Geophys. J. R. astr. Soc. (1983) **72**, 417–433

Gaussian beams in two-dimensional elastic inhomogeneous media

V. Červený *Institute of Geophysics, Charles University, Ke Karlovu 3, 121 16 Praha 2, Czechoslovakia*
I. Pšenčík *Geophysical Institute, Czechoslovakian Academy of Sciences, Boční II, 141 31 Praha 4, Czechoslovakia*

Received 1982 July 16; in original form 1982 April 29

Summary. Asymptotic high-frequency solutions of elastodynamic equations in two-dimensional laterally inhomogeneous media concentrated close to rays of P- and S-waves are investigated. From a physical point of view, these vectorial solutions correspond to Gaussian beams; the amplitude distribution of their principal components is bell-shaped along the direction perpendicular to the ray. The principal component of the elastodynamic Gaussian beam is controlled by the parabolic equation, which has exactly the same form as the parabolic equation for scalar Gaussian beams. The elastodynamic Gaussian beams are regular everywhere, including caustics.

1 Introduction

Gaussian beams have found many applications in various wave propagation problems. Among others, they provide the possibility of computing the wavefields generated by point sources in complex laterally inhomogeneous structures. As in the ray method, the whole wavefield is decomposed into contributions corresponding to individual rays. These contributions, however, represent Gaussian beams; the distribution of their amplitudes along the phase front is bell-shaped. Contrary to ray solutions, the Gaussian beams are regular everywhere, even at caustics. Consequently, the final expressions for the wavefield are uniformly valid. For a detailed treatment of the application of Gaussian beams to the solution of the scalar wave equation see Červený, Popov & Pšenčík (1982), where many other references are also given. In this paper, the above-mentioned Gaussian beams will be called scalar Gaussian beams.

In seismology, however, we are interested in the solution of the vectorial elastodynamic equation, not in the solution of a scalar wave equation. In applying the procedure of Gaussian beams to the evaluation of seismic wavefields and computation of synthetic seismograms in complex laterally inhomogeneous structures (with curved interfaces and blocks), we must solve several problems: (a) we must derive Gaussian beams as time-harmonic solutions of the vectorial elastodynamic equation concentrated close to the rays of P- and S-waves for smooth elastic inhomogeneous media – elastodynamic Gaussian beams; (b) we must generalize the expressions for Gaussian beams for layered media, containing curved interfaces; (c) we must find expansions of various types of waves, such as plane waves or waves generated by a line or a point source, into elastodynamic Gaussian beams; (d) we must generalize the above time-harmonic Gaussian beams to evaluate synthetic seismograms. (In the time domain, the Gaussian beams can be replaced, e.g. by Gaussian wave packets.)

In this paper, we shall concentrate our attention on the first problem, i.e. on elastodynamic Gaussian beams in a smooth infinite inhomogeneous elastic medium without interfaces. As many problems in present theoretical seismology are solved in two dimensions, we shall also consider only two-dimensional media, where all the properties of Gaussian beams are considerably simpler than in three-dimensional media. The other problems listed above will be solved elsewhere.

Note that the problem listed first was also solved in a 3-D medium by Kirpichnikova (1971). Kirpichnikova was interested in the problem mainly from a mathematical point of view; in our investigations, we shall emphasize the seismological aspects and relations to the ray method. To begin with, we shall formulate the problem in an orthogonal curvilinear coordinate system, more commonly used in seismology (see Aki & Richards 1980) and not in the general non-orthogonal curvilinear system, used by Kirpichnikova. From this point of view, we believe that our presentation will be more familiar to seismologists. We shall also correct some mistakes in Kirpichnikova's paper (e.g. one term missing in the parabolic equation), which are not of principal importance from the methodological point of view but, unfortunately, lead to incorrect final results.

Several results derived in this paper were presented earlier in Červený (1981a, b), without a detailed derivation. The latter reference includes examples of synthetic seismograms for simple models of laterally varying layered structures computed by the Gaussian wave packet approach. At present, the computer program to calculate synthetic seimograms for general 2-D laterally varying structures with curved interfaces, block structures and isolated bodies, based on the Gaussian wave packet approach, is available to the authors. (The point or line source may be situated anywhere in the medium.) The program and the results of the test computations will be described elsewhere.

2 Two-dimensional elastodynamic equations in ray-centred coordinates

In this section, we shall introduce 2-D ray-centred coordinates s, n and express the elastodynamic equations in terms of these coordinates.

The elastodynamic equations in a general orthogonal curvilinear coordinate system are given by Aki & Richards (1980, section 2.6). We shall use these formulae and specify them for a 2-D case. Consider a right-handed orthogonal system c_1, c_2, c_3 and denote the corresponding scaling factors by h_1, h_2, h_3. (We shall write all indices as subscripts.) Let us assume that the medium and the wavefield do not depend on one coordinate, say c_2, and that the corresponding scaling factor $h_2 = 1$. This situation corresponds, e.g. to a medium in which the parameters do not depend on one Cartesian coordinate and the wavefield is generated by a line source parallel to the considered coordinate axis. The displacement components in a coordinate system c_1, c_2, c_3 are denoted by u_1, u_2, u_3. The elastodynamic equations then read, see Aki & Richards (1980),

$$\rho \frac{\partial^2 u_1}{\partial t^2} = \frac{1}{h_1 h_3}\left[\frac{\partial}{\partial c_1}(\tau_{11} h_3) + \frac{\partial}{\partial c_3}(\tau_{13} h_1)\right] + \frac{\tau_{13}}{h_1 h_3}\frac{\partial h_1}{\partial c_3} - \frac{\tau_{33}}{h_1 h_3}\frac{\partial h_3}{\partial c_1},$$

$$\rho \frac{\partial^2 u_2}{\partial t^2} = \frac{1}{h_1 h_3}\left[\frac{\partial}{\partial c_3}(\tau_{23} h_1) + \frac{\partial}{\partial c_1}(\tau_{12} h_3)\right], \tag{1}$$

$$\rho \frac{\partial^2 u_3}{\partial t^2} = \frac{1}{h_1 h_3}\left[\frac{\partial}{\partial c_3}(\tau_{33} h_1) + \frac{\partial}{\partial c_1}(\tau_{13} h_3)\right] + \frac{\tau_{13}}{h_1 h_3}\frac{\partial h_3}{\partial c_1} - \frac{\tau_{11}}{h_1 h_3}\frac{\partial h_1}{\partial c_3}.$$

Here τ_{ij} are the stress components, which are expressed in terms of displacement components u_1, u_2, u_3 in the following form

$$\tau_{11} = (\lambda + 2\mu)\left(\frac{1}{h_1}\frac{\partial u_1}{\partial c_1} + \frac{u_3}{h_1 h_3}\frac{\partial h_1}{\partial c_3}\right) + \lambda\left(\frac{1}{h_3}\frac{\partial u_3}{\partial c_3} + \frac{u_1}{h_1 h_3}\frac{\partial h_3}{\partial c_1}\right),$$

$$\tau_{22} = \lambda\left(\frac{1}{h_1}\frac{\partial u_1}{\partial c_1} + \frac{1}{h_3}\frac{\partial u_3}{\partial c_3} + \frac{u_3}{h_1 h_3}\frac{\partial h_1}{\partial c_3} + \frac{u_1}{h_1 h_3}\frac{\partial h_3}{\partial c_1}\right),$$

$$\tau_{33} = (\lambda + 2\mu)\left(\frac{1}{h_3}\frac{\partial u_3}{\partial c_3} + \frac{u_1}{h_1 h_3}\frac{\partial h_3}{\partial c_1}\right) + \lambda\left(\frac{1}{h_1}\frac{\partial u_1}{\partial c_1} + \frac{u_3}{h_1 h_3}\frac{\partial h_1}{\partial c_3}\right), \tag{2}$$

$$\tau_{12} = \mu\frac{1}{h_1}\frac{\partial u_2}{\partial c_1},$$

$$\tau_{13} = \mu\frac{1}{h_3}\frac{\partial u_1}{\partial c_3} + \mu\frac{1}{h_1}\frac{\partial u_3}{\partial c_1} - \mu\frac{u_1}{h_1 h_3}\frac{\partial h_1}{\partial c_3} - \mu\frac{u_3}{h_1 h_3}\frac{\partial h_3}{\partial c_1},$$

$$\tau_{23} = \mu\frac{1}{h_3}\frac{\partial u_2}{\partial c_3}.$$

The symbols λ and μ in (2) denote Lame's elastic parameters. It is assumed that λ, μ are smooth functions of the coordinates c_1, c_3. Body forces are not considered in (1).

As we can easily see from (1) and (2), the system of three elastodynamic equations (1) can be separated into two systems. The first system consists of the first and the third equation in (1) and contains only the displacement components u_1 and u_3. The second system reduces to one equation only (the second equation) and only contains the displacement component u_2.

In the following, we shall only consider the coordinate plane $c_2 = 0$. (The wavefield in any other plane $c_2 = \text{constant} \neq 0$ is the same as in plane $c_2 = 0$.) We shall introduce orthogonal coordinates $c_1 = s$ and $c_3 = n$ in the following way: We shall select an arbitrary ray Ω (corresponding to a P- or an S-wave) situated in plane $c_2 = 0$. The orthogonal coordinate system s, n is connected with ray Ω as shown in Fig. 1. The coordinate s measures the arclength along the ray from an arbitrary reference point, n represents a length coordinate in the direction perpendicular to Ω at s. The equation of the ray Ω in the coordinates s, n thus becomes $n = 0$. The basis of the new 2-D coordinate system is formed

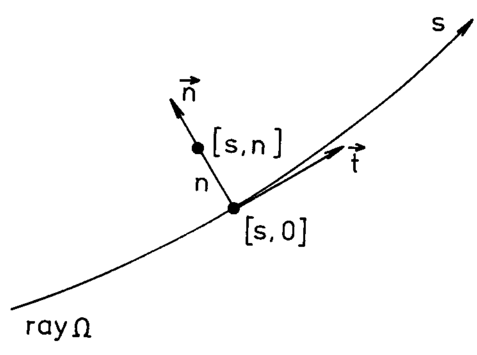

Figure 1. Ray-centred coordinate system (s, n); s is the arclength along the ray, n is the length along the line perpendicular to the ray.

by two unit vectors **t** and **n**, where **t** is a unit tangent and **n** a unit vector perpendicular to the ray Ω. The vector **n** always points to the same side of the ray. For completeness, we shall introduce a unit vector **b** perpendicular to plane $c_2 = 0$ and chosen in such a way that the system of vectors **t**, **n** and **b** is right-handed.

Let us now introduce the following notation:

$$V_P(s, n) = \left(\frac{\lambda + 2\mu}{\rho}\right)^{1/2}, \qquad V_S(s, n) = \left(\frac{\mu}{\rho}\right)^{1/2}. \tag{3}$$

V_P and V_S denote the velocities of compressional and shear waves, respectively. V_P and V_S are (in general) smooth functions of both coordinates s, n. Now we introduce $\alpha(s)$ and $\beta(s)$ as follows

$$\alpha(s) = V_P(s, 0), \qquad \beta(s) = V_S(s, 0), \tag{4}$$

and, similarly, their derivatives

$$\alpha_{,n}(s) = \left[\frac{\partial V_P(s, n)}{\partial n}\right]_{n=0}, \qquad \beta_{,n}(s) = \left[\frac{\partial V_S(s, n)}{\partial n}\right]_{n=0}. \tag{5}$$

Thus, α, β, $\alpha_{,n}$ and $\beta_{,n}$ are functions of s only, but not of n. α and β again denote the velocities of the compressional and shear waves, but directly at ray Ω.

We shall also use the notation v and $v_{,n}$ for the velocity and its derivative measured directly at Ω, which we introduce in the following way: If the ray Ω corresponds to a P-wave, we put

$$v = \alpha, \qquad v_{,n} = \alpha_{,n}, \tag{6}$$

if the ray Ω corresponds to an S-wave, we put

$$v = \beta, \qquad v_{,n} = \beta_{,n}. \tag{7}$$

Now we shall return to the coordinate system s, n and show that the system is orthogonal. For the infinitesimal length element dr we have in this system

$$dr^2 = d\mathbf{r} \cdot d\mathbf{r} = h^2\, ds^2 + dn^2, \tag{8}$$

where h is given by the formula

$$h = 1 + v^{-1} v_{,n} n. \tag{9}$$

Formula (8) shows that the system is orthogonal, with the scale factors

$$h_s = h, \qquad h_n = 1. \tag{10}$$

This coordinate system was first introduced into seismology by Popov & Pšenčík (1978), see also Červený and Hron (1980). Following Červený & Hron (1980), we shall call the system a ray-centred coordinate system. For a detailed discussion of its properties and for a simple derivation of relation (8) refer to Červený (1981a).

In the case when the ray Ω is curved, the normals to Ω intersect at certain distances from Ω. In other words, the ray-centred coordinate system (s, n) is not regular at large distances from Ω. In the following, we shall only consider a region along Ω, in which the ray-centred coordinate system (s, n) is regular, and we shall call it the regularity region.

We shall now return to the elastodynamic equations (1) and specify them for a ray-centred coordinate system (s, n); for this purpose we use equation (10). To simplify the equations, we shall use commas between subscripts to denote the derivatives of the vector components with respect to coordinates and time, e.g. $u_{s,n} = \partial u_s/\partial n$, $u_{s,tt} = \partial^2 u_s/\partial t^2$, $u_{n,nn} = \partial^2 u_n/\partial n^2$, etc. Similar notation for derivatives will also be used for the derivatives of scalar quantities, e.g. $h_{,n} = \partial h/\partial n$.

On inserting (2) into (1) we obtain the following systems:

(a) Elastodynamic equations for the displacement components $u_s = u_1$ and $u_n = u_3$ (P–SV case):

$$\begin{aligned}\rho u_{s,tt} &= h^{-2}(\lambda + 2\mu)u_{s,ss} + \mu u_{s,nn} + h^{-1}(\lambda + \mu)u_{n,sn} + u_{s,s}A^1_{ss} + u_{s,n}A^1_{sn} + u_{n,s}A^1_{ns} \\ &\quad + u_{n,n}A^1_{nn} + u_s B^1_s + u_n B^1_n, \\ \rho u_{n,tt} &= h^{-2}\mu u_{n,ss} + (\lambda + 2\mu)u_{n,nn} + h^{-1}(\lambda + \mu)u_{s,sn} + u_{s,s}A^2_{ss} + u_{s,n}A^2_{sn} \\ &\quad + u_{n,s}A^2_{ns} + u_{n,n}A^2_{nn} + u_s B^2_s + u_n B^2_n.\end{aligned} \tag{11}$$

In the preceding equations, we have used the following notation,

$$\begin{aligned}
A^1_{ss} &= h^{-1}[h^{-1}(\lambda + 2\mu)]_{,s}, & A^2_{ss} &= -h^{-2}(\lambda + 3\mu)h_{,n} + h^{-1}\lambda_{,n},\\
A^1_{sn} &= h^{-1}(\mu h)_{,n}, & A^2_{sn} &= h^{-1}\mu_{,s},\\
A^1_{ns} &= h^{-2}(\lambda + 3\mu)h_{,n} + h^{-1}\mu_{,n}, & A^2_{ns} &= h^{-1}(h^{-1}\mu)_{,s},\\
A^1_{nn} &= h^{-1}\lambda_{,s}, & A^2_{nn} &= h^{-1}[h(\lambda + 2\mu)]_{,n},\\
B^1_s &= -h^{-1}(\mu h_{,n})_{,n} - h^{-2}(h_{,n})^2, & B^2_s &= -h^{-1}[h^{-1}\mu h_{,n}]_{,s},\\
B^1_n &= h^{-1}[h^{-1}h_{,n}(\lambda + 2\mu)]_{,s}, & B^2_n &= h^{-1}(\lambda h_{,n})_{,n} - h^{-2}(\lambda + 2\mu)(h_{,n})^2.
\end{aligned} \tag{12}$$

(b) Elastodynamic equation for the displacement component u_2 (SH case)

$$\rho u_{2,tt} = \mu(u_{2,nn} + h^{-2}u_{2,ss}) + u_{2,n}h^{-1}(h\mu)_{,n} + u_{2,s}h^{-1}(h^{-1}\mu)_{,s}. \tag{13}$$

Equations (11)–(13) give the final form of the elastodynamic equations we shall use in the following.

3 High-frequency approximation of elastodynamic equations

It is well known that the high-frequency elastic wavefield propagates mostly along rays. To find the solution of the elastodynamic equations if the wave propagates mostly in some preferred direction, it is useful to use the parabolic equation method. We shall use this method to find the solutions of the elastodynamic equations concentrated close to Ω. For the time being, we shall not specify whether the ray corresponds to P- or S-waves; this will be done in the next section. Thus, the velocity v along a ray may be α or β, see (6) and (7). We shall only consider time-harmonic solutions and denote the circular frequency by ω.

The basic step in the parabolic equation method is the following substitution, see Červený *et al.* (1982),

$$u_j(s, n, \omega, t) = \exp\left\{-i\omega\left[t - \int_0^s v^{-1}(s)\, ds\right]\right\} U_j(s, n, \omega), \tag{14}$$

where j may be s, n or 2. The integral in (14) is taken along the ray Ω so that $v(s)$ is considered to be a function of one coordinate s only, see (4)–(7). The lower limit in the integral may be an arbitrary real-valued constant; without loss to generality we put this constant equal to zero. Inserting (14) into (11) and (13) yields a new system of elastodynamic equations for the variables U_s, U_n and U_2. For U_s and U_n (P–SV case) (11) yields

$$-\rho\omega^2 U_s = h^{-2}(\lambda+2\mu)\{[-v^{-2}\omega^2 + i\omega(1/v)_{,s}]U_s + 2i\omega v^{-1}U_{s,s} + U_{s,ss}\} + \mu U_{s,nn} + h^{-1}(\lambda+\mu)(i\omega v^{-1}U_{n,n} + U_{n,sn}) + (i\omega v^{-1}U_s + U_{s,s})A^1_{ss} + U_{s,n}A^1_{sn} + (i\omega v^{-1}U_n + U_{n,s})A^1_{ns} + U_{n,n}A^1_{nn} + U_sB^1_s + U_nB^1_n, \tag{15a}$$

$$-\rho\omega^2 U_n = h^{-2}\mu\{[-v^{-2}\omega^2 + i\omega(1/v)_{,s}]U_n + 2i\omega v^{-1}U_{n,s} + U_{n,ss}\} + (\lambda+2\mu)U_{n,nn} + h^{-1}(\lambda+\mu)(i\omega v^{-1}U_{s,n} + U_{s,sn}) + (i\omega v^{-1}U_s + U_{s,s})A^2_{ss} + U_{s,n}A^2_{sn} + (i\omega v^{-1}U_n + U_{n,s})A^2_{ns} + U_{n,n}A^2_{nn} + U_sB^2_s + U_nB^2_n. \tag{15b}$$

For U_2 (SH case) equation (13) yields

$$-\rho\omega^2 U_2 = h^{-2}\mu\{[-\beta^{-2}\omega^2 + i\omega(1/\beta)_{,s}]U_2 + 2i\omega\beta^{-1}U_{2,s} + U_{2,ss}\} + \mu U_{2,nn} + h^{-1}(h\mu)_{,n}U_{2,n} + (i\omega\beta^{-1}U_2 + U_{2,s})h^{-1}(\mu h^{-1})_{,s}. \tag{16}$$

In deriving the parabolic equations, we shall assume that $n = 0(\omega^{-\eta})$ with $\eta = 1/2$. This assumption expresses the fact that, for large ω, the investigation can be restricted to a thin 'boundary layer' along Ω. It is then suitable to introduce a new coordinate ν instead of n ($\nu = 0(1)$):

$$\nu = \sqrt{\omega}\, n. \tag{17}$$

On substituting ν into (15), for the P–SV case we obtain

$$-\omega^2[h^{-2}v^{-2}(\lambda+2\mu) - \rho]U_s + i\omega^{3/2}v^{-1}(\lambda+\mu)h^{-1}U_{n,\nu} + \omega\{U_s[iv^{-1}A^1_{ss} + ih^{-2}(1/v)_{,s}(\lambda+2\mu)] + U_n iv^{-1}A^1_{ns} + 2iv^{-1}h^{-2}(\lambda+2\mu)U_{s,s} + \mu U_{s,\nu\nu}\} + \omega^{1/2}[h^{-1}(\lambda+\mu)U_{n,s\nu} + U_{s,\nu}A^1_{sn} + U_{n,\nu}A^1_{nn}] + h^{-2}(\lambda+2\mu)U_{s,ss} + U_{s,s}A^1_{ss} + U_{n,s}A^1_{ns} + U_sB^1_s + U_nB^1_n = 0, \tag{18a}$$

$$-\omega^2[h^{-2}v^{-2}\mu - \rho]U_n + i\omega^{3/2}h^{-1}(\lambda+\mu)v^{-1}U_{s,\nu} + \omega\ U_n[iv^{-1}A^2_{ns} + ih^{-2}(1/v)_{,s}\mu] + iv^{-1}U_sA^2_{ss} + 2iv^{-1}h^{-2}\mu U_{n,s} + (\lambda+2\mu)U_{n,\nu\nu}\} + \omega^{1/2}[ih^{-1}(\lambda+\mu)U_{s,s\nu} + U_{s,\nu}A^2_{sn} + U_{n,\nu}A^2_{nn}] + h^{-2}\mu U_{n,ss} + U_{s,s}A^2_{ss} + U_{n,s}A^2_{ns} + U_sB^2_s + U_nB^2_n = 0. \tag{18b}$$

Similarly, from (16) for the SH case we obtain

$$-\omega^2[h^{-2}\beta^{-2}\mu - \rho]U_2 + \omega\{U_2[i(1/\beta)_{,s}h^{-2}\mu + ih^{-1}\beta^{-1}(\mu h^{-1})_{,s}] + 2i\mu h^{-2}\beta^{-1}U_{2,s} + \mu U_{2,\nu\nu}\} + \omega^{1/2}h^{-1}(h\mu)_{,n}U_{2,\nu} + U_{2,ss}\mu h^{-2} + U_{2,s}h^{-1}(\mu h^{-1})_{,s} = 0. \tag{19}$$

Now we shall start to look for the asymptotics of equations (18)–(19) for $\omega \to \infty$. We shall expand U_s, U_n and U_2 into asymptotic series in inverse powers of $\omega^{1/2}$,

$$U_s \sim U^0_s + \omega^{-1/2}U^1_s + \omega^{-1}U^2_s + \dots,$$
$$U_n \sim U^0_n + \omega^{-1/2}U^1_n + \omega^{-1}U^2_n + \dots, \tag{20}$$
$$U_2 \sim U^0_2 + \omega^{-1/2}U^1_2 + \omega^{-1}U^2_2 + \dots.$$

We shall also expand the coefficients in (18) and (19) which are functions of the coordinate n and consequently of ω, since $n = \omega^{-1/2}\nu$. These expansions will, of course, be different for P-waves and for S-waves; they will be discussed in the next section.

In (18) and (19), we shall retain only the terms of the order $\omega^{\eta}, \eta \geqslant 1$ and neglect all the terms of the order $\omega^{\eta}, \eta < 1$. The final form of the elastodynamic equations valid for both P- and S-waves, with the accuracy to ω^1 is as follows.

For a P–SV case, (18a), (18b) and (20) yield

$$-\omega^2[h^{-2}v^{-2}(\lambda+2\mu)-\rho](U_s^0+\omega^{-1/2}U_s^1+\omega^{-1}U_s^2)+i\omega^{3/2}(\lambda+\mu)v^{-1}h^{-1}$$
$$\times(U^0_{n,\nu}+\omega^{-1/2}U^1_{n,\nu})+\omega\{ih^{-1}[h^{-1}v^{-1}(\lambda+2\mu)]_{,s}U_s^0+iv^{-1}A^1_{ns}U_n^0+\mu U^0_{s,\nu\nu}$$
$$+2iv^{-1}h^{-2}(\lambda+2\mu)U^0_{s,s}\}=0, \tag{21a}$$

$$-\omega^2(h^{-2}v^{-2}\mu-\rho)(U_n^0+\omega^{-1/2}U_n^1+\omega^{-1}U_n^2)+i\omega^{3/2}(\lambda+\mu)h^{-1}v^{-1}(U^0_{s,\nu}+\omega^{-1/2}U^1_{s,\nu})$$
$$+\omega[ih^{-1}(h^{-1}v^{-1}\mu)_{,s}U_n^0+iv^{-1}U_s^0A^2_{ss}+2iv^{-1}h^{-2}\mu U^0_{n,s}+(\lambda+2\mu)U^0_{n,\nu\nu}]=0. \tag{21b}$$

Similarly, for SH-waves (19) and (20) yield

$$-\omega^2(h^{-2}\beta^{-2}\mu-\rho)(U_2^0+\omega^{-1/2}U_2^1+\omega^{-1}U_2^2)+$$
$$+\omega[ih^{-1}(h^{-1}\beta^{-1}\mu)_{,s}U_2^0+2i\mu h^{-2}\beta^{-1}U^0_{2,s}+\mu U^0_{2,\nu\nu}]=0. \tag{22}$$

4 Two-dimensional parabolic equations

In this section, we shall discuss equations (21a), (21b) and (22) independently for solutions concentrated close to rays of P-, SV- and SH-waves.

For simplicity, we shall use here the term 'component of the displacement vector' both for u_s, u_n, u_2 and for U_s, U_n, U_2, see (14).

4.1 PARABOLIC EQUATION FOR A P-WAVE

The vectorial solutions of elastodynamic equations concentrated close to a ray Ω of a P-wave have generally two non-vanishing components in a 2-D case: U_s and U_n. Here U_s denotes the component of the displacement vector in the direction of the tangent to the ray Ω and it will be called the principal component (see Fig. 2a). Similarly, U_n denotes the component of the displacement vector in the direction perpendicular to the ray Ω and it will be called the additional component. In this section, we shall find asymptotic expressions for both these components (for $\omega \to \infty$).

To find these solutions, we must specify the equations (21a) and (21b) for P-waves (i.e. insert $v = \alpha$) and expand the coefficients in terms of n. As $n = \omega^{-1/2}\nu$, these expansions automatically represent the expansions in $\omega^{-1/2}$. In all cases, we shall retain only the terms which do not produce terms of an order lower than ω^1 in (21a) and (21b).

For illustration, we shall derive only the expansion for one coefficient; the procedure is quite similar in all other cases.

Let us consider the coefficient with ω^2 in (21a) $-[h^{-2}\alpha^{-2}(\lambda+2\mu)-\rho]$, and rewrite it as follows, $-[h^{-2}\alpha^{-2}(\lambda+2\mu)-\rho]=\rho[1-h^{-2}\alpha^{-2}V_P^2(s,n)]$, see (6). For $V_P(s,n)$ we can write

$$V_P(s,n)\sim\alpha(s)+\alpha_{,n}(s)n+\frac{1}{2}\alpha_{,nn}(s)n^2, \tag{23}$$

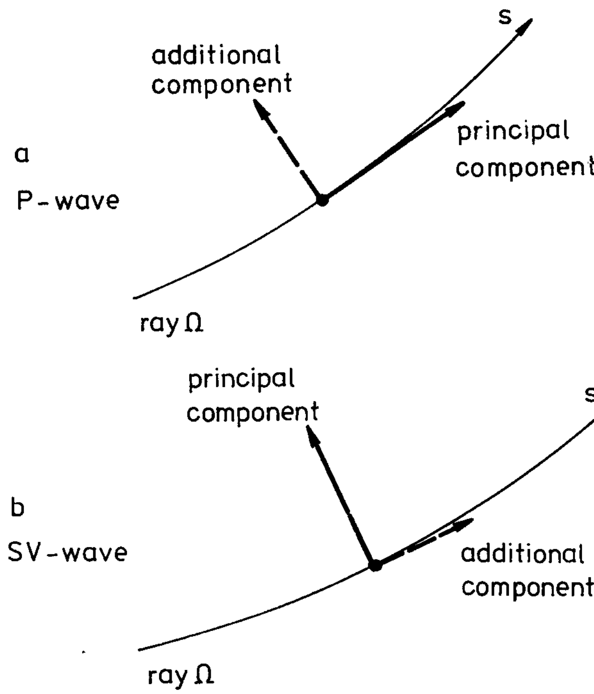

Figure 2. Principal and additional components for: (a) a *P*-wave, and (b) an *S*-wave.

where $\alpha_{,nn}$ is given by the relation

$$\alpha_{,nn}(s) = \left[\frac{\partial^2 V_P(s,n)}{\partial n^2}\right]_{n=0}. \tag{24}$$

Thus we obtain

$$-[h^{-2}\alpha^{-2}(\lambda + 2\mu) - \rho] \sim \rho\{1 - h^{-2}[1 + \alpha^{-1}\alpha_{,n}n + \tfrac{1}{2}\alpha^{-1}\alpha_{,nn}n^2]^2\}$$
$$= \rho\{1 - h^{-2}[h + \tfrac{1}{2}\alpha^{-1}\alpha_{,nn}n^2]^2\}.$$

From this we simply obtain the final result

$$-[h^{-2}\alpha^{-2}(\lambda + 2\mu) - \rho] \sim -\rho\alpha^{-1}\nu^2\alpha_{,nn}\omega^{-1}. \tag{25}$$

Similarly, for $-(h^{-2}\alpha^{-2}\mu - \rho)$ we obtain

$$-(h^{-2}\alpha^{-2}\mu - \rho) \sim \rho\alpha^{-2}(\alpha^2 - \beta^2) + 2\rho\omega^{-1/2}\alpha^{-2}\beta^2(\alpha^{-1}\alpha_{,n} - \beta^{-1}\beta_{,n})\nu + \dots. \tag{26}$$

We also denote

$$\beta_{,nn}(s) = \left[\frac{\partial^2 V_S(s,n)}{\partial n^2}\right]_{n=0}, \tag{27}$$

see (3).

In other terms of (21a) and (21b), it is sufficient to consider the following approximations

$$h \sim 1, \qquad \lambda + 2\mu \sim \rho\alpha^2, \qquad \mu \sim \rho\beta^2. \tag{28}$$

Inserting (25), (26) and (28) into (21a) and (21b) yields

$$i\omega^{3/2}(\alpha^2-\beta^2)\alpha^{-1}U^0_{n,\nu}+\omega[\![U^0_s[i\rho^{-1}(\rho\alpha)_{,s}-\alpha^{-1}\nu^2\alpha_{,nn}]+2i\alpha U^0_{s,s}+\beta^2U^0_{s,\nu\nu}$$
$$+i(\alpha^2-\beta^2)\alpha^{-1}U^1_{n,\nu}+i\rho^{-1}\alpha^{-1}A^1_{ns}U^0_n]\!]=0, \qquad (29a)$$

$$\alpha^{-2}(\alpha^2-\beta^2)\omega^2U^0_n+\omega^{3/2}(\alpha^2-\beta^2)[i\alpha^{-1}U^0_{s,\nu}+\alpha^{-2}U^1_n]=0. \qquad (29b)$$

In (29b), we have not written the last term (with ω), as we shall not need it in the following. From (29b) it is simple to obtain

$$U^0_n=0, \qquad U^1_n=-i\alpha U^0_{s,\nu}. \qquad (30)$$

Thus, the first term in the asymptotic series for the additional component U_n (see 20) vanishes and the additional component is given by the relation

$$U_n\sim-i\alpha\omega^{-1/2}U^0_{s,\nu}. \qquad (31)$$

Let us now return to (29a). We immediately obtain $U^0_{n,\nu}=0$. Inserting (30) into the second term of (29a) (with ω) yields

$$2i\alpha^{-1}U^0_{s,s}+U^0_{s,\nu\nu}+U^0_s[\![i\alpha^{-1}[\ln(\rho\alpha)]_{,s}-\alpha^{-3}\alpha_{,nn}\nu^2]\!]=0. \qquad (32)$$

This is the parabolic equation for U^0_s. Equation (32) can be simplified by the substitution

$$U^0_s(s,\nu)=[\alpha(s)\rho(s)]^{-1/2}W^\alpha(s,\nu). \qquad (33)$$

By inserting (33) into (32), we obtain the final form of the parabolic equation,

$$2i\alpha^{-1}W^\alpha_{,s}+W^\alpha_{,\nu\nu}-\alpha^{-3}\alpha_{,nn}\nu^2W^\alpha=0. \qquad (34)$$

Thus, the vectorial time-harmonic solution of 2-D elastodynamic equations, concentrated close to a ray of a P-wave reads

$$\mathbf{u}(s,n,\omega,t)=u_s(s,n,\omega,t)\mathbf{t}+u_n(s,n,\omega,t)\mathbf{n}, \qquad (35)$$

where

$$u_s(s,n,\omega,t)=\frac{1}{\sqrt{\alpha(s)\rho(s)}}\exp\left\{-i\omega\left[t-\int_0^s\frac{ds}{\alpha(s)}\right]\right\}W^\alpha(s,\nu), \qquad (36a)$$

$$u_n(s,n,\omega,t)=-\frac{i}{\sqrt{\omega}}\sqrt{\frac{\alpha(s)}{\rho(s)}}\exp\left\{-i\omega\left[t-\int_0^s\frac{ds}{\alpha(s)}\right]\right\}\frac{\partial W^\alpha(s,\nu)}{\partial\nu}, \qquad (36b)$$

and where $W^\alpha(s,\nu)$ is a solution of the parabolic wave equation (34). Note that $W^\alpha(s,\nu)$ is a function of frequency ω, as $\nu=\sqrt{\omega}n$.

4.2 PARABOLIC EQUATION FOR AN SV-WAVE

The vectorial solutions of 2-D elastodynamic equations concentrated close to a ray Ω of an SV-wave have again generally two non-vanishing components, U_s and U_n. The component U_n will be now referred to as the principal component, the component U_s as the additional component, see Fig. 2(b).

The expressions for the components U_n and U_s can be obtained in the same way as in the case of the P-waves. As in Section 4.1, we can write

$$h^{-2}\beta^{-2}\mu - \rho \sim \omega^{-1}\beta^{-1}\rho\nu^2\beta_{,nn}, \tag{37}$$

$$h^{-2}\beta^{-2}(\lambda + 2\mu) - \rho \sim \rho\beta^{-2}(\alpha^2 - \beta^2) + 2\rho\alpha^2\beta^{-2}\omega^{-1/2}(\alpha^{-1}\alpha_{,n} - \beta^{-1}\beta_{,n})\nu + \ldots. \tag{38}$$

We insert (37) and (38) into (21a) and (21b). We can again use the approximations (28) in the last terms (with ω):

$$-\omega^2\beta^{-2}(\alpha^2 - \beta^2)U_s^0 + \omega^{3/2}(\alpha^2 - \beta^2)(i\beta^{-1}U_{n,\nu}^0 - \beta^{-2}U_s^1) = 0, \tag{39a}$$

$$i\omega^{3/2}(\alpha^2 - \beta^2)\beta^{-1}U_{s,\nu}^0 + \omega[\![U_n^0[i\rho^{-1}(\rho\beta)_{,s} - \beta^{-1}\beta_{,nn}\nu^2] + i\beta^{-1}\rho^{-1}U_s^0A_{ss}^2 + 2i\beta U_{n,s}^0 + \alpha^2U_{n,\nu\nu}^0 + i(\alpha^2 - \beta^2)\beta^{-1}U_{s,\nu}^1]\!] = 0. \tag{39b}$$

As in (29b), we do not write the last term (with ω) in (39a), as we do not need it in the following.

The solution of equations (39a,b) is as follows

$$U_s^0 = 0, \qquad U_s^1 = i\beta U_{n,\nu}^0, \tag{40}$$

and

$$U_n^0(s,\nu) = [\beta(s)\rho(s)]^{-1/2}W^\beta(s,\nu),$$

where $W^\beta(s,\nu)$ is a solution of the parabolic equation

$$2i\beta^{-1}W_{,s}^\beta + W_{,\nu\nu}^\beta - \beta^{-3}\beta_{,nn}\nu^2W^\beta = 0. \tag{41}$$

To summarize, we can again write the vectorial time-harmonic solution of 2-D elastodynamic equations, concentrated close to the ray of SV-wave, in the form of (35), where

$$u_n(s,n,\omega,t) = \frac{1}{\sqrt{\beta(s)\rho(s)}}\exp\left\{-i\omega\left[t - \int_0^s \frac{ds}{\beta(s)}\right]\right\}W^\beta(s,\nu), \tag{42a}$$

$$u_s(s,n,\omega,t) = \frac{i}{\sqrt{\omega}}\sqrt{\frac{\beta(s)}{\rho(s)}}\exp\left\{-i\omega\left[t - \int_0^s \frac{ds}{\beta(s)}\right]\right\}\frac{\partial W^\beta(s,\nu)}{\partial\nu}, \tag{42b}$$

where $W^\beta(s,\nu)$ is a solution of the parabolic equation (41). Note again that $W^\beta(s,\nu)$ is a function of frequency, as $\nu = \sqrt{\omega}n$.

4.3 PARABOLIC EQUATION FOR AN SH-WAVE

The SH component of the solution of a 2-D elastodynamic equation concentrated close to a ray Ω of an S-wave is fully separated from the P and SV components. In other words, the solution has in this case only one non-vanishing component, U_2. U_2 now has the properties of the principal component. The expression for U_2 is simply obtained from (20) and (22). Using (37), (22) yields

$$2i\beta^{-1}U_{2,s}^0 + U_{2,\nu\nu}^0 + U_2^0[\![i\beta^{-1}[\ln(\rho\beta)]_{,s} - \beta^{-3}\beta_{,nn}\nu^2]\!] = 0. \tag{43}$$

As in the preceding cases, we finally obtain

$$\mathbf{u}(s,n,\omega,t) = u_2(s,n,\omega,t)\mathbf{b}, \tag{44a}$$

where

$$u_2(s,n,\omega,t)=\frac{1}{\sqrt{\beta(s)\rho(s)}}\exp\left\{-i\omega\left[t-\int_0^s\frac{ds}{\beta(s)}\right]\right\}W^{\beta}(s,\nu). \tag{44b}$$

Here $W^{\beta}(s,\nu)$ is again a solution of the parabolic equation (41), $\nu=\sqrt{\omega}\,n$.

5 Solutions of a 2-D parabolic equation

In all the three cases (P-, SV- and SH-waves) we arrived at a parabolic equation. The parabolic equations for the individual cases differ only in the velocities appearing in their coefficients (α for a P-wave and β for SV- and SH-waves). We shall again use v to denote the velocity in the sense of (6) and (7) and consider the parabolic equation in a form general for all the three waves:

$$2iv^{-1}W_{,s}+W_{,\nu\nu}-v^{-3}v_{,nn}\nu^2W=0. \tag{45}$$

The parabolic equation (45) has exactly the same form as that obtained by Červený *et al.* (1982) for the solutions of a wave equation concentrated close to rays. Since the method of solving the parabolic equation (45) is derived in detail in the mentioned reference, we shall only deal with it briefly here.

We seek the solution of (45) in the following form (see also Babich & Kirpichnikova 1974):

$$W(s,\nu)=A(s)\exp\left(\frac{i}{2}\nu^2M\right), \tag{46}$$

where $A=A(s)$ and $M=M(s)$ are unknown complex-valued functions. By inserting (46) into (45) we find that (45) is satisfied if

$$M_{,s}+vM^2+v^{-2}v_{,nn}=0, \tag{47}$$

and

$$A_{,s}+\frac{1}{2}vAM=0. \tag{48}$$

The equation (47) is an ordinary non-linear differential equation of the first order of the Riccati type. It can be rewritten into a system of two linear differential equations of the first order specifying

$$M=p/q, \qquad p=v^{-1}q_{,s}, \tag{49}$$

where $q=q(s), p=p(s)$. We then obtain

$$q_{,s}=vp, \qquad p_{,s}=-v^{-2}v_{,nn}q. \tag{50}$$

Now it is easy to find the solution of (48). It is sufficient to realize that M can also be written as $M=v^{-1}\,d(\ln q)/ds$. Thus

$$A(s)=\frac{\Psi}{\sqrt{q(s)}}, \tag{51}$$

where Ψ is a complex constant. The constant Ψ remains the same along the whole ray, but may differ for different rays.

In the following, we shall be interested mainly in the cases in which the quantity $q(s)$ is complex-valued. Under $q^{1/2}(s)$ we shall understand the following function. At the initial point $s = s_0$ we choose the principal branch of the square root, i.e.

$$q^{1/2}(s_0) = |q(s_0)|^{1/2} \exp\left\{\frac{i}{2} \arg [q(s_0)]\right\},$$

$-\pi < \arg q(s_0) \leqslant \pi$. It will be shown later that $q(s) \neq 0$ for $s \neq s_0$. Therefore, the function $\arg [q(s)]$ is continuous and for arbitrary $s \neq s_0$ we have

$$q^{1/2}(s) = |q(s)|^{1/2} \exp\left\{\frac{i}{2} \arg [q(s)]\right\}.$$

Inserting (46), (49) and (51) into (35) and (36) and reverting to the original coordinate n, for P-waves we obtain

$$\mathbf{u}(s, n, \omega, t) = \frac{\Psi_P}{\sqrt{\alpha(s)\rho(s)q(s)}} \left[\mathbf{t} + \frac{np(s)\alpha(s)}{q(s)}\mathbf{n}\right] \times \exp\left\{-i\omega\left[t - \int_0^s \frac{ds}{\alpha(s)} - \frac{1}{2}\frac{p(s)}{q(s)}n^2\right]\right\}. \quad (52)$$

Similarly for SV-waves,

$$\mathbf{u}(s, n, \omega, t) = \frac{\Psi_{SV}}{\sqrt{\beta(s)\rho(s)q(s)}} \left[\mathbf{n} - \frac{np(s)\beta(s)}{q(s)}\mathbf{t}\right] \times \exp\left\{-i\omega\left[t - \int_0^s \frac{ds}{\beta(s)} - \frac{1}{2}\frac{p(s)}{q(s)}n^2\right]\right\}. \quad (53)$$

Finally for SH-waves

$$\mathbf{u}(s, n, \omega, t) = \frac{\Psi_{SH}}{\sqrt{\beta(s)\rho(s)q(s)}} \mathbf{b} \exp\left\{-i\omega\left[t - \int_0^s \frac{ds}{\beta(s)} - \frac{1}{2}\frac{p(s)}{q(s)}n^2\right]\right\}. \quad (54)$$

Let us now derive another useful equation for $\mathbf{u}(s, n, \omega, t)$. This equation will be especially suitable for the continuation of the wavefield of Gaussian beams along the central ray Ω. To be brief, we shall consider only the P-wave, see (52). We can rewrite it in this form:

$$\mathbf{u}(s, n, \omega, t) = U_P(s, \omega) \exp\left\{-i\omega\left[t - \frac{1}{2}\frac{p(s)}{q(s)}n^2\right]\right\}\left[\mathbf{t} + \frac{np(s)\alpha(s)}{q(s)}\mathbf{n}\right], \quad (55)$$

where

$$U_P(s, \omega) = \frac{\Psi_P}{\sqrt{\alpha(s)\rho(s)q(s)}} \exp\left\{i\omega \int_0^s \frac{ds}{\alpha(s)}\right\}. \quad (56)$$

Here $U_P(s, \omega)$ denotes the component of the displacement vector into the direction of the central ray Ω, measured directly at the central ray Ω (factor $\exp(-i\omega t)$ not considered). It is easy to show that

$$U_P(s, \omega) = U_P(s_0, \omega) \left[\frac{\alpha(s_0)\rho(s_0)q(s_0)}{\alpha(s)\rho(s)q(s)}\right]^{1/2} \exp\left\{i\omega \int_{s_0}^s \frac{ds}{\alpha(s)}\right\}. \quad (57)$$

Inserting (57) into (55) finally yields

$$\mathbf{u}(s,n,\omega,t) = U_P(s_0,\omega)\left[\frac{\alpha(s_0)\rho(s_0)q(s_0)}{\alpha(s)\rho(s)q(s)}\right]^{1/2}\left[\mathbf{t}+\frac{np(s)\alpha(s)}{q(s)}\,\mathbf{n}\right]$$

$$\times\exp\left\{-i\omega\left[t-\int_{s_0}^{s}\frac{ds}{\alpha(s)}-\frac{1}{2}\frac{p(s)}{q(s)}\,n^2\right]\right\}. \tag{58}$$

Equations (53) and (54) can, of course, be rewritten in a similar way.

If we express $U_P(s,\omega)$ as $A_P(s)\exp[i\omega\tau(s)]$, where

$$\tau(s) = \int_0^s \alpha^{-1}\,ds,$$

we get from (57),

$$A_P(s) = A_P(s_0)\left[\frac{\alpha(s_0)\rho(s_0)q(s_0)}{\alpha(s)\rho(s)q(s)}\right]^{1/2}, \qquad \tau(s) = \tau(s_0) + \int_{s_0}^{s}\frac{ds}{\alpha(s)}. \tag{57$'$}$$

The relations (57′) are well known in the ray method, see Červený, Molotkov & Pšenčík (1977). Note, however, that the quantity q in (57′) can be complex-valued.

Let us note that the solution (46) is not the only solution of the parabolic equation. On introducing the following notation

$$W^0(s,\nu) = \frac{1}{\sqrt{q(s)}}\exp\left(\frac{i}{2}\nu^2\frac{p}{q}\right), \tag{59}$$

it would be possible to construct from (59) an infinite number of other solutions $W^k(s,\nu)$, $k = 1, 2, \ldots,$ concentrated close to rays. All these solutions will be of zero order with respect to ω. In the following, however, we shall deal only with the 'basic mode' (59), but for completeness, we also present an infinite system of linearly independent solutions of (45). The kth member of this system reads

$$W^k(s,\nu) = \frac{1}{\sqrt{q(s)}}\left(\frac{q^*}{q}\right)^{k/2} H_k\left(\nu\sqrt{\operatorname{Im}\left(\frac{p}{q}\right)}\right)\exp\left(\frac{i}{2}\nu^2\frac{p}{q}\right). \tag{60}$$

Here H_k denotes the Hermite polynomial of the kth order.

6 Properties of 2-D elastodynamic Gaussian beams

In this section, we shall investigate the properties of elastodynamic Gaussian beams. Elastodynamic Gaussian beams have many common properties with 'scalar' Gaussian beams, investigated in detail by Červený *et al.* (1982). Therefore, we shall be brief and only emphasize the differences between both types of Gaussian beams. It was shown by Červený *et al.* (1982) that the quantities q, p must be complex-valued to obtain the Gaussian beams (for details see below). We shall investigate the solutions of the parabolic equation with complex-valued quantities q and p in Section 6.2. To get a better insight into the whole problem, we shall first pay attention to the solutions of the parabolic equation with real-valued q and p.

6.1 PROPERTIES OF SOLUTIONS OF THE PARABOLIC EQUATION WITH REAL-VALUED q AND p

We can immediately see that for real-valued q and p and for $n = 0$ (52)–(58) become standard ray formulae applicable directly to the ray Ω. The quantity $q(s)$ has the meaning of geometrical spreading (see Červený *et al.* 1977). Let us note that the function $q(s)$ vanishes at caustics and, consequently causes infinite amplitudes there. This corresponds to the well-known singularity of the ray method.

Equations (52)–(58) also give us the possibility of studying the wavefield in some neighbourhood of the ray Ω (for $n \neq 0$), not only directly at it. Thus, to investigate the wavefield in the vicinity of Ω, it is not necessary to shoot new rays with ray parameters close to the ray parameter of Ω, but we can use (52)–(58) directly.

There are two factors which depend on n in (52)–(54).

First, (52)–(54) contain the factor $\exp(i\omega pn^2/2q)$. It can be easily shown (see, e.g. Červený & Hron 1980; Hubral 1980, etc.) that the factor $pn^2/2q$ corresponds to the second-order term in the expansion of the wavefront in the plane perpendicular to the ray Ω. Consequently,

$$M(s) = p(s)/q(s) = v^{-1}(s)K_w(s), \tag{61}$$

where $K_w(s)$ is the curvature of the wavefront. The quantities $M(s)$ or $q(s)$ and $p(s)$ can be determined from differential equations (47) or (50). These equations are known in the ray theory as dynamic ray tracing equations. They can be used for various purposes, e.g. for the computation of geometrical spreading, for the evaluation of second derivatives of the travel-time field or for the approximate computation of rays in the vicinity of the specified ray Ω.

Secondly, the final directions of the displacement vectors corresponding to P- or SV-waves are determined by the vectors

$$\mathbf{t} + nK_w\mathbf{n}, \qquad \mathbf{n} - nK_w\mathbf{t}, \tag{62}$$

respectively. The vectors (62) approximately correspond to the normal and tangent to the wavefront at point (s, n). Thus, the additional terms in (52) and (53) represent corrections to the principal terms, which keep the displacement vector perpendicular or tangent to the corresponding wavefront.

It is interesting to notice that the other amplitude factors (e.g. geometrical spreading) do not depend on n. In other words, the changes in these factors with n are of a higher order and can be neglected when we are interested in the wavefield in the close neighbourhood of the ray Ω. Thus, in the same way as dynamic ray tracing can be used to compute the travel-time field and other kinematic characteristics of the wavefield in the neighbourhood of Ω, the equations (52)–(54) can serve to evaluate the displacement vector at any point close to Ω.

Both the dynamic ray tracing system (50) and equations (52)–(54) can then be used to evaluate the wavefield not only on the ray Ω, but also in its vicinity. For this purpose, it is sufficient to know the ray Ω and the solutions of the dynamic ray tracing system (50) along it. Taking into account these properties of (52)–(54) with real-valued q and p, we can call them the *paraxial ray solutions* of the elastodynamic equation.

Any solution of system (50) can be expressed as a linear combination of two linearly independent real solutions of (50). Let us denote by $\pi(s)$ the fundamental matrix of (50),

$$\pi(s) = \begin{bmatrix} q_1(s) & q_2(s) \\ p_1(s) & p_2(s) \end{bmatrix}, \tag{63}$$

and specify it by the initial conditions for $s = s_0$,

$$\pi(s_0) = \begin{bmatrix} 1 & 0 \\ 0 & v^{-1}(s_0) \end{bmatrix}. \tag{64}$$

Any real solution of (50) can then be written in the following form

$$q = A_1 q_1 + A_2 q_2, \qquad p = A_1 p_1 + A_2 p_2, \tag{65}$$

where A_1 and A_2 are arbitrary real constants. It is easy to see that the first solution (q_1, p_1) corresponds to 'plane wave' type initial conditions at $s = s_0$ (zero curvature of the wavefront at $s = s_0$), and the second solution (q_2, p_2) corresponds to 'line source' type initial conditions at $s = s_0$ (infinite curvature of the wave front at $s = s_0$). Thus, any solution with arbitrary real-valued initial conditions can be evaluated from two known solutions, which correspond to 'plane wave' and 'line source' initial conditions.

6.2 GAUSSIAN BEAMS

Let us now consider complex-valued quantities q, p. We shall assume that these quantities satisfy the condition $\text{Im}(p/q) > 0$ along the whole ray. (For more details on this condition see below.) The exponential functions in (52)–(54) can then be expressed in an alternative way

$$\exp\left[-i\omega[t - \tau(s)] + \frac{i\omega n^2}{2v} K_B(s) - \frac{n^2}{L^2(s)}\right], \tag{66}$$

where

$$K_B(s) = v(s)\,\text{Re}\left[\frac{p(s)}{q(s)}\right], \qquad L(s) = \left\{\frac{\omega}{2}\,\text{Im}\left[\frac{p(s)}{q(s)}\right]\right\}^{-1/2}, \tag{67}$$

$$\tau(s) = \int_0^s v^{-1}(s)\,ds.$$

The expression (66) decreases exponentially with increasing distance n from the ray Ω. The decrease is Gaussian. Due to this property, the solutions (53)–(54) may be called Gaussian beams. The quantities $K_B(s)$, $L(s)$ and $\tau(s)$ have the following physical meaning. $K_B(s)$ is the curvature of the phase front of the beam, $L(s)$ is some frequency-dependent effective half-width of the beam and $\tau(s)$ is the travel-time of the beam along the ray Ω.

The dynamic ray tracing system (50) plays a very important role also in the computation of Gaussian beams. Any complex solution of this system can again be expressed as a linear combination of two linearly independent real solutions, with complex coefficients.

As was shown by Červený *et al.* (1982), it is useful to look for complex solutions of (50) in the form

$$q = \epsilon q_1 + q_2, \qquad p = \epsilon p_1 + p_2, \tag{68}$$

where

$$\epsilon = S_0 - i\frac{\omega_0}{2v(s_0)} L_0^2. \tag{69}$$

Here $\omega_0 = 2\pi$ Hz. The complex-valued constant ϵ has a simple meaning in a homogeneous medium with a constant velocity $v_0 = v(s_0)$. In a homogeneous medium, the effective half-width of the beam $L(s)$ changes hyperbolically along the ray, with a minimum at some point $s = s_M$. L_0 then represents the effective half-width of the beam for the frequency $f = 1$ Hz at the point $s = s_M$ and S_0 is the distance of this point from the reference point $s = s_0$. The dimension of L_0 is the length. Notice a slight difference between the quantity L_0 introduced here and the quantity L_M introduced by Červený *et al.* (1982). The quantity L_M is frequency-dependent, whereas L_0 introduced here corresponds to the frequency $f = 1$ Hz. If we specify the initial conditions at the point $s = s_0$ and use $S_0 = 0$, the quantity L_0 determines the effective half-width of the beam for the frequency $f = 1$ Hz directly at the reference point $s = s_0$, even in inhomogeneous media.

If the curvature of the phase front of the beam $K_B(s_0)$ and the effective half-width of the beam $L(s_0)$ are known at $s = s_0$, the real-valued constants S_0 and L_0 in (69) are given by the relations

$$S_0 = \frac{\omega^2 L^4(s_0) K_B(s_0)}{4v^2(s_0) + \omega^2 K_B^2(s_0) L^4(s_0)}, \qquad L_0 = \frac{2v(s_0)(\omega/\omega_0)^{1/2} L(s_0)}{[4v^2(s_0) + K_B^2(s_0)\omega^2 L^4(s_0)]^{1/2}}.$$

Note that the expression $\omega L^2(s_0)$ does not depend on frequency (see 67), so that S_0 and L_0 are also frequency-independent. Varying the quantities S_0, L_0, the properties of the Gaussian beams at an arbitrary point close to the ray Ω can be influenced. A detailed discussion of these problems is given in Červený *et al.* (1982), where it is also shown that it is sufficient to consider $L_0 \neq 0$ to guarantee the fulfilment of the two important conditions along the whole ray Ω:

Condition I:

$$q(s) \neq 0. \tag{70}$$

The condition guarantees the regularity of the Gaussian beam along the ray (with finite amplitudes at caustics).

Condition II:

$$\mathrm{Im}\,(p/q) > 0. \tag{71}$$

The condition guarantees the concentration of the solutions close to rays; see (66) and (67).

Let us now return to equations (52)–(54) for the Gaussian beams. The additional components in (52) and (53) can be divided into two parts, real and imaginary. The real parts of the additional components together with principal components, as in the case of real q and p, form the vectors

$$\mathbf{t} + nK_B\mathbf{n}, \qquad \mathbf{n} - nK_B\mathbf{t}, \tag{72}$$

which represent the normal and the tangent to the phase front of the beam at point (s, n). The remaining parts of the additional components are phase-shifted (by $\pi/2$) and generally depend on the width of the beam. The wider the beam, the weaker the component.

References

Aki, K. & Richards, P., 1980. *Quantitative Seismology,* Freeman, San Francisco.

Babich, V. M. & Kirpichnikova, N. J., 1974. *The Boundary Layer Method in Diffraction Problems,* Leningrad University Press (in Russian; English translation by Springer-Verlag, 1980).

Červený, V., 1981a. Seismic wave fields in structurally complicated media (ray and Gaussian beam approaches) *Lecture Notes, Rijksuniversiteit Utrecht,* Vening-Meinesz Laboratory, Utrecht.

Červený, V., 1981b. Ray tracing by Gaussian beams, a paper presented at *51st Annual Meeting of the SEG,* Los Angeles, October 11–15.

Červený, V. & Hron, F., 1980. The ray series method and dynamic ray tracing systems for 3-D inhomogeneous media, *Bull. seism. Soc. Am.,* **70**, 47–77.

Červený, V., Molotkov, I. A. & Pšenčík, I., 1977. *Ray Method in Seismology,* Karlova Universita, Praha.

Červený, V., Popov, M. M. & Pšenčík, I., 1982. Computation of wave fields in inhomogeneous media – Gaussian beam approach, *Geophys. J. R. astr. Soc.,* **70**, 109–128.

Hubral, P., 1980. Wave front curvatures in 3-D laterally inhomogeneous media with curved interfaces, *Geophysics,* **45**, 905–913.

Kirpichnikova, N. J., 1971. Construction of solutions concentrated close to rays for the equations of elasticity theory in an inhomogeneous isotropic space, *Mathematical Problems of Theory of Diffraction and Propagation of Waves, Vol. 1,* Nauka, Leningrad (in Russian; English translation by American Mathematical Society, 1974).

Popov, M. M. & Pšenčík, I., 1978. Computation of ray amplitudes in inhomogeneous media with curved interfaces, *Studia geophys. geod.,* **22**, 248–258.