary to shoot other additional rays from O_0. For details see Červený et al. (1987b). A similar procedure can be applied even for the evaluation of Kirchhoff integrals of various types.

3.9. *Diffraction problems*

For certain diffraction problems, it is useful to know the analytic continuation of the travel-time field from the illuminated region to the shadow zone. For example, such an analytic continuation is needed in the computation of edge waves by the Klem-Musatov and Aizenberg (1984) method. If we apply dynamic ray tracing and determine the ray propagator matrix along the boundary ray (or along some other ray close to the boundary ray), the analytic continuation is obtained immediately by means of the paraxial travel-time equations, see section 3.5.

3.10. *Phase shift due to caustics*

To evaluate the complex-valued amplitudes of the displacement vector, we must know the phase shift due to caustics $\delta\tau(O_0, O_s)$ along Ω between O_0 and O_s. The phase shift $\delta\tau(O_0, O_s)$ can be expressed in terms of the index of the ray trajectory between O_0 and O_s, $k(O_0, O_s)$, as follows: $\delta\tau(O_0,O_s) = -\frac{1}{2}\pi k(O_0, O_s)$. Note that the index of the ray trajectory $k(O_0, O_s)$, also known as the KMAH index, is defined as the sum of the caustic points along Ω between O_0 and O_s; caustic points of the second order (point caustics) being considered twice in this sum. The index of the ray trajectory $k(O_0, O_s)$ can be determined by dynamic ray tracing, following changes in the matrix $\mathbf{Q}$ along the ray. Specifically, at the caustic point of the first order (line caustic), $\det \mathbf{Q} = 0$, $\mathbf{Q} \neq \mathbf{O}$. At the caustic point of the second order, $\mathbf{Q} = \mathbf{O}$.

3.11. *Paraxial displacement vector*

Using the ray propagator matrix, we can approximately determine not only the paraxial travel times, but also the paraxial displacement vector at a point S outside the ray Ω. Assume that we know the ray from O_0 to O_s, the ray propagator matrix $\pi(O_s, O_0)$ and the displacement vector at O_s. Then we can determine approximately the displacement vector at a point S close to O_s, and we call it the paraxial ray approximation. The paraxial ray approximation plays an important role in many problems of great seismological significance. For example, if we use the paraxial ray approximation formulae, we do not need to perform the two-point ray tracing. It is sufficient to evaluate the displacement vector along rays passing in the vicinity of the point S of interest, not necessarily through S.

3.12. Gaussian beams

Dynamic ray tracing is the basic procedure in Gaussian beam computation. In the expressions for Gaussian beams, however, complex-valued 2×2 matrices **M** and **Q** must be used instead of the real-valued **M** and **Q** used in the standard ray method. Otherwise both the expressions are the same. If $Im\ \mathbf{M}(s)$ is positive definite, the amplitude profile of the solution in the plane perpendicular to the ray Ω is Gaussian, with its maximum on the ray. In this sense, the solution represents 'a solution concentrated close to the ray Ω', also called the Gaussian beam. To evaluate the complex-valued **M** and **Q**, we must know, in general, the complete ray propagator matrix. As soon as $Im\ \mathbf{M}(s)$ is positive definite at one point of the ray Ω, it is positive definite along the whole ray Ω. (Note that the paraxial ray approximation represents infinitely broad Gaussian beams.)

3.13. Summation of Gaussian beams

The high-frequency seismic wave field generated either by a point source or by a source of a finite extent may be approximately expanded into Gaussian beams. As the computation of Gaussian beams requires the knowledge of the ray propagator matrix, these matrices must be determined along all rays used in the summation. For details regarding the superposition formulae see Klimeš (1984) and Červený (1985b, c).

In the superposition of Gaussian beams, we can also use infinitely broad Gaussian beams. Then, we express the wave field as a superposition of paraxial ray approximations. The knowledge of complete ray propagator matrices is again useful along all rays used in the summation. The summation equations obtained in this way are close to those obtained by the Maslov method, see Chapman and Drummond (1982).

3.14. High-frequency elastodynamic Green's function

The components of the Green's function $G_{mn}(O_s, t; O_0, t_0)$, $m,n = 1,2,3$ are defined in the well-known way: $G_{mn}(O_s, t; O_0, t_0)$ is the mth Cartesian component of the displacement vector at O_s at time t, caused by the application of a single force unit impulse in the direction of the nth Cartesian axis at the point O_0 and time t_0.

In the ray theory, the complete wave field at O_s due to a point source at O_0 is composed of contributions which travel from O_0 to O_s along various ray trajectories Ω. To evaluate the individual contributions corresponding to different elementary seismic body waves, we must determine the relevant geometrical spreading. Thus, dynamic ray tracing is required in the computation of the ray theoretical elastodynamic Green's function. See Červený et al. (1987b) and Červený (1987a,c).

Assume now that the rays Ω pass through O_0 and $O_{\bar{s}}$ and we wish to determine the Green's function $G_{mn}(S, t; O_0, t_0)$; S being situated close to $O_{\bar{s}}$. Then we can use the paraxial ray approximation for individual elementary waves and obtain the paraxial ray approximation elastodynamic Green's function. Thus, the ray propagator matrix must be known for each elementary wave.

The elastodynamic Green's function can also be computed by a summation of Gaussian beams or by a summation of paraxial ray approximations. Both these methods, of course, require the evaluation of the ray propagator matrices along all rays of all elementary waves used in the summation.

All the relevant equations can be found in Červený et al. (1987b) and Červený (1987a,c).

3.15. Inverse problems: location of hypocentres

The dynamic ray tracing and ray propagator matrices are sure to find broad applications in seismic inverse problems. For example, the solution of the structural inverse problem usually requires two-point ray tracing. Instead of standard two-point ray tracing algorithms we can use, considerably more efficiently, two-point ray tracing for paraxial rays wherever possible, see Section 3.7. This will lead to a numerically more efficient algorithm. Similarly, by using the ray propagator matrices, it is not difficult to find the first and second derivatives of the travel-time field at a fixed receiver point O_s with respect to the coordinates of the source O_0, for any type of a 3-D laterally varying layered structure and for any type of wave. Such expressions may be used directly in location procedures.

Klimeš (1987) has proposed the following algorithm for a kinematic hypocentre location in general 3-D structures: a sufficiently dense system of rays leaving given receivers is computed. The results of the kinematic and dynamic ray tracing at a sufficiently dense system of points along each computed ray are stored in a file. The travel times from any point of the model to the receivers may then be obtained by an interpolation in the irregular net of points stored in the file. The Gaussian packet summation method is used to yield the interpolation and the smoothing of the travel times. In this way, the computation of the travel times at a set of points does not much increase the computational time in comparison with a single-point computation. The complete solution of the inverse problem (i.e. the whole probability density function of hypocentral coordinates and time) may be obtained by the method described by Tarantola and Valette (1982).

3.16. Ray perturbation theory

Let us consider a reference medium with a given velocity distribution, and another medium in which the velocity distribution is only slightly different from that in the reference medium. Consider a (reference) ray Ω connecting points O_0 and O_s in the unperturbed reference medium. We can perform dynamic ray tracing along Ω in the reference medium, supplemented by certain numerical quadratures along Ω, which take into account the perturbation of the medium. This yields a paraxial approximation of the perturbed ray with respect to the reference ray. Moreover, it also yields the perturbation of the propagator matrix (i.e. the difference between the propagator matrix along a perturbed ray in a perturbed medium and the propagator matrix along a reference ray in a reference medium).

In this way, this type of dynamic ray tracing gives us an opportunity to solve effectively both the direct and inverse problems for the perturbed medium. An especially efficient procedure would be the combination of the perturbation theory with the Gaussian beams. For more details and for numerical examples see Farra and Madariaga (1987).

The application of dynamic ray tracing in ray perturbation theory is extremely promising and this approach is sure to be used in future in many important seismological problems.

3.17. Determination of spreading-free amplitudes

In certain seismological applications, e.g. in studies of the absorption of seismic body waves, in true amplitude studies in seismic prospecting for oil, etc., it is very useful to exclude the relative geometrical spreading factor $J(O_s, O_0) = |\det \mathbf{Q}_2(O_s, O_0)|$ from the measured amplitudes of seismic body waves. The problem whether the relative geometrical spreading can be determined from purely kinematic measurements at O_0 and O_s, without any knowledge of the structure and of the actual type of the body wave, is one of the basic problems in ray theory and is broadly discussed in the seismological literature. Here, we present quite a general solution of this problem. For more details see Červený (1987c).

Let us consider an arbitrary multiply reflected, possibly converted, seismic body wave propagating in a general 3-D laterally varying layered structure. We perform the three following experiments (see Fig. 3).

(1) For a point source at O_0, we determine the matrix of the second derivatives of the travel-time field at O_s, $\mathbf{M}(O_s, O_0)$, from travel-time measurements at O_s.

(2) For a point source at O_s, we determine the matrix of the second derivatives of the travel-time

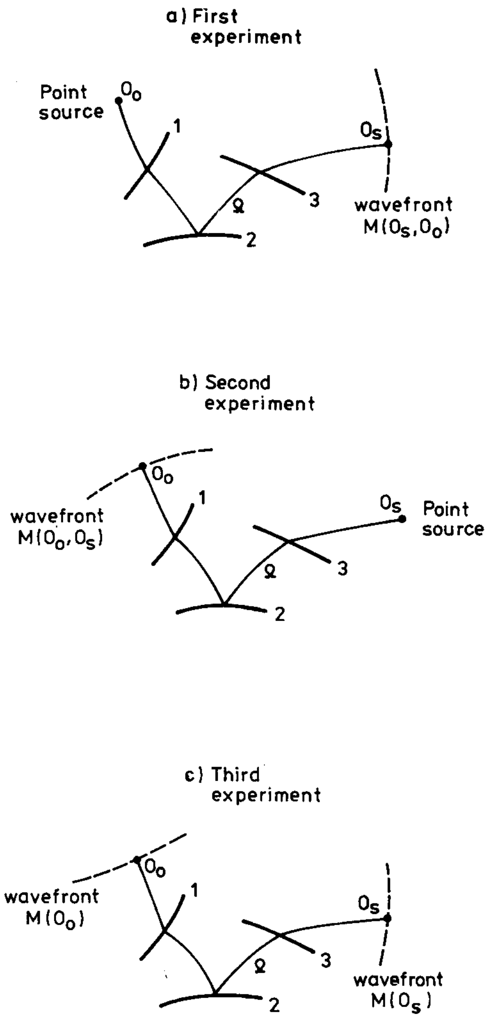

Fig. 3. Three experiments to determine the relative geometrical spreading. First experiment: point source at O_0, $\mathbf{M}$ is measured at O_s (denoted by $\mathbf{M}(O_s, O_0)$), Second experiment: point source at O_s, $\mathbf{M}$ is measured at O_0 (denoted by $\mathbf{M}(O_0, O_s)$). Third experiment: a non-point source at O_0, with $\mathbf{M}(O_0) \neq \infty$, $\mathbf{M}$ is measured at O_s (denoted by $\mathbf{M}(O_s)$). For coinciding O_0 and O_s, only two experiments are required. Numbers 1, 2, and 3 denote interfaces.

field at O_0, $\mathbf{M}(O_0, O_s)$, from the travel-time measurements at O_0.

(3) In the third experiment, the source may be situated either at O_0 or at O_s, but it must be different from a point source (i.e. the wavefront generated by the source must be sufficiently different from the wavefront generated by a point source). We can consider, e.g., a plane wavefront source.

Such a source, of course, may be simulated by an array of point sources. Assume that such a source is situated at O_0. Denote the matrix of the second derivatives of the travel-time field corresponding to this source at O_0 by $\mathbf{M}(O_0)$, with det $\mathbf{M}(O_0) \neq 0$, and assume that $\mathbf{M}(O_0)$ is known. Then we determine the matrix of the second derivatives of the travel-time field at O_s, $\mathbf{M}(O_s)$, from the travel-time measurements at O_s.

Thus, from the above three experiments, we know four matrices: $\mathbf{M}(O_s, O_0)$, $\mathbf{M}(O_0, O_s)$, $\mathbf{M}(O_0)$ and $\mathbf{M}(O_s)$. The relative geometrical spreading $J(O_s, O_0)$ is then given by the relationship

$$J^2(O_s, O_0) = [\det \mathbf{Q}_2(O_s, O_0)]^2 = \{\det[\mathbf{M}(O_s) - \mathbf{M}(O_s, O_0)] \times \det[\mathbf{M}(O_0) - \mathbf{M}(O_0, O_s)]\}^{-1} \quad (23)$$

The derivation of this equation can be found in Červený (1987c). Some special cases of the above equation are known from the literature, see e.g. Hubral (1983).

Equation 23 is invariant with respect to the local orientation of the unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$ at O_0 and O_s.

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