HERMITE-GAUSSIAN BEAMS IN INHOMOGENEOUS ELASTIC MEDIA

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Резюме: *Эрмит-Гауссевы пучки в трехмерной упругой неоднородной среде получены как асимптотические высокочастотоные решения уравнений движения упругой среды, сосредоточенные в окрестности лучей P и S волн. Уравнения движения упругой среды в этом случае приводит к параболическому уравнению (ур. Шредингера). Построены явные выражения для полной системы линейно-независимых решений параболического уравнения. При выводе используется метод операторов рождения и уничтожения, известный из квантовой механики.*

1. INTRODUCTION

Seismic research into the structure of the Earth's crust and upper mantle is now mostly being conducted in regions which are important from a geodynamic point of view. In these regions, the structure of the medium is usually very complicated, the velocity of propagation of seismic waves varies in dependence on all three coordinates, and the structural boundaries are strongly curved. One of the main trends of present theoretical seismology, therefore, is concentrated on developing a method of computing seismic wave fields in laterally inhomogeneous media with curved interfaces. The demand for similar methods is now also increasing in seismic oil prospection, because seismic prospection is being conducted in increasingly complicated areas and at larger depths.

Mainly ray methods are now being used in numerical modelling of seismic wave fields in structurally complicated media, whose dimensions exceed the predominant wavelength considerably. Ray methods yield approximate asymptotic solutions of the elasto-dynamic equations for high frequencies. These methods have found a number of applications in several fields of seismology, however, they do have their disadvantages in actual calculations carried out for a certain fixed predominant frequency. They do not provide satisfactory results in singular regions such as the caustic region, or the transition region between the illuminated region and the ray shadow zone. Another disadvantage of ray methods is their sensitivity to the fine details of medium structure.

These disadvantages of ray methods are eliminated, to a certain extent, by the method of expanding the wave field into Gaussian beams, which represents a particular generalization of the ray methods. The principal role in the said method is played by Gaussian beams as an approximate solution of the wave or elasto-dynamic equations, concentrated in the neighbourhood of rays. The amplitudes of Gaussian beams decrease exponentially with the square of the distance from the central ray and, therefore, their amplitude profile in this direction is bell-shaped. It is because of this property that the beams are referred to as Gaussian. Gaussian beams were first derived as high-frequency asymptotic solutions of the wave equation by V. M. Babich in 1968 [1] in studying solutions concentrated in the neighbourhood of rays, see also [2]. In this case, the wave (elasto-dynamic) equation reduces to a parabolic (Schrödinger) equation. Since then, Gaussian beams have been the subject of a number of papers; for a review of various applications and lists of references, the reader is referred to [3—5] particularly.

However, Gaussian beams are not the only solution to the parabolic equation. There exists a complete system of linearly independent solutions to the parabolic equation, among which the Gaussian beam corresponds to the zero mode. Higher modes contain Hermite polynomials. That is why we may refer to these solutions of the parabolic equation as Hermite-Gaussian beams.

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In determining the complete system of Hermite-Gaussian beams in inhomogeneous media, we use the method of ladder operators (anihilation and creation operators, spectrum-generating operators) developed in treating some of the problems of quantum physics [7, 8]. The indicated procedure of deriving the system of linearly independent solutions to the parabolic equation has been adopted from the paper of N. J. Kirpichnikova [6]. However, the particular paper [6] does not go as far as the explicit form of Hermite-Gaussian beams.

It should be mentioned that Hermite-Gaussian beams may find applications in a number of topical seismic-wave problems. For example, any beam (with a non-Gaussian amplitude profile) may be constructed as a superposition of Hermite-Gaussian beams. Hermite-Gaussian beams will apparently also find applications in studying beam diffraction on a screen, cuspidal points of a boundary, etc.

2. PARABOLIC EQUATION

In investigating the high-frequency solutions of the elastodynamic equations, concentrated close to the rays of P- and S-waves, it is very convenient to use the orthogonal ray-centered coordinates (σ, q^1, q^2). Here σ is the parameter along the ray, related to the arclength s as

$$\sigma(s) = \sigma(s_0) + \int_{s_0}^{s} v(\xi)\, \mathrm{d}\xi\,,$$

v being the velocity of propagation of the corresponding wave; q^1 and q^2 denoting the Cartesian coordinates in the plane perpendicular to the ray at σ, with the origin at the ray.

Using these coordinates and introducing the substitution

$$u = (\varrho v)^{-1/2} \exp\left[\mathrm{i}\omega(\tau - t)\right] W\,, \tag{1}$$

where ϱ is the density and τ is the propagation time along the ray, we arrive asymptotically, at high frequencies ω, at the following equation for W [6, 4]:

$$\left(\mathrm{i}\frac{\partial}{\partial\sigma} + \tfrac{1}{2}\omega^{-1}\frac{\partial}{\partial q^B}\frac{\partial}{\partial q^B} - \tfrac{1}{2}\omega q^A v_{AB} q^B v^{-3}\right) W = 0\,. \tag{2}$$

Here v and v_{AB} are the functions of the coordinate σ only, v_{AB} denotes the second derivative of v with respect to q^A and q^B, with $v_{AB} = v_{BA}$. The capital-letter indices take the values 1 and 2. Pairs of identical indices indicate the summation.

The Gaussian beams, discussed earlier, may thus be described by Eq. (1) in which the complex solution of parabolic equation (2) has been substituted for W. Simultaneously, however, Eq. (2) may also be interpreted as a two-dimensional Schrödinger equation in the potential field $\frac{1}{2}q^A\, v_{AB}(\sigma)\, q^B\, v^{-3}(\sigma)$, where σ is time, and q^1 and q^2 are spatial coordinates; this problem and solution may easily be generalized for more dimensions.

The objective of this paper is to derive the complete system of solutions of parabolic equation (2) and thus to find the complete system of Hermite-Gaussian beams which may propagate in an elastic inhomogeneous medium. These solutions will be transformed into their final explicit form which can be simply realized by computer. Considerable attention is also devoted to the most general initial conditions for which Eq. (2) yields Hermite-Gaussian beams as the solution.

3. THE PARABOLIC OPERATOR AND OPERATORS COMMUTING WITH IT

3.1. Parabolic operator

To simplify Eq. (2) formally, we shall use new coordinates,

$$\xi^A = q^A \sqrt{\omega}\,, \tag{3}$$

and then solve the equation

$$\left[\mathrm{i} \frac{\partial}{\partial \sigma} + \tfrac{1}{2}\Delta - \tfrac{1}{2}\xi^A v_{AB} v^{-3} \xi^B \right] W = 0\,, \tag{4}$$

in which we have put

$$\Delta = \frac{\partial}{\partial \xi^B} \frac{\partial}{\partial \xi^B}\,. \tag{5}$$

Solving Eq. (4) represents finding the complete orthogonal system of functions from the operator kernel

$$T = \mathrm{i} \frac{\partial}{\partial \sigma} + \tfrac{1}{2}\Delta - \tfrac{1}{2}\xi^A v_{AB} v^{-3} \xi^B\,, \tag{6}$$

i.e. the maximum set of mutually perpendicular functions W satisfying the equation

$$TW = 0\,, \tag{7}$$

which is only Eq. (4) expressed formally in another way. We shall now refer to operator T, defined by Eq. (6), as the parabolic operator.

We shall seek the solution of Eq. (7) in space $L^2(\langle \sigma_0, \sigma_1 \rangle \times R^2)$ of Lebesgue square integrable functions, the size of the limited interval $\langle \sigma_0, \sigma_1 \rangle$ being arbitrary. In this space, operator (6) is defined densely.

Let us mention some of the properties of the operators and their commutators which are useful in seeking the complete system of solutions of Eq. (7). For more details see [8].

1. If the symmetric operators A and B commute, i.e. if

$$[A, B] = AB - BA = 0\,, \tag{8}$$

every eigen-subspace of operator B can be expressed as the direct sum of its intersections with the eigen-subspaces of operator A.

2. If

$$[H, A] = \alpha A \tag{9}$$

holds for operatos H and A, and if ψ is the eigenvector of operator H corresponding to the eigenvalue λ, $A\psi$ is (if $A\psi \neq 0$) the eigenvector of operator H corresponding to the eigenvalue $\lambda + \alpha$, since

$$H(A\psi) = ([H, A] + AH)\psi = \alpha A\psi + A\lambda\psi = (\alpha + \lambda) A\psi .$$

We then say that operator A is the ladder (spectrum-generating, creation) operator to H. In particular for commuting operators, every eigen-subspace of one operator is an invariant subspace of the other operator, which can be used to prove property 1.

3. By resolving the following equation, it is easy to prove that

$$[AB, C] = A[B, C] + [A, C] B \tag{10}$$

holds for the commutator of operators.

Note: The properties 1 to 3 holds in any subspace of the intersection of the domains of all performing operators and their products. It can be proved, that the kernel of the operator (6) is the subspace of the domains of all operators used below.

3.2. Differential operators of the first order commuting with the parabolic operator

Since the parabolic operator (6) is quadratic in terms $\mathrm{i}\,\partial/\partial\xi^A$ and ξ^A, it is difficult to find the complete system of solutions of equation (7) directly. We shall, therefore, attempt to find operators Λ_k linearly compiled of $\mathrm{i}\,\partial/\partial\xi^A$ and ξ^A which would commute with operator T (see (6)). We shall then be able to find the complete system of solutions of Eq. (7) by discreet solving the equation in the intersections of the eigen-subspaces of the commuting operators compiled of $\Lambda_1, \ldots, \Lambda_n$.

The general form of the desired operators Λ_k is

$$\Lambda_k = -\xi^A P_{Ak} - \mathrm{i}\frac{\partial}{\partial\xi^A} Q_{Ak} , \tag{11}$$

where P_{Ak} and Q_{Ak} are complex functions of the parameter σ.

We now compute the commutator of operators T and Λ_k,

$$[T, \Lambda_k] = -\frac{\partial}{\partial\xi^A} \cdot P_{Ak} + \tfrac{1}{2}\mathrm{i}(-v_{AB}v^{-3}\xi^B - \xi^B v_{BA}v^{-3})\, Q_{Bk} - \mathrm{i}\xi^A P'_{Ak} + \\ + \frac{\partial}{\partial\xi^A} Q'_{Ak} = \frac{\partial}{\partial\xi^A}(-P_{Ak} + Q'_{Ak}) + \mathrm{i}\xi^A(-v_{AB}v^{-3}Q_{Bk} - P'_{Ak}) , \tag{12}$$

in which we have put

$$P'_{Ak} = \mathrm{d}P_{Ak}/\mathrm{d}\sigma , \quad Q'_{Ak} = \mathrm{d}Q_{Ak}/\mathrm{d}\sigma \tag{13}$$

and made use of the symmetry

$$v_{AB} = v_{BA} . \tag{14}$$

We can see that, for the for the operators T and Λ_k to commute, the complex functions $P_{Ak}(\sigma)$ and $Q_{Ak}(\sigma)$ must satisfy the ordinary differential equations

(15) $$Q'_{Ak} = P_{Ak}\,, \quad P'_{Ak} = -v_{AB}v^{-3}Q_{Bk}$$

with any initial conditions.

Therefore, we shall now assume that the coefficients P_{Ak} and Q_{Ak} satisfy Eq. (15), the initial conditions being hitherto unspecified. We now compute the commutator of the operators Λ_k and Λ_l,

(16) $$[\Lambda_k, \Lambda_l] = \mathrm{i}(Q_{Ak}P_{Al} - P_{Ak}Q_{Al})\,,$$

and check that, Eq. (15) being satisfied, they do not depend on σ:

$$\frac{\mathrm{d}}{\mathrm{d}\sigma}[\Lambda_k, \Lambda_l] = \mathrm{i}\left(P_{Ak}P_{Al} - Q_{Ak}\frac{v_{AB}}{v^3}Q_{Bl} + Q_{Bk}\frac{v_{AB}}{v^3}Q_{Al} - P_{Ak}P_{Al}\right) = 0\,.$$

It is, therefore, sufficient to investigate the commutators of operators Λ_k and Λ_l at the initial point σ_0:

(17) $$[\Lambda_k, \Lambda_l] = \mathrm{i}\{Q_{Ak}(\sigma_0)\,P_{Al}(\sigma_0) - P_{Ak}(\sigma_0)\,Q_{Al}(\sigma_0)\}\,.$$

In virtue of Eq. (17), we cannot select the initial values $P_{Ak}(\sigma_0)$ and $Q_{Ak}(\sigma_0)$ to obtain three independent commuting operators Λ_k, but we can select $P_{A1}(\sigma_0)$, $Q_{A1}(\sigma_0)$, $P_{A2}(\sigma_0)$ and $Q_{A2}(\sigma_0)$ to render operators Λ_1 and Λ_2 independent and make them commute. Moreover, for each such selection we can adopt $P_{A3}(\sigma_0)$, $Q_{A3}(\sigma_0)$, $P_{A4}(\sigma_0)$ and $Q_{A4}(\sigma_0)$ to render the operators Λ_1, Λ_2, Λ_3 and Λ_4 independent and, in addition to the relation

(18) $$[\Lambda_1, \Lambda_2] = 0\,,$$

to have the following commutative relations valid:

(19) $$[\Lambda_1, \Lambda_4] = 0\,, \quad [\Lambda_2, \Lambda_3] = 0\,, \quad [\Lambda_3, \Lambda_4] = 0\,.$$

For convenience we shall now introduce

(20) $$\tilde{\Lambda}_1 = \Lambda_3\,, \quad \tilde{\Lambda}_2 = \Lambda_4\,,$$

(21) $$[\Lambda_1, \tilde{\Lambda}_1] = \lambda_1\,, \quad [\Lambda_2, \tilde{\Lambda}_2] = \lambda_2\,.$$

3.3. Other operators commuting with the parabolic operator

If we introduce the operators

(22) $$L_1 = \tilde{\Lambda}_1\Lambda_1\,, \quad L_2 = \tilde{\Lambda}_2\Lambda_2\,,$$

the following commutative relations will hold for them:

(23) $$[L_1, \Lambda_1] = -\lambda_1\Lambda_1\,, \quad [L_1, \tilde{\Lambda}_1] = \lambda_1\tilde{\Lambda}_1\,,$$
$$[L_2, \Lambda_2] = -\lambda_2\Lambda_2\,, \quad [L_2, \tilde{\Lambda}_2] = \lambda_2\tilde{\Lambda}_2\,,$$
$$[L_1, \Lambda_2] = 0\,, \quad [L_1, \tilde{\Lambda}_2] = 0\,, \quad [L_2, \Lambda_1] = 0\,, \quad [L_2, \tilde{\Lambda}_1] = 0\,, \quad [L_1, L_2] = 0\,.$$

We now have operators L_1 and L_2 commuting with operator T and with each other, We also have the ladder (or the spectrum-generating) operators Λ_1, $\tilde{\Lambda}_1$, Λ_2 and $\tilde{\Lambda}_2$.

For the eigenfunctions of operators L_1 and L_2, corresponding to different eigenvalues, to be mutually perpendicular, the operators L_1 and L_2 must be self-adjoint. Reverting to definitions (22), we can see that the operators Λ_A and $\tilde{\Lambda}_A$ have to be adjoint to one another:

$$\tilde{\Lambda}_A = \Lambda_A^+ . \tag{24}$$

Equation (11) then yields

$$\tilde{\Lambda}_A = -\xi^B P_{BA}^* - \mathrm{i}\, \partial/\partial\xi^B \,.\, Q_{BA}^* \tag{25}$$

the asterisk indicating complex conjugacy.

3.4. Fundamental eigenfunctions of operators L_1 and L_2

Since operators (22) are positively semi-definitive, the set of eigenvalues of operators L_1 and L_2 is limited from below by the eigenvalues μ_1 and μ_2, respectively. Assume the operator Λ_A to shift downwards to the eigenvalue μ_A. If this were not so, the operator $\tilde{\Lambda}_A$ would have to do so, and it would be sufficient to effect the change

$$\begin{gathered} \Lambda_A \to \tilde{\Lambda}_A\,, \quad \tilde{\Lambda}_A \to \Lambda_A\,, \\ L_A = \tilde{\Lambda}_A \Lambda_A \to L_A = \Lambda_A \tilde{\Lambda}_A\,, \quad \mu_A \to \mu_A + \lambda_A \end{gathered} \tag{26}$$

(no summation over A). We shall refer to the eigenvectors of operators L_1 and L_2, corresponding to the lowest eigenvalues μ_1 and μ_2, as fundamental eigenfunctions.

Assume W_{00} to be the common eigenvector of operators L_1 and L_2, corresponding to the minimum eigenvalues μ_1 and μ_2. $\Lambda_1 W_{00}$ and $\Lambda_2 W_{00}$ can then no longer be the eigenfunctions of the operators L_1 and L_2 and, therefore, necessarily

$$\Lambda_1 W_{00} = 0\,, \quad \Lambda_2 W_{00} = 0\,. \tag{27}$$

Since we have introduced the operators Λ_A in the form of (11), we may develop (27) as follows:

$$\xi^B P_{BA}(\sigma)\, W_{00} = -\mathrm{i}\, \partial W_{00}/\partial \xi^B\, Q_{BA}(\sigma)\,. \tag{28}$$

The solution to Eq. (28) reads

$$W_{00} = A_{00}(\sigma) \exp\left[\tfrac{1}{2}\mathrm{i}\xi^A M_{AB}(\sigma)\, \xi^B\right], \tag{29}$$

where function $M_{AB}(\sigma)$ must satisfy the equation

$$\tfrac{1}{2}(M_{AB} + M_{BA})\, Q_{BC} = P_{AC}\,, \tag{30}$$

as can easily be proved by substituting (29) into (28).

Components M_{12} and M_{21} do not occur in (30) on their own, but only the sum $M_{12} + M_{21}$ has the meaning of a coefficient function with the product $\xi^1\xi^2$. For the sake of simplicity and uniqueness we shall put $M_{12} = M_{21}$ as is usual with matrices of quadratic forms. M_{AB} is then a symmetric matrix in its real and imaginary part, and (30) will simplify to

$$M_{AB}Q_{BC} = P_{AC}\,. \tag{31}$$

We can see that the columns $\mathbf{Q}_1 = \begin{pmatrix} Q_{11} \\ Q_{21} \end{pmatrix}$ and $\mathbf{Q}_2 = \begin{pmatrix} Q_{12} \\ Q_{22} \end{pmatrix}$ cannot be linearly dependent, because the columns of matrix P_{AB} and, therefore, also the operators Λ_1 and Λ_2, defined by Eq. (11), would be linearly dependent in the same way. Q_{AB} must thus be a regular matrix and we are able to rewrite (31) to read

$$M_{AB} = P_{AC}Q_{CB}^{-1}\,. \tag{32}$$

Moreover, we must also require the imaginary part of matrix M_{AB} to be positively definitive for positive frequencies ω, because otherwise function W_{00}, defined by (29), would not belong to the space of square integrable functions involved.

We shall now make use of (31) to develop the initial values for the integration of Eqs (15) into the following form:

$$\begin{pmatrix} P_{11}(\sigma_0), & P_{12}(\sigma_0) \\ P_{21}(\sigma_0), & P_{22}(\sigma_0) \\ Q_{11}(\sigma_0), & Q_{12}(\sigma_0) \\ Q_{21}(\sigma_0), & Q_{22}(\sigma_0) \end{pmatrix} = \begin{pmatrix} M_{11}(\sigma_0), & M_{12}(\sigma_0) \\ M_{21}(\sigma_0), & M_{22}(\sigma_0) \\ 1, & 0 \\ 0, & 1 \end{pmatrix} \begin{pmatrix} Q_{11}(\sigma_0), & Q_{12}(\sigma_0) \\ Q_{21}(\sigma_0), & Q_{22}(\sigma_0) \end{pmatrix}. \tag{33}$$

We can express the complex conjugate matrix which yields the adjoint operators $\tilde{\Lambda}_1$ and $\tilde{\Lambda}_2$ similarly, and from Eq. (17) for the commutators of operators $\Lambda_1, \Lambda_2, \tilde{\Lambda}_1$ and $\tilde{\Lambda}_2$ we shall find that, for the commutative relations (18) and (19) to be satisfied, the commutator

$$\begin{aligned} [\Lambda_A, \Lambda_B] &= \mathrm{i}(Q_{CA}P^*_{CB} - P_{CA}Q^*_{CB}) = \\ &= \mathrm{i}(Q_{CA}M^*_{CD}Q^*_{DB} - Q_{DA}M_{CD}Q^*_{CB}) = \\ &= Q_{CA}\,2\,\mathrm{Im}\,(M_{CD})\,Q^*_{DB} = Q^*_{DB}\,2\,\mathrm{Im}\,(M_{DC})\,Q_{CA} \end{aligned} \tag{34}$$

must equal zero for $A \neq B$.

General initial conditions for matrices **P** and **Q** may be selected to render commutator (34) zero for $A \neq B$, for instance in the following manner. The following matrices (2×2) are chosen arbitrarily: **R** a real symmetric matrix (3 real constants), **T** a real diagonal matrix (2 real constants), **V** a real unitary matrix (1 real constant), **S** a real diagonal matrix (2 real constants), **U** a complex unitary matrix (4 real constants). The initial conditions we shall adopt are

$$\mathbf{P}(\sigma_0) = \mathbf{V}^T\mathbf{T}^T\mathbf{RUS} + \mathrm{i}\mathbf{V}^T\mathbf{T}^T\mathbf{US}\,, \quad \mathbf{Q}(\sigma_0) = \mathbf{V}^{-1}\mathbf{T}^{-1}\mathbf{US}\,. \tag{35}$$

In this case, the initial conditions are given by the choice of 12 real constants, the first 6 of which determine the initial value of matrix (32).

4. FUNDAMENTAL SOLUTION TO THE PARABOLIC EQUATION. GAUSSIAN BEAM

We still assume $P_{AB}(\sigma)$ and $Q_{AB}(\sigma)$ to be solutions of Eqs (15) with the initial conditions (35) and matrix $M_{AB}(\sigma)$ to be given by relation (32). Let us now solve Eq. (7) (which is the same as Eq. (4)) in the common eigen-subspace of operators L_1 and L_2 corresponding to the lowest eigenvalues $\mu_1 = 0$ and $\mu_2 = 0$. This subspace is the space of functions having the form of (29).

If we substitute a function of form (29) into Eq. (4), we arrive at

$$(36) \qquad (\mathrm{i}A'_{00}A_{00}^{-1} - \tfrac{1}{2}\xi^A M'_{AB}\xi^B - \tfrac{1}{2}\xi^A M_{AC}M_{CB}\xi^B + \\ + \tfrac{1}{2}\mathrm{i}M_{AA} - \tfrac{1}{2}\xi^A v_{AB} v^{-3}\xi^B)\, A_{00} \exp(\tfrac{1}{2}\mathrm{i}\xi^A M_{AB}\xi^B) = 0\,,$$

which is for $A_{00} \neq 0$ equivalent to equations

$$(37), (38) \qquad M'_{AB} + M_{AC}M_{CB} + v_{AB}v^{-3} = 0\,, \quad (\ln A_{00})' + \tfrac{1}{2}M_{AA} = 0\,.$$

If we multiply (37) from the right by matrix Q_{BD}, we can see that the system of equations we have thus obtained is, in virtue of Eq. (31), equivalent to Eqs (15) and, therefore, already identically satisfied.

Equation (38) yields the solution

$$(39) \qquad A_{00}(\sigma) = A_{00}(\sigma_0) \exp\left[-\tfrac{1}{2}\int_{\sigma_0}^{\sigma} M_{AA}(\sigma)\, \mathrm{d}\sigma\right],$$

or, using Eq. (32),

$$(40) \qquad A_{00}(\sigma) = A_{00}(\sigma_0)\,\{\det[Q_{AB}(\sigma)]\}^{-1/2}\,,$$

where we have made use of

$$(41) \qquad [\det(Q_{AB})]' = \det(Q_{CD})\,\mathrm{tr}\,(Q'_{EF}Q_{FG}^{-1})\,,$$

being true for any matrix Q_{AB}.

In the common eigen-subspace of operators L_1 and L_2, corresponding to the lowest eigenvalues, the solution to Eq. (4) is the function

$$(42) \qquad W_{00} = C_{00}\, J^{-1/2}(\sigma) \exp(\tfrac{1}{2}\mathrm{i}\xi^A\, M_{AB}(\sigma)\, \xi^B)\,,$$

where we have put

$$(43) \qquad J(\sigma) = \det[Q_{AB}(\sigma)] = \exp\left\{\int_{\sigma_0}^{\sigma} \mathrm{tr}\,[M_{AB}(\sigma)]\, \mathrm{d}\sigma\right\}.$$

In (42) we wrote C_{00} instead of $A_{00}(\sigma_0)\sqrt{J(\sigma_0)}$.

5. OTHER SOLUTIONS TO THE PARABOLIC EQUATION. HERMITE-GAUSSIAN BEAMS

Since the other eigen-subspaces of operators L_1 and L_2 are obtained as a result of the ladder (or creation) operators $\tilde{\Lambda}_1$ and $\tilde{\Lambda}_2$ acting on the subspace corresponding to the lowest eigenvalues, and since operators $\tilde{\Lambda}_1$ and $\tilde{\Lambda}_2$ commute with operator T, we obtain the solution to Eq. (7) in these subspaces by operators $\tilde{\Lambda}_1$ and $\tilde{\Lambda}_2$ acting on function W_{00}, given by Eq. (42) [7, 8]:

$$W_{mn} = (\tilde{\Lambda}_1)^m (\tilde{\Lambda}_2)^n W_{00} . \tag{44}$$

To modify Eq. (44) we shall make use of

$$\tilde{\Lambda}_A \exp\left[\tfrac{1}{2}\mathrm{i}\xi^B M^*_{BC}(\sigma)\,\xi^C\right] = 0 , \tag{45}$$

which is an analogy of Eqs (27). If we put

$$\tilde{W}_{00} = \exp\left[\tfrac{1}{2}\mathrm{i}\xi^A M^*_{AB}(\sigma)\,\xi^B\right] , \tag{46}$$

the following will hold for every function W,

$$\tilde{\Lambda}_A \tilde{W}_{00} W = \left(-\xi^B P^*_{BA} - \mathrm{i}\frac{\partial}{\partial \xi^B} Q^*_{BA}\right)\tilde{W}_{00} W = \tilde{W}_{00}\left(-\mathrm{i}\frac{\partial}{\partial \xi^B} Q^*_{BA}\right) W , \tag{47}$$

and we may express (44) as

$$W_{mn} = \tilde{W}_{00}\left(-\mathrm{i}\frac{\partial}{\partial \xi^B} Q^*_{B1}\right)^m \left(-\mathrm{i}\frac{\partial}{\partial \xi^C} Q^*_{C2}\right)^n \tilde{W}_{00}^{-1} W_{00} = C_{00}\, J^{-1/2}(\sigma) \times \tag{48}$$

$$\times \exp\left(\tfrac{1}{2}\mathrm{i}\xi^A M^*_{AB}\xi^B\right)\left(-\mathrm{i}\frac{\partial}{\partial \xi^C} Q^*_{C1}\right)^m \left(-\mathrm{i}\frac{\partial}{\partial \xi^D} Q^*_{D2}\right)^n \exp\left[\tfrac{1}{2}\mathrm{i}\xi^E (M_{EF} - M^*_{EF})\,\xi^F\right] .$$

If we now use I_{AB} to denote the imaginary part of matrix M_{AB},

$$W_{mn} = W_{00} \exp\left(\xi^A I_{AB}\xi^B\right)\left(-\mathrm{i}\frac{\partial}{\partial \xi^C} Q^*_{C1}\right)^m \left(-\mathrm{i}\frac{\partial}{\partial \xi^D} Q^*_{D2}\right)^n \exp\left(-\xi^E I_{EF}\xi^F\right) , \tag{49}$$

and, therefore,

$$W_{mn} = R_{mn}(\sigma, \xi^1, \xi^2)\, W_{00} , \tag{50}$$

where R_{mn} are polynomials in ξ^1 and ξ^2 with coefficients dependent on σ:

$$R_{mn} = \exp\left(\xi^A I_{AB}\xi^B\right)\left(-\mathrm{i}\frac{\partial}{\partial \xi^C} Q^*_{C1}\right)^m \left(-\mathrm{i}\frac{\partial}{\partial \xi^D} Q^*_{D2}\right)^n \exp\left(-\xi^E I_{EF}\xi^F\right) . \tag{51}$$

6. EXPLICIT EXPRESSION OF HERMITE-GAUSSIAN BEAMS USING HERMITE POLYNOMIALS

Using $\boldsymbol{l}_1 = (l_{11}, l_{12})$, $\boldsymbol{l}_2 = (l_{21}, l_{22})$ for unit eigenvectors and λ_1, λ_2 for eigenvalues of the imaginary part I_{AB} of matrix M_{AB}, we construct the matrix

$$J_{AB} = \begin{pmatrix} l_{11}\sqrt{\lambda_1}, & l_{12}\sqrt{\lambda_1} \\ l_{21}\sqrt{\lambda_2}, & l_{22}\sqrt{\lambda_2} \end{pmatrix}, \tag{52}$$

for which

$$J_{CA} J_{CB} = I_{AB}\,. \tag{53}$$

Instead of the coordinates ξ^1 and ξ^2, introduced by (3), we shall now use the coordinates

$$\eta^A = J_{AB}\xi^B = J_{AB} q^B \sqrt{\omega}\,. \tag{54}$$

This relation may be expressed in vectorial form:

$$\eta^1 = (\boldsymbol{l}_1 \boldsymbol{q})\sqrt{(\omega\lambda_1)}\,, \quad \eta^2 = (\boldsymbol{l}_2 \boldsymbol{q})\sqrt{(\omega\lambda_2)}\,, \tag{55}$$

where $\boldsymbol{q} = (q_1, q_2)$. In transforming to the new coordinate system, the following holds:

$$\frac{\partial}{\partial \xi^A} = \frac{\partial \eta^B}{\partial \xi^A}\frac{\partial}{\partial \eta^B} = J_{BA}\frac{\partial}{\partial \eta^B}\,. \tag{56}$$

We now substitute from (54) and (56) into (51),

$$R_{mn} = \exp(\eta^A\eta^A)\left(-\mathrm{i}Q^*_{C1} J_{BC}\frac{\partial}{\partial \eta^B}\right)^m \left(-\mathrm{i}Q^*_{E2} J_{DE}\frac{\partial}{\partial \eta^D}\right)^n \exp(-\eta^F\eta^F)\,.$$

If we denote by

$$B_{AB} = J_{BC} Q^*_{CA} \tag{57}$$

the components of the matrix Hermite-adjoint to matrix

$$B^+_{AB} = J_{AC} Q_{CB}\,, \tag{58}$$

we can then write

$$R_{mn} = \exp(\eta^A\eta^A)\left(-\mathrm{i}B_{1B}\frac{\partial}{\partial \eta^B}\right)^m \left(-\mathrm{i}B_{2C}\frac{\partial}{\partial \eta^C}\right)^n \exp(-\eta^D\eta^D)\,.$$

After carrying out the multiplications,

$$R_{mn} = (-\mathrm{i})^{m+n}\exp(\eta^A\eta^A)\left[\sum_{j=0}^{m}\binom{m}{j}\left(B_{11}\frac{\partial}{\partial \eta^1}\right)^j \left(B_{12}\frac{\partial}{\partial \eta^2}\right)^{m-j}\right] \times$$

$$\times \left[\sum_{l=0}^{n}\binom{n}{l}\left(B_{21}\frac{\partial}{\partial \eta^1}\right)^l \left(B_{22}\frac{\partial}{\partial \eta^2}\right)^{n-l}\right]\exp(-\eta^B\eta^B) =$$

$$= (-\mathrm{i})^{m+n} \exp(\eta^A \eta^A) \left[\sum_{k=0}^{m+n} \sum_{j=0}^{k} \binom{m}{j} \binom{n}{k-j} B_{11}^j B_{12}^{m-j} \times \right.$$
$$\left. \times B_{21}^{k-j} B_{22}^{n-(k-j)} \left(\frac{\partial}{\partial \eta^1} \right)^k \left(\frac{\partial}{\partial \eta^2} \right)^{m+n-k} \right] \exp(-\eta^B \eta^B),$$

which we can adjust to read

$$(59) \qquad R_{mn} = (\mathrm{i})^{m+n} \sum_{k=0}^{m+n} \left[\sum_{j=0}^{k} \binom{m}{j} \binom{n}{k-j} B_{11}^j B_{21}^{k-j} B_{12}^{m-j} B_{22}^{n-(k-j)} \right] \times$$
$$\times H_k(\eta^1) \, H_{m+n-k}(\eta^2),$$

where

$$(60) \qquad H_k(z) = (-1)^k \exp(z^2) \frac{\mathrm{d}^k}{\mathrm{d}z^k} \exp(-z^2)$$

are the Hermite polynomials of k-th degree. In (59) we put $\binom{m}{j} = 0$ for $j > m$ and we substitute from (54) or (55) for η^A into the Hermite polynomials.

The functions

$$W_{mn}(\sigma, q^1, q^2) = R_{mn}(\sigma, q^1, q^2) \, W_{00}(\sigma, q^1, q^2),$$

where W_{00} is defined by (42), are orthogonal in space $L^2(\langle \sigma_0, \sigma_1 \rangle \times R^2)$ of Lebesgue square integrable functions and their norm is

$$(61) \qquad \|W_{mn}\| = |C_{00}| \, |\sigma_1 - \sigma_0| \sqrt{(\pi \cdot 2^{m+n} m! \, n! \, L_{11}^{m-1/2} L_{22}^{n-1/2} \omega^{-1})}.$$

L_{11} and L_{22} are diagonal elements of the matrix

$$(62) \qquad \mathbf{L} = \mathbf{BB}^+,$$

which is constant along the ray (see Eq. (34)) and, with a view to the initial conditions (35), diagonal:

$$(63) \qquad \mathbf{L} = \mathbf{SS}.$$

7. SPECIAL CASES OF HERMITE-GAUSSIAN BEAMS

In certain special cases, the explicit relations (59) for R_{mn} will simplify considerably. We shall mention two of these cases.

a) If matrix $\mathbf{B}$ is diagonal along the whole ray, which may be the case in certain special types of media, e.g., in a 1-D medium in which the velocity of wave propagation only depends on one of the three spatial coordinates, Eq. (59) will reduce to

$$(64) \qquad R_{mn} = (\mathrm{i})^{m+n} B_{11}^m B_{22}^n H_m(\eta^1) \, H_n(\eta^2).$$

b) We shall arrive at the 2-D case by replacing all the 2-D matrices in the 3-D case with ordinary functions. The solution, given by Eqs (42), (50) and (51), will then simplify to

$$W_m = (\mathrm{i}I^{1/2}Q^*)^m H_m(I^{1/2}\xi)\, C_0 Q^{-1/2} \exp\left(\tfrac{1}{2}\mathrm{i}\xi M\xi\right). \tag{65}$$

The relation derived for 2-D Hermite-Gaussian beams has already been given in [5].

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