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Linearized Solutions of Kinematic Problems of Seismic Body Waves in Inhomogeneous Slightly Anisotropic Media

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Abstract. The linearization approach to the evaluation of travel-times of seismic body waves propagating in inhomogeneous, slightly anisotropic media is discussed. General linearization equations are specified both for quasi-compressional and quasi-shear waves. Various situations of seismological interest are investigated in detail. This applies, e.g., to the situation where the unperturbed medium is isotropic and to the case where the unperturbed ray is a plane curve. The numerical examples presented suggest that the linearization approach gives travel-times of seismic body waves with a accuracy sufficient to solve direct and inverse kinematic problems for inhomogeneous anisotropic models of the Earth's crust and the uppermost mantle.

Key words: Elastic anisotropy – Direct kinematic problems – Inverse kinematic problems

Introduction

The ray method was first applied to inhomogeneous anisotropic media by Babich (1961). In principle, the ray tracing and travel-time computations can be performed in arbitrary inhomogeneous anisotropic media where all the 21 elastic parameters change with all the three coordinates. The ray tracing systems for general anisotropic media are derived and discussed in detail in Červený (1972) (see also Červený et al., 1977; Crampin, 1981). The evaluation of rays and travel-times of seismic body waves in inhomogeneous anisotropic media is now a well-understood problem. Various computer programs for simpler anisotropic media (e.g., transversely isotropic) have been written. Some numerical examples can be found in Červený and Pšenčík (1972), Červený et al. (1977), and in Jech (in press 1982). The computations are straightforward, but rather lengthy. They may be simply applied to the solution of direct kinematic problems, but it would be rather time-consuming to try to use these methods to solve some inverse kinematic problems for inhomogeneous anisotropic media.

A simpler procedure for the solution of both direct and inverse kinematic problems in inhomogeneous, slightly anisotropic media is based on a linearization. A linearization procedure for the solution of direct and inverse kinematic problems in laterally inhomogeneous isotropic media was suggested by Romanov (1972, 1978). The procedure has been used successfully in the inversion of travel-time data for laterally heterogeneous isotropic structures (Alekseev et al., 1970; Firbas, 1981; Novotný, 1981). Romanov (1978) was also the first to derive the linearized equation for one special simple case of an anisotropic medium (which can be applied, e.g., to quasi-shear SH waves in a transversely isotropic medium). For general anisotropic media, the linearization equations are derived in Červený (in press 1981). This reference, however, gives only the derivation of general equations; they are not discussed there from a seismological point of view. Similar linearization equations have been obtained independently by Hanyga (personal communication 1981).

In this paper, the linearized approach to the evaluation of travel-times of seismic body waves in a generally inhomogeneous, slightly anisotropic medium is investigated in greater detail. Attention is devoted to both quasi-compressional and quasi-shear waves. Explicit solutions are written for several situations of practical interest in seismology. This applies, e.g., in case where the unperturbed medium is isotropic, and where the ray in a unperturbed medium is a plane curve.

Two numerical examples are presented. They suggest that the accuracy of the linearization formulae will be high enough to solve both the direct and inverse kinematical problems in an inhomogeneous, slightly anisotropic Earth's crust and in the uppermost mantle.

The proposed method can be used in 3D laterally inhomogeneous, slightly anisotropic media. The unperturbed medium may be either isotropic or anisotropic. In this sense, the method gives some generalization of the well-known approach suggested by Backus (1965) which has been broadly used to investigate anisotropy in the uppermost mantle. Backus' method can be applied to homogeneous, slightly anisotropic media and starts from an isotropic unperturbed medium.

Linearization of Travel Times in Inhomogeneous Anisotropic Media

Let us consider an inhomogeneous anisotropic medium described by 21 elastic parameters c_{ijkl} and by the density ρ. The elastic parameters c_{ijkl}, density ρ and their derivatives are assumed to be continuous functions of Cartesian coordinates x_i, $i=1,2,3$. Instead of the elastic parameters c_{ijkl} we shall use the parameters

$$a_{ijkl}=c_{ijkl}/\rho, \tag{1}$$

and we shall also call them elastic parameters, for simplicity.

We shall investigate the propagation for an elastic body wave in the medium described above. We describe its wavefront by the equation

$$t=\tau(x_i). \tag{2}$$

A very important role in the investigation of anisotropic media is played by the 3×3 symmetric matrix $\boldsymbol{\Gamma}$, whose elements are given by the expressions

$$\Gamma_{jk}=p_i\, p_l\, a_{ijkl}, \tag{3}$$

where p_1, p_2, p_3 are components of the slowness vector $\mathbf{p}$, $p_i=\partial\tau/\partial x_i$. In Eq. (3) and throughout this paper, the Einstein summation convention is used.

The matrix $\boldsymbol{\Gamma}$ has three eigenvalues, G_1, G_2, G_3. They are solutions of the characteristic equation

$$\operatorname{Det}(\Gamma_{jk}-G_m\,\delta_{jk})=0, \tag{4}$$

where δ_{jk} is the Kronecker delta, $\delta_{jk}=1$ for $j=k$, $\delta_{jk}=0$ for $j\neq k$.

Three independent wavefronts can propagate in the anisotropic medium. The propagation of any of these wavefronts is controlled by a non-linear partial differential equation of the first order (also called the "eikonal equation")

$$G_m(p_1, p_2, p_3, x_1, x_2, x_3)=1, \tag{5}$$

$m=1,2,3$, $p_i=\partial\tau/\partial x_i$. One of the wavefronts (say, $m=1$) corresponds to the so-called quasi-compressional wave, the other two wave-fronts ($m=2,3$) to two different quasi-shear waves. In the degenerate case of two identical eigenvalues, there will be only two independent wavefronts. This applies, e.g., to the isotropic media, where $G_1=\alpha^2 p_i p_i$, $G_2=G_3=\beta^2 p_i p_i$; α and β being the velocities of the compressional and shear waves, respectively.

If the eikonal equations are known, it is not difficult to write the ray tracing system for any of the three waves propagating in an inhomogeneous anisotropic medium, see Červený (1972). The ray tracing system can be used to evaluate the trajectory of the ray, the components of the slowness vector p_i and the travel-times at any point of the ray. For general types of anisotropic media, however, the ray tracing systems are rather cumbersome; the rays are mostly 3D spatial curves, with a non-zero torsion.

A simpler procedure for the evaluation of travel-times in slightly anisotropic inhomogeneous media was suggested by Červený (in press 1981). The procedure is based on linearization. Here we shall only shortly summarize the results, without deriving them.

We shall consider a medium H^0 with elastic parameters a^0_{ijkl} and call it the "unperturbed" medium. We now change the elastic parameters slightly. The new medium, denoted by H, will be described by the elastic parameters a_{ijkl},

$$a_{ijkl}=a^0_{ijkl}+a^1_{ijkl}, \tag{6}$$

where a^1_{ijkl} represent "small corrections" to or "small perturbations" of a^0_{ijkl}.

Let us now specify two points, M^0 and M, with coordinates x^0_i and x_i. We select one of the three waves propagating in the anisotropic medium and denote by L^0 the ray connecting the points M^0 and M and by $\tau^0(x^0_i, x_i)$ the travel-time along L^0 from M^0 to M in the H^0 (unperturbed) medium. The following expression can then be written for the travel-time $\tau(x^0_i, x_i)$ from M^0 to M in the H medium (described by the parameters a_{ijkl}),

$$\tau(x^0_i, x_i)=\tau^0(x^0_i, x_i)+\tau^1(x^0_i, x_i), \tag{7}$$

where $\tau^1(x^0_i, x_i)$ is a small correction to the travel-time $\tau^0(x^0_i, x_i)$ due to the perturbations in the elastic parameters, a^1_{ijkl}.

The basic linearization formula, derived by Červený (in press 1981), gives a linear relation between $\tau^1(x^0_i, x_i)$ and a^1_{ijkl},

$$\tau^1(x^0_i, x_i)=-\tfrac{1}{2}\int_{L^0}\left(\frac{\partial G_m}{\partial a_{ijkl}}\right)_0 a^1_{ijkl}\, d\tau^0. \tag{8}$$

The integral is taken along the ray L^0, $d\tau^0$ is the infinitesimal time increment along L^0. The derivative $\partial G_m/\partial a_{ijkl}$ is determined in the unperturbed medium H^0, and contains only the unperturbed slowness vector components.

Thus, we have obtained a very important result: To determine the travel-time correction τ^1, we can just integrate the small perturbations of elastic parameters a^1_{ijkl} (multiplied by some weighting function) along the unperturbed ray L^0; we do not need to evaluate the new ray in the perturbed medium. A similar result is well known for isotropic media, see Romanov (1972), Gubbins (1981).

Equation (8) is quite general. The expressions for G_m are, however, rather complicated, except for cases when the characteristic equation (4) factorizes (see below). When the eigenvalue G_m does not coincide with any of the two remaining eigenvalues, the derivatives can be found using the theorem of implicit functions directly from (4). We then obtain

$$\left(\frac{\partial G_m}{\partial a_{ijkl}}\right)_0=(p_i\, p_l\, D_{jk}/D)_0, \tag{9}$$

where

$$\begin{aligned}
&D_{11}=(\Gamma_{22}-1)(\Gamma_{33}-1)-\Gamma^2_{23},\\
&D_{22}=(\Gamma_{11}-1)(\Gamma_{33}-1)-\Gamma^2_{13},\\
&D_{33}=(\Gamma_{11}-1)(\Gamma_{22}-1)-\Gamma^2_{12},\\
&D=D_{11}+D_{22}+D_{33},\\
&D_{12}=D_{21}=\Gamma_{13}\Gamma_{23}-\Gamma_{12}(\Gamma_{33}-1),\\
&D_{13}=D_{31}=\Gamma_{12}\Gamma_{23}-\Gamma_{13}(\Gamma_{22}-1),\\
&D_{23}=D_{32}=\Gamma_{12}\Gamma_{13}-\Gamma_{23}(\Gamma_{11}-1),\\
&\Gamma_{jk}=p_i\, p_l\, a_{ijkl}.
\end{aligned} \tag{10}$$

Again, p_i $(i=1,2,3)$ denote the components of the slowness vector in the H^0 medium, along the ray L^0.

These are known, being obtained as a by-product in the ray tracing of L^0.

Inserting (9) into (8) yields

$$\tau^1(x_i^0, x_i) = -\tfrac{1}{2}\int_{L^0} (p_i p_l D_{jk}/D)_0\, a_{ijkl}^1\, d\tau^0. \tag{11}$$

Let us compare Eqs. (8) and (11). Equation (8) can be used generally, but it leads to applicable results only in the case when analytical expressions for G_m are simple. Equation (11) gives the results analytically, in a closed form, for an arbitrary anisotropic medium. Some complications in the application of (11), however, appear when the unperturbed medium H^0 is isotropic. For quasi-P waves, Eq. (11) can be used generally even in the isotropic case. For S waves, its application is not strictly permitted, as the two eigen-values corresponding to shear waves coincide. But the equation yields certain interesting results even in this case, as shown below.

Isotropic Unperturbed Medium

Equations (8) and (11) give especially simple results when the unperturbed medium H^0 is isotropic. This situation has great practical importance. The evaluation of rays and travel-times in an inhomogeneous isotropic medium is now a well-understood problem, even for laterally inhomogeneous media. A large number of programs is available for computing rays in isotropic media. Moreover, the computation of rays in isotropic media is fast and not so time consuming as in anisotropic media. In this section, therefore, we shall specify the linearization formulae for the case where the unperturbed medium H^0 is isotropic. We shall also try to show what can be obtained for S waves in this case.

To do this, we must first find the meaning of $(D_{jk}/D)_0$ in (11) for isotropic media. Let us denote Lame's elastic parameters by λ and μ; $\alpha = (\lambda + 2\mu)^{\frac{1}{2}}/\rho^{\frac{1}{2}}$ and $\beta = (\mu/\rho)^{\frac{1}{2}}$ being the velocities of compressional (P) and shear (S) waves. To simplify the following equations, we shall also use the notations

$$\Lambda = \lambda/\rho, \qquad M = \mu/\rho. \tag{12}$$

We can then write

$$a_{ijkl} = \Lambda\,\delta_{ij}\,\delta_{kl} + M(\delta_{ik}\,\delta_{jl} + \delta_{il}\,\delta_{jk}), \tag{13}$$

This yields

$$\Gamma_{jk} = (\Lambda + M)\,p_j p_k + M(p_i p_i)\,\delta_{jk}. \tag{14}$$

For $D = D_{11} + D_{22} + D_{33}$ we obtain

$$D = (1 - M p_i p_i)(3 - (2\Lambda + 5M)\,p_k p_k). \tag{15}$$

As we can see from (15), the function D vanishes for $p_i p_i = 1/M = 1/\beta^2$, i.e. along the ray of the S wave. Fortunately, the factor $(1 - M p_i p_i)$ will also appear in the expression for D_{jk} and D_{jk}/D remains fully determined, even for $p_i p_i \to 1/\beta^2$. After some manipulation, we obtain

$$\frac{D_{jk}}{D} = \delta_{jk} + \frac{(\Lambda + M)\,p_j p_k - \delta_{jk}[2 - (\Lambda + 3M)\,T]}{3 - (2\Lambda + 5M)\,T}, \tag{16}$$

where $T = p_k p_k$.

For a P wave, $T = 1/\alpha^2$ and (16) yields

$$D_{jk}/D = \alpha^2 p_j p_k. \tag{17}$$

Similarly, for an S wave, $T = 1/\beta^2$ and (16) yields

$$D_{jk}/D = \tfrac{1}{2}(\delta_{jk} - \beta^2 p_j p_k). \tag{18}$$

Let us now assume that the ray L^0 in the medium H^0 corresponds to a P wave. For the travel time of a quasi-compressional wave $\tau_P(x_i^0, x_i)$ in H we can then write the following expression (see Eq. 7),

$$\tau_P(x_i^0, x_i) = \tau_P^0(x_i^0, x_i) + \tau_P^1(x_i^0, x_i), \tag{19}$$

where τ_P^0 is the travel time of the P wave in the H^0 medium along the ray L^0 and τ_P^1 is given by the relation

$$\tau_P^1(x_i^0, x_i) = -\tfrac{1}{2}\int_{L^0} \alpha^2\, p_i p_j p_k p_l\, a_{ijkl}^1\, d\tau^0. \tag{20}$$

Here α denotes the velocity of the P waves in the H^0 medium. The components p_i correspond to a ray L^0 in the unperturbed medium H^0, $p_i p_i = 1/\alpha^2$.

The situation is more complicated if the ray in H^0 corresponds to an S wave. Even though two quasi-shear waves propagate in the anisotropic medium, Eq. (11) gives only one travel-time correction $\tau^1(x_i^0, x_i)$. It is not clear yet what the meaning of this correction is. We shall denote it by $\tau_S^1(x_i^0, x_i)$. We can then write

$$\tau_S(x_i^0, x_i) = \tau_S^0(x_i^0, x_i) + \tau_S^1(x_i^0, x_i), \tag{21}$$

where

$$\tau_S^1(x_i^0, x_i) = -\tfrac{1}{4}\int_{L^0} (\delta_{jk} - \beta^2 p_j p_k)\, p_i p_l\, a_{ijkl}^1\, d\tau^0. \tag{22}$$

Here β again denotes the velocity of S waves in the H^0 medium and the components p_i correspond to a ray L^0 in the unperturbed medium H^0, with $p_i p_i = 1/\beta^2$.

To decide the meaning of $\tau_S^1(x_i^0, x_i)$ in (22), if H medium is anisotropic, we can use a slightly different approach. Let us denote the travel times from M^0 to M of two quasi-shear waves by $\tau_{S1}(x_i^0, x_i)$ and $\tau_{S2}(x_i^0, x_i)$, and the corresponding eigenvalues by G_{S1} and G_{S2}. Then see (7),

$$\begin{aligned} \tau_{S1}(x_i^0, x_i) &= \tau_S^0(x_i^0, x_i) + \tau_{S1}^1(x_i^0, x_i), \\ \tau_{S2}(x_i^0, x_i) &= \tau_S^0(x_i^0, x_i) + \tau_{S2}^1(x_i^0, x_i), \end{aligned} \tag{23}$$

where $\tau_S^0(x_i^0, x_i)$ is the travel time of the S wave in the medium H^0 along the ray L^0. The functions τ_{S1}^1 and τ_{S2}^1 are given by (8), where G_m equals G_{S1} and G_{S2}, respectively. Equation (23) yields

$$\begin{aligned} &\tau_{S1}(x_i^0, x_i) + \tau_{S2}(x_i^0, x_i) \\ &= 2\tau_S^0(x_i^0, x_i) + \tau_{S1}^1(x_i^0, x_i) + \tau_{S2}^1(x_i^0, x_i), \end{aligned} \tag{24}$$

where

$$\begin{aligned} &\tau_{S1}^1(x_i^0, x_i) + \tau_{S2}^1(x_i^0, x_i) \\ &= -\frac{1}{2}\int_{L^0} \left\{\frac{\partial(G_{S1} + G_{S2})}{\partial a_{ijkl}}\right\}_0 a_{ijkl}^1\, d\tau^0. \end{aligned} \tag{25}$$

Thus we evaluate the superposition of travel-time corrections for both quasi-shear waves. We can find the derivatives for $G_{S1}+G_{S2}$ more simply than for G_{S1} and G_{S2} independently. We find them analytically and specify them for the anisotropic medium. Finally we obtain

$$\tau^1_{S1}(x^0_i, x_i)+\tau^1_{S2}(x^0_i, x_i) = -\tfrac{1}{2}\int_{L^0} (\delta_{jk}-\beta^2 p_j p_k)\, p_i p_l a^1_{ijkl}\, d\tau^0. \tag{26}$$

Comparing (22) and (26) we find that $\tau^1_S(x^0_i, x_i)$ computed in (22) is an average value of the travel-time correction for both quasi-shear waves.

When the two quasi-shear waves separate, we can write the linearizations for $S1$ and $S2$ independently. This situation will be considered below where we find that Eq. (26) is valid in all cases we investigate.

In the preceding equations, we used the notation a_{ijkl} for the elastic parameters (divided by the density), with four indices i, j, k, l. This form is particularly suitable in equations for a general anisotropic medium because of various symmetries and the Einstein summation convention. In practical applications, however, it is more common to use the notation A_{mn} for elastic parameters, with two indices m, n. The A_{mn} are derived from the a_{ijkl} in the well-known way: m corresponds to the first pair of indices, i, j, and n to the second pair, k, l. The correspondence $i, j\to m$ and $k, l\to n$ is as follows: $1,1\to 1$; $2,2\to 2$; $3,3\to 3$; $1,2\to 6$; $2,1\to 6$; $1,3\to 5$; $3,1\to 5$; $2,3\to 4$; $3,2\to 4$.

We shall now rewrite the basic equations (20) and (26) using the constants A_{mn} instead of a_{ijkl}. We shall again use the notation

$$A_{mn}=A^0_{mn}+A^1_{mn}, \tag{27}$$

where A^0_{mn} corresponds to the H^0 medium, and A_{mn} to the H medium.

When the ray L^0 in the isotropic H^0 medium corresponds to a P wave, we can rewrite the expression (20) for the travel-time correction of the quasi-compressional wave $\tau^1_P(x^0_i, x_i)$ in the following form

$$\begin{aligned}\tau^1_P(x^0_i, x_i) = &-\tfrac{1}{2}\int_{L^0} \alpha^2\{A^1_{11}p_1^4+A^1_{22}p_2^4+A^1_{33}p_3^4\\ &+2(A^1_{12}+2A^1_{66})p_1^2p_2^2+2(A^1_{13}+2A^1_{55})p_1^2p_3^2\\ &+2(A^1_{23}+2A^1_{44})p_2^2p_3^2+4A^1_{16}p_1^3p_2+4A^1_{15}p_1^3p_3\\ &+4(A^1_{14}+2A^1_{56})p_1^2p_2p_3+4A^1_{26}p_2^3p_1+4A^1_{24}p_2^3p_3\\ &+4(A^1_{25}+2A^1_{46})p_2^2p_1p_3+4A^1_{35}p_3^3p_1+4A^1_{34}p_3^3p_2\\ &+4(A^1_{36}+2A^1_{45})p_3^2p_1p_2\}\, d\tau^0.\end{aligned} \tag{28}$$

The symbols p_i again correspond to the components of the slowness vector in the unperturbed medium H^0, $p_i p_i=1/\alpha^2$.

Similarly, when the ray L^0 in the isotropic medium H^0 corresponds to an S wave, we get from (26), for the sum of the travel-time corrections of quasi-shear waves,

$$\begin{aligned}\tau^1_{S1}(x^0_i, x_i)&+\tau^1_{S2}(x^0_i, x_i)\\ = &-\tfrac{1}{2}\int_{L^0} \{[(A^1_{11}+A^1_{55}+A^1_{66})p_1^2+(A^1_{22}+A^1_{44}+A^1_{66})p_2^2\\ &+(A^1_{33}+A^1_{44}+A^1_{55})p_3^2+2(A^1_{16}+A^1_{26}+A^1_{45})p_1p_2\\ &+2(A^1_{15}+A^1_{35}+A^1_{46})p_1p_3+2(A^1_{24}+A^1_{34}+A^1_{56})p_2p_3]\\ &-\beta^2[A^1_{11}p_1^4+A^1_{22}p_2^4+A^1_{33}p_3^4+2(A^1_{12}+2A^1_{66})p_1^2p_2^2\\ &+2(A^1_{13}+2A^1_{55})p_1^2p_3^2+2(A^1_{23}+2A^1_{44})p_2^2p_3^2+4A^1_{16}p_1^3p_2\\ &+4A^1_{15}p_1^3p_3+4(A^1_{14}+2A^1_{56})p_1^2p_2p_3+4A^1_{26}p_2^3p_1\\ &+4A^1_{24}p_2^3p_3+4(A^1_{25}+2A^1_{46})p_2^2p_1p_3+4A^1_{35}p_3^3p_1\\ &+4A^1_{34}p_3^3p_2+4(A^1_{36}+2A^1_{45})p_3^2p_1p_2]\}\, d\tau^0.\end{aligned} \tag{29}$$

Equation (29) can also be written in a different form, which might be more useful in certain situations. If we insert $p_i p_i=1/\beta^2$ into (29), we obtain

$$\begin{aligned}\tau^1_{S1}(x^0_i, x_i)&+\tau^1_{S2}(x^0_i, x_i)\\ = &-\tfrac{1}{2}\int_{L^0} \beta^2\{(A^1_{55}+A^1_{66})p_1^4+(A^1_{44}+A^1_{66})p_2^4+(A^1_{44}+A^1_{55})p_3^4\\ &+(A^1_{11}+A^1_{22}+A^1_{44}+A^1_{55}-2A^1_{12}-2A^1_{66})p_1^2p_2^2\\ &+(A^1_{11}+A^1_{33}+A^1_{44}+A^1_{66}-2A^1_{13}-2A^1_{55})p_1^2p_3^2\\ &+(A^1_{22}+A^1_{33}+A^1_{55}+A^1_{66}-2A^1_{23}-2A^1_{44})p_2^2p_3^2\\ &+2(A^1_{26}+A^1_{45}-A^1_{16})p_1^3p_2+2(A^1_{35}+A^1_{46}-A^1_{15})p_1^3p_3\\ &+2(A^1_{24}+A^1_{34}-2A^1_{14}-3A^1_{56})p_1^2p_2p_3\\ &+2(A^1_{16}+A^1_{45}-A^1_{26})p_1p_2^3+2(A^1_{34}+A^1_{56}-A^1_{24})p_2^3p_3\\ &+2(A^1_{15}+A^1_{35}-2A^1_{25}-3A^1_{46})p_2^2p_1p_3\\ &+2(A^1_{15}+A^1_{46}-A^1_{35})p_1p_3^3+2(A^1_{24}+A^1_{56}-A^1_{34})p_2p_3^3\\ &+2(A^1_{16}+A^1_{26}-2A^1_{36}-3A^1_{45})p_1p_2p_3^2\}\, d\tau^0.\end{aligned} \tag{30}$$

Let us again emphasize that p_i denote the components of the slowness vector in the unperturbed H^0 medium. The quantities p_i may be replaced by the direction cosines n_i of the direction of the slowness vector (perpendicular to the wavefront). In the integrals for quasi-compressional P waves, we have $p_i=n_i/\alpha$, and in the integrals for quasi-shear waves $p_i=n_i/\beta$.

Especially simple formulae are obtained if the medium H (perturbed) is also isotropic. Using (13) for a^1_{ijkl} and inserting it into (20) and (22), we obtain without difficulty the following simple formulae

$$\tau^1_P(x^0_i, x_i)=\int_{L^0}\delta(1/\alpha)\, ds, \qquad \tau^1_S(x^0_i, x_i)=\int_{L^0}\delta(1/\beta)\, ds.$$

Here $\delta(1/\alpha)$ and $\delta(1/\beta)$ denote small corrections to the quantities $1/\alpha$ and $1/\beta$, respectively. The quantity ds is an elementary arc-length along the ray L^0.

These equations are well-known in seismology (Romanov, 1972; 1978). They have been used successfully in the solution of 2D inverse kinematic problems in inhomogeneous 2D isotropic media (Alekseev et al., 1970; Firbas, 1981; Novotný, 1981). They have also been used in the joint determination of velocity structure and hypocenter location (Gubbins, 1981).

Linearized Expressions for Ray Velocities

In this section, we shall present one simple application of the preceding theory – the linearized expressions for ray velocities. By the term ray velocities we understand the velocities with which the wave propagates along a ray (group velocities). It is not difficult to show (Backus, 1965) that the linearized expressions for ray velocities are equivalent to linearized expressions for the phase velocities; the differences between both the velocities are of a higher order. In strongly anisotropic media, of course, the ray velocities may be rather different from the phase velocities.

In this paper, we consider only slightly anisotropic media. We shall start with Eq. (7) which gives the travel-time from any point M^0 to a point M in a slightly anisotropic inhomogeneous medium. The expression (8) for the travel-time correction can generally be written in the following form

$$\tau^1(x_i^0, x_i) = -\tfrac{1}{2}\int_{L^0} W^0(x_i, p_i)\, d\tau^0. \tag{31}$$

Specific expressions for W^0 for various situations can be obtained from equations presented in the preceding section.

Let us now assume that both the media H^0 and H are homogeneous. Denote the distance between M^0 and M by r, the ray velocity in H^0 by v_0 and the same velocity in H by v. Equations (7) and (8) then yield

$$\frac{r}{v} = \frac{r}{v_0} - \frac{r}{2v_0} W^0, \tag{32}$$

and, consequently,

$$v = v_0(1-\tfrac{1}{2}W^0)^{-1}, \qquad v - v_0 = v_0\{(1-\tfrac{1}{2}W^0)^{-1} - 1\}. \tag{33}$$

As W^0 is small,

$$v - v_0 \sim \frac{v_0}{2} W^0. \tag{34}$$

Similarly, we can write

$$v^2 - v_0^2 \sim v_0^2 W^0. \tag{35}$$

When the medium H^0 is isotropic, we can write the following expressions for the velocity v_P of a quasi-compressional wave and the velocities of both quasi-shear waves v_{S1} and v_{S2}:

$$v_P^2 - \alpha^2 \sim \alpha^2 W^0, \qquad v_{S1}^2 + v_{S2}^2 - 2\beta^2 \sim \beta^2 W^0, \tag{36}$$

where W^0 can be obtained, e.g., from (20) and (26). Inserting $W^0 = \alpha^2 p_i p_j p_k p_l a_{ijkl}^1$ for P waves and $W^0 = (\delta_{jk} - \beta^2 p_j p_k) p_i p_l a_{ijkl}^1$ for S waves, see (20) and (26), we obtain

$$v_P^2 - \alpha^2 \sim n_i n_j n_l n_k a_{ijkl}^1, \tag{37'}$$

$$v_{S1}^2 + v_{S2}^2 - 2\beta^2 \sim n_i n_l a_{ijkl}^1 (\delta_{jk} - n_j n_k). \tag{37''}$$

The Eq. (37′) corresponds fully to that obtained by Backus (1965) for the phase velocities of quasi-compressional waves (Eq. (17) in Backus, 1965).

Another simple equation is obtained by the combination of both formulae (37′) and (37″). We take into account that the rays of P and S waves in a homogeneous isotropic medium are the same (a straight line between M^0 and M). From (37′) and (37″), we then obtain

$$v_P^2 + v_{S1}^2 + v_{S2}^2 - 2\beta^2 - \alpha^2 \sim n_i n_l a_{ijjl}^1. \tag{38}$$

An equivalent equation was also obtained by Backus (Eq. (28), Backus, 1965).

Plane Curve Unperturbed Ray

The formulae (12)–(30) are applicable when the unperturbed medium H^0 is isotropic. Now we return to a general case of an anisotropic medium H^0. The formulae (8) and (11) remain valid, both for quasi-compressional and quasi-shear waves. To specify these formulae for some special cases of practical interest, it will be useful here to write the expressions for Γ_{ij} explicitely. For a general anisotropic medium, we have

$$\begin{aligned}
\Gamma_{11} &= A_{11}p_1^2 + A_{66}p_2^2 + A_{55}p_3^2 \\
&\quad + 2A_{16}p_1p_2 + 2A_{15}p_1p_3 + 2A_{56}p_2p_3, \\
\Gamma_{22} &= A_{66}p_1^2 + A_{22}p_2^2 + A_{44}p_3^2 \\
&\quad + 2A_{26}p_1p_2 + 2A_{46}p_1p_3 + 2A_{24}p_2p_3, \\
\Gamma_{33} &= A_{55}p_1^2 + A_{44}p_2^2 + A_{33}p_3^2 \\
&\quad + 2A_{45}p_1p_2 + 2A_{35}p_1p_3 + 2A_{34}p_2p_3, \\
\Gamma_{12} &= \Gamma_{21} = A_{16}p_1^2 + A_{26}p_2^2 + A_{45}p_3^2 + (A_{12}+A_{66})p_1p_2 \\
&\quad + (A_{14}+A_{56})p_1p_3 + (A_{46}+A_{25})p_2p_3, \\
\Gamma_{13} &= \Gamma_{31} = A_{15}p_1^2 + A_{46}p_2^2 + A_{35}p_3^2 + (A_{14}+A_{56})p_1p_2 \\
&\quad + (A_{13}+A_{55})p_1p_3 + (A_{36}+A_{45})p_2p_3, \\
\Gamma_{23} &= \Gamma_{32} = A_{56}p_1^2 + A_{24}p_2^2 + A_{34}p_3^2 + (A_{25}+A_{46})p_1p_2 \\
&\quad + (A_{36}+A_{45})p_1p_3 + (A_{23}+A_{44})p_2p_3.
\end{aligned} \tag{39}$$

Let us now assume that the ray L^0 in the unperturbed medium H^0 is a plane curve and is fully situated in a plane Σ. We choose the Cartesian co-ordinate system x_1, x_2, x_3 so that the plane is described by the equation $x_2 = 0$. We shall assume that the plane $x_2 = 0$ is a vertical plane and that the x_3 axis corresponds to the depth axis, as is common in seismological applications. (The following investigations may be applied, however, to an arbitrary orientation of the plane Σ.) If the elastic constants do not depend locally on x_2 in the vicinity of L^0, the component p_2 of the slowness vector $\mathbf{p}$ does not change along the ray. We shall consider the case $p_2 = 0$. In addition, we assume

$$A_{14} = A_{16} = A_{34} = A_{36} = A_{45} = A_{56} = 0, \tag{40}$$

so that the matrix of elastic parameters is as follows

$$\begin{pmatrix}
A_{11} & A_{12} & A_{13} & 0 & A_{15} & 0 \\
 & A_{22} & A_{23} & A_{24} & A_{25} & A_{26} \\
 & & A_{33} & 0 & A_{35} & 0 \\
 & & & A_{44} & 0 & A_{46} \\
 & & & & A_{55} & 0 \\
 & & & & & A_{66}
\end{pmatrix}. \tag{41}$$

This is the most general selection of elastic parameters for which the ray L^0 is fully situated in the plane $x_2 = 0$. The model (41) includes, e.g., the monoclinic system with the symmetry plane coinciding with the plane $x_2 = 0$.

For Γ_{ij} (39) then yields

$$\begin{aligned}
\Gamma_{11} &= A_{11}p_1^2 + A_{55}p_3^2 + 2A_{15}p_1p_3, \\
\Gamma_{22} &= A_{66}p_1^2 + A_{44}p_3^2 + 2A_{46}p_1p_3, \\
\Gamma_{33} &= A_{55}p_1^2 + A_{33}p_3^2 + 2A_{35}p_1p_3,
\end{aligned}$$

$$\Gamma_{13}=\Gamma_{31}=A_{15}p_1^2+A_{35}p_3^2+(A_{13}+A_{55})p_1p_3,$$
$$\Gamma_{12}=\Gamma_{21}=\Gamma_{23}=\Gamma_{32}=0. \quad (42)$$

The characteristic equation (4) can be factorized in this case. We can rewrite it in the form

$$(\Gamma_{22}-G)\{(\Gamma_{11}-G)(\Gamma_{33}-G)-\Gamma_{13}^2\}=0. \quad (43)$$

Thus, two eigenvalues are solutions of the equation

$$(\Gamma_{11}-G)(\Gamma_{33}-G)-\Gamma_{13}^2=0. \quad (44)$$

If the plane $x_2=0$ is a vertical plane, the two solutions of (44) correspond to a quasi-compressional P and a quasi-SV waves. We can denote them by G_P and G_{SV}. The remaining eigenvalue G_{SH}, corresponding to a quasi-SH wave, is given by the equation

$$G_{SH}=\Gamma_{22}. \quad (45)$$

Thus, the derivatives of the eigenvalues can be obtained simply for quasi-SH waves. For G_P and G_{SV}, using (5) and the theorem on implicit functions, (44) yields

$$\frac{\partial G}{\partial A_{ij}}=\frac{1}{\Gamma_{11}+\Gamma_{33}-2}\,\frac{\partial[(\Gamma_{11}-1)(\Gamma_{33}-1)-\Gamma_{13}^2]}{\partial A_{ij}}. \quad (46)$$

By inserting (45) into (8), for the travel-time corrections of the quasi-SH wave we obtain

$$\tau_{SH}^1(x_i^0,x_i)=-\tfrac{1}{2}\int_{L^0}(A_{66}^1p_1^2+A_{44}^1p_3^2+2A_{46}^1p_1p_3)\,d\tau^0. \quad (47)$$

For both the quasi-compressional P and quasi-SV waves, we have formally the same equation,

$$\begin{aligned}\tau_{P,SV}^1(x_i^0,x_i)=&-\tfrac{1}{2}\int_{L^0}(\Gamma_{11}^0+\Gamma_{33}^0-2)^{-1}\\
&\cdot\{p_1^2(A_{55}^0p_1^2+A_{33}^0p_3^2+2A_{35}^0p_1p_3-1)A_{11}^1\\
&+p_3^2(A_{11}^0p_1^2+A_{55}^0p_3^2+2A_{15}^0p_1p_3-1)A_{33}^1\\
&+(A_{33}^0p_3^4+A_{11}^0p_1^4-2A_{13}^0p_1^2p_3^2-p_1^2-p_3^2)A_{55}^1\\
&+2p_1(p_3^3A_{33}^0+p_1p_3^2A_{35}^0-p_1^3A_{15}^0-p_1^2p_3A_{13}^0-p_3)A_{15}^1\\
&+2p_3(p_1^3A_{11}^0+p_1^2p_3A_{15}^0-p_1p_3^2A_{13}^0-p_3^3A_{35}^0-p_1)A_{35}^1\\
&-2p_1p_3(A_{15}^0p_1^2+A_{35}^0p_3^2+(A_{13}^0+A_{55}^0)p_1p_3)A_{13}^1\}\,d\tau^0.\end{aligned} \quad (48)$$

Here p_i are components of the slowness vector in the H^0 medium; they correspond to the relevant waves. They are different for a quasi-compressional P and a quasi-SV wave.

Thus, we have obtained separated expressions for the travel-time corrections corresponding to quasi-SV and quasi-SH waves.

Let us now specify Eqs. (47) and (48) for the isotropic H^0 medium. Then

$$A_{11}^0=A_{22}^0=A_{33}^0=\alpha^2,$$
$$A_{44}^0=A_{55}^0=A_{66}^0=\beta^2,$$
$$A_{12}^0=A_{13}^0=A_{23}^0=\alpha^2-2\beta^2,$$

with all other elastic constants vanishing. For quasi-shear waves we put $p_1^2+p_3^2=1/\beta^2$, for a quasi-compressional wave $p_1^2+p_3^2=1/\alpha^2$. From (48) we then obtain

$$\begin{aligned}\tau_P^1(x_i^0,x_i)&=-\tfrac{1}{2}\int_{L^0}\alpha^2\{A_{11}^1p_1^4+A_{33}^1p_3^4+2(A_{13}^1+2A_{55}^1)p_1^2p_3^2\\
&\quad+4A_{15}^1p_1^3p_3+4A_{35}^1p_1p_3^3\}\,d\tau^0,\end{aligned} \quad (49)$$

$$\begin{aligned}\tau_{SV}^1(x_i^0,x_i)=&-\tfrac{1}{2}\int_{L^0}\beta^2\{A_{55}^1p_1^4+A_{55}^1p_3^4\\
&+(A_{11}^1+A_{33}^1-2A_{55}^1-2A_{13}^1)p_1^2p_3^2\\
&+2(A_{15}^1-A_{35}^1)p_1p_3(p_3^2-p_1^2)\}\,d\tau^0.\end{aligned} \quad (50)$$

Together with (47), Eqs. (49) and (50) give a complete system of linearization equation for all the three waves propagating in the isotropic inhomogeneous medium under study, when the ray L^0 is situated in the plane $x_2=0$. From (47) and (50), we also easily obtain an expressions for the time difference between the travel-times of both quasi-shear waves. If we multiply the integrand of (47) by $\beta^2(p_1^2+p_3^2)=1$, we obtain

$$\begin{aligned}&\tau_{SV}^1(x_i^0,x_i)-\tau_{SH}^1(x_i^0,x_i)\\
&=-\tfrac{1}{2}\int_{L^0}\beta^2\{(A_{55}^1-A_{66}^1)p_1^4+(A_{55}^1-A_{44}^1)p_3^4\\
&\quad+(A_{11}^1+A_{33}^1-A_{44}^1-A_{66}^1-2A_{13}^1-2A_{55}^1)p_1^2p_3^2\\
&\quad+2(A_{15}^1-A_{46}^1-A_{35}^1)p_1p_3^3\\
&\quad+2(A_{35}^1-A_{46}^1-A_{15}^1)p_1^3p_3\}\,d\tau^0.\end{aligned} \quad (51)$$

Similarly we can check that Eqs. (47) and (50) yield a general formula (30) when we put $p_2=0$ and insert (40).

Similar formulae can easily be derived also for cases when the ray L^0 is situated completely in the plane $x_1=0$ or in the plane $x_3=0$. They can either be derived directly, or obtained by rotating the coordinate system. We shall consider here only one of these two cases, when the ray L^0 is situated in the plane $x_3=0$ (constant depth). A similar situation has often been considered in seismological applications and was also discussed in detail by Backus (1965). This may also be considered as a "model situation" for refraction shooting measurements, in which all the rays are "horizontal". (They are assumed to propagate just below the Mohorovičić discontinuity, which is assumed to be a surface of constant depth.)

We shall assume $p_3=0$ and put

$$A_{14}=A_{15}=A_{24}=A_{25}=A_{46}=A_{56}=0. \quad (52)$$

As in (40), this is the most general selection of elastic constants for which the ray L^0 is situated completely in the plane $x_3=0$. The model (52) includes, e.g., the monoclinic system with the symmetry plane coinciding with the plane $x_3=0$.

The characteristic Eq. (4) can then be factorized. The separated quasi-shear wave corresponds to the quasi-SV wave in this case, not to the quasi-SH wave as in the previous case. The quasi-SH wave is coupled with the quasi-compressional P wave. For the travel-time corrections we obtain general equations, similar to (47) and (48)

$$\tau_{SV}^1(x_i^0,x_i)=-\tfrac{1}{2}\int_{L^0}(A_{55}^1p_1^2+A_{44}^1p_2^2+2A_{45}^1p_1p_2)\,d\tau^0, \quad (53)$$

$$\tau_{P,SH}^1(x_i^0,x_i)=-\tfrac{1}{2}\int_{L^0}(\Gamma_{11}^0+\Gamma_{22}^0-2)^{-1}$$

$$\cdot \{p_1^2(A_{66}^0 p_1^2 + A_{22}^0 p_2^2 + 2A_{26}^0 p_1 p_2 - 1) A_{11}^1 + p_2^2(A_{11}^0 p_1^2 + A_{66}^0 p_2^2 + 2A_{16}^0 p_1 p_2 - 1) A_{22}^1 + (p_1^4 A_{11}^0 + p_2^4 A_{22}^0 - 2A_{12}^0 p_1^2 p_2^2 - p_1^2 - p_2^2) A_{66}^1 + 2p_1(p_2^3 A_{22}^0 + p_1 p_2^2 A_{26}^0 - p_1^2 p_2 A_{12}^0 - A_{16}^0 p_1^3 - p_2) A_{16}^1 + 2p_2(A_{11}^0 p_1^3 - A_{12}^0 p_1 p_2^2 + A_{16}^0 p_1^2 p_2 - A_{26}^0 p_2^3 - p_1) A_{26}^1 - 2p_1 p_2(A_{16}^0 p_1^2 + A_{26}^0 p_2^2 + (A_{12}^0 + A_{66}^0) p_1 p_2) A_{12}^1\} \, d\tau^0. \tag{54}$$

The time differences corresponding to the quasi-compressional P and quasi-SH waves in (54) are distinguished by different p_i and $d\tau^0$. If the unperturbed H^0 medium is isotropic,

$$\tau_P^1(x_i^0, x_i) = -\tfrac{1}{2}\int_{L^0} \alpha^2 \{A_{11}^1 p_1^4 + A_{22}^1 p_2^4 + 2(A_{12}^1 + 2A_{66}^1) p_1^2 p_2^2 + 4A_{16}^1 p_1^3 p_2 + 4A_{26}^1 p_2^3 p_1\} \, d\tau^0, \tag{55}$$

$$\tau_{SH}^1(x_i^0, x_i) = -\tfrac{1}{2}\int_{L^0} \beta^2 \{A_{66}^1 p_1^4 + A_{66}^1 p_2^4 + (A_{11}^1 + A_{22}^1 - 2A_{12}^1 - 2A_{66}^1) p_1^2 p_2^2 + 2(A_{26}^1 - A_{16}^1) p_1 p_2 (p_1^2 - p_2^2)\} \, d\tau^0. \tag{56}$$

For the difference $\tau_{SV}^1 - \tau_{SH}^1$ we obtain

$$\tau_{SV}^1(x_i^0, x_i) - \tau_{SH}^1(x_i^0, x_i) = -\tfrac{1}{2}\int_{L^0} \beta^2 \{(A_{55}^1 - A_{66}^1) p_1^4 + (A_{44}^1 - A_{66}^1) p_2^4 + (A_{44}^1 + A_{55}^1 - A_{11}^1 - A_{22}^1 + 2A_{12}^1 + 2A_{66}^1) p_1^2 p_2^2 + 2(A_{45}^1 + A_{16}^1 - A_{26}^1) p_1^3 p_2 + 2(A_{45}^1 + A_{26}^1 - A_{16}^1) p_1 p_2^3\} \, d\tau^0. \tag{57}$$

The discussion and application of all the formulae presented above is straightforward.

Let us mention one exceptionally simple application of all the above formulae. If the unperturbed medium H^0 is homogeneous, the rays L^0 are straight lines. This, of course, does not mean that we can remove the integrals from the formulae presented above, as the medium H is inhomogeneous. However, it would be possible to replace the integrals by sums and specify the corrections of the elastic parameters A_{ij}^1 is some rectangular network. The values of A_{ij}^1 at the grid points of the network could then be found by some modification of the method suggested by Aki et al. (1977) for 3D isotropic media.

Numerical Examples

It is not simple to express analytically the accuracy of linearized equations. It would be necessary to devote a more detailed and extensive numerical study to this problem.

In this section, we shall present two simple numerical examples. As the computer programs to evaluate exact travel-times are now available for a vertically inhomogeneous, transversely isotropic medium (Červený et al., 1977), we shall use such a medium for our numerical experiments. For simplicity, we shall only consider quasi-compressional waves, not quasi-shear waves. For two selected models of vertically inhomogeneous transversely isotropic media, we shall compute the travel time both exactly and by linearization. We shall use Cartesian co-ordinates so that the ray L^0 is situated in the plane $x_2=0$. For computation of travel-time corrections, we shall use Eq. (49) with $A_{15}^1 = A_{35}^1 = 0$. The exact travel times are evaluated by numerical ray tracing, directly for the perturbed model of the medium specified by elastic parameters A_{mn}. In the linearization approach, we start with some unperturbed isotropic medium specified by parameters A_{mn}^0 and determine the unperturbed travel times τ^0 for this medium by well-known methods. Then we determine the small perturbations of elastic parameters $A_{mn}^1 = A_{mn} - A_{mn}^0$ and evaluate the travel time correction τ^1 by the linearization equation (49). Then, the travel time τ is obtained by (7), $\tau = \tau^0 + \tau^1$.

Table 1. Model of transversely isotropic medium used for computing the travel-times in Fig. 2. A_{11}, A_{33}, A_{55}, A_{66} and A_{13} are the depth-dependent elastic parameters divided by density (in km²/s²), δ is the coefficient of anisotropy

Depth (km)	A_{11}	A_{33}	A_{55}	A_{66}	A_{13}	δ
0	7.84	4.00	1.33	2.61	2.84	29%
1.5	12.25	7.24	2.43	4.08	4.48	23%
4.0	23.04	17.64	5.88	7.68	8.33	13%
18.0	42.25	38.44	12.81	14.08	14.66	5%
25.0	53.29	51.04	17.28	17.76	18.00	2%
50.0	74.82	62.41	21.06	24.92	26.53	9%

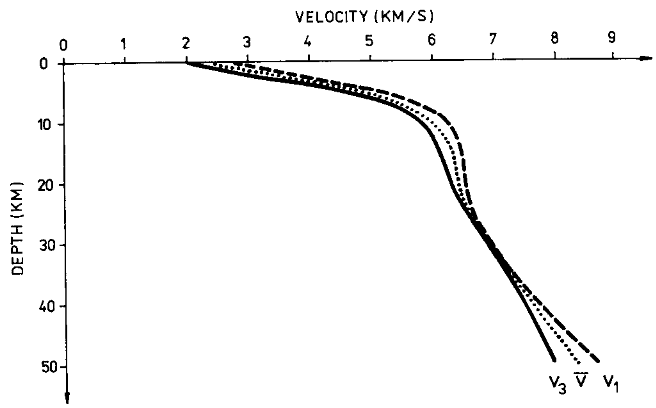

Fig. 1. Model used for computing the travel-time curves presented in Fig. 2. $V_1 = \sqrt{A_{11}}$ corresponds to the horizontal velocity, $V_3 = \sqrt{A_{33}}$ to the vertical velocity, $\bar{V}$ to the average velocity, $\bar{V} = \frac{1}{2}(V_1 + V_3)$

First Example

The model of the vertically inhomogeneous, transversely isotropic medium used for the computations is specified in Table 1 by the values of the elastic parameters A_{11}, A_{33}, A_{55}, A_{66}, A_{13} at depths of 0 km, 1.5 km, 4 km, 18 km, 35 km and 50 km. These values are interpolated by cubic splines to arbitrary depths. The quantity $V_1 = \sqrt{A_{11}}$ corresponds to the quasi-compressional velocity in the horizontal direction, $V_3 = \sqrt{A_{33}}$ in the vertical direction. These velocities are shown in Fig. 1. The coefficient of anisotropy δ (introduced here

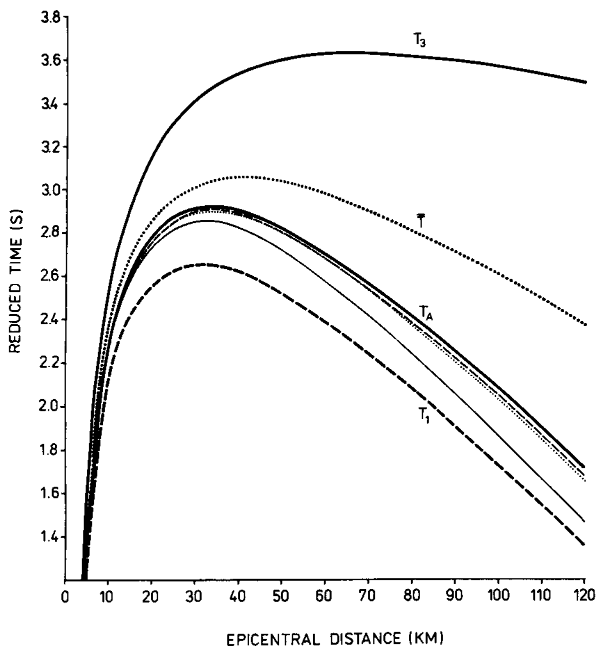

Fig. 2. Reduced travel-time curves computed for the models specified in Table 1 (see also Fig. 1). Reduction velocity is 6.0 km/s. The curve T_A is exact and corresponds to a transversely isotropic medium. The bold curves T_1, T_3 and $\bar{T}$ correspond to isotropic media with the velocities V_1, V_3 and $\bar{V}$, respectively. The remaining three curves (thin) correspond to a transversely isotropic medium and are obtained by linearization; the thin continuous curve from T_3, the thin *dashed curve* from T_1 and the *thin dotted curve* from $\bar{T}$

as a ratio $\delta = 100 \cdot (V_1 - V_3)/V_3$ varies with depth; it is very large close to the Earth's surface ($\delta \sim 30\%$) and smaller at larger depths, see Table 1 and Fig. 1.

The travel-time curve for this model of a transversely isotropic medium is shown in Fig. 2, by a bold line (denoted by T_A). The same figure also shows the travel-time curves for three isotropic media. The first corresponds to the medium described by the P velocity $V_1 = \sqrt{A_{11}}$ (denoted by T_1), the second by the P velocity $V_3 = \sqrt{A_{33}}$ (denoted by T_3) and the third by the average value $\bar{V} = \frac{1}{2}(\sqrt{A_{11}} + \sqrt{A_{33}})$ (denoted by $\bar{T}$). The differences between the travel-time curve corresponding to the anisotropic medium and the travel-time curves corresponding to the three isotropic media are rather large.

Figure 2 also shows three travel-time curves for the transversely isotropic medium specified by Table 1 obtained by linearization starting from the three isotropic media discussed above. We can see that these travel-time curves obtained by linearization are very close to the exact travel-times. For example, if we start with the "average" isotropic medium, the travel-time differences with respect to the exact travel-time do not exceed 0.04 s in the range of epicentral distances 10–120 km. In absolute terms, the travel-time differences at an epicentral distance of 120 km is less than 0.2%.

Second Example

The model is specified in Table 2, as in the first example. The horizontal and vertical P velocities V_1 and V_3

Table 2. Model of transversely isotropic medium used for computing the travel times in Fig. 4. A_{11}, A_{33}, A_{55}, A_{66} and A_{13} are the depth-dependent elastic parameters divided by density (in km^2/s^2), δ is the coefficient of anisotropy

Depth (km)	A_{11}	A_{33}	A_{55}	A_{66}	A_{13}	δ
0	4.84	4.00	1.33	1.61	1.73	10%
1.5	8.82	7.29	2.43	2.94	3.14	10%
4.0	27.34	17.64	5.88	7.11	7.60	10%
18.0	46.51	38.44	12.81	15.50	16.58	10%
25.0	62.73	51.04	17.28	20.91	21.89	10%
50.0	75.52	62.41	21.06	25.17	26.39	10%

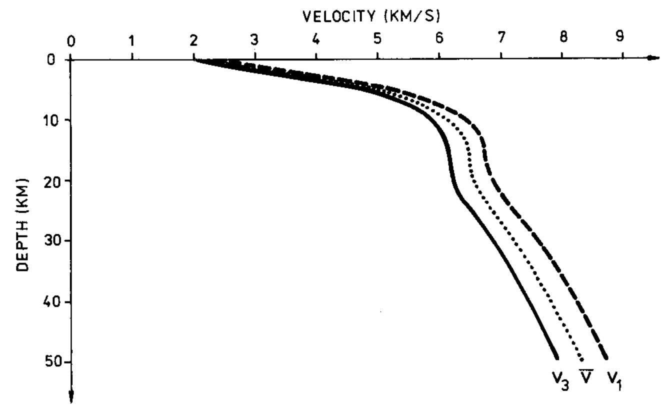

Fig. 3. Model used for computing the travel-time curves presented in Fig. 4. V_1, V_3 and $\bar{V}$ correspond to the horizontal, vertical and average velocity, respectively

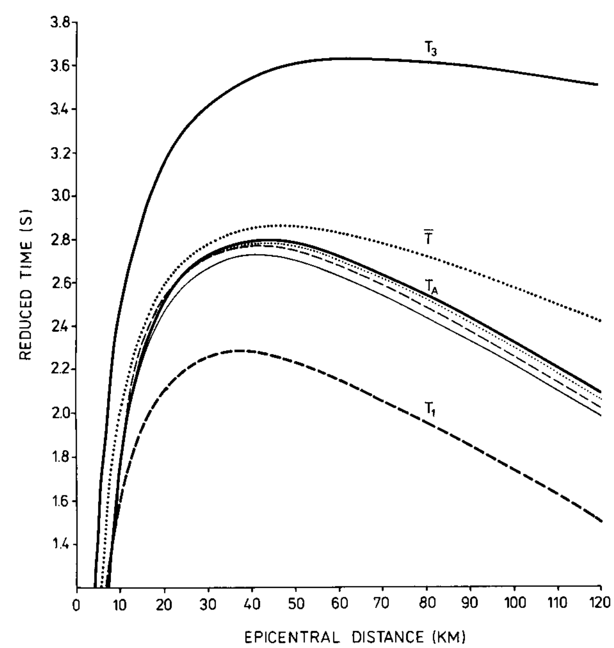

Fig. 4. Reduced travel-time curves computed for the models specified in Table 2 (see also Fig. 3). The reduction velocity is 6.0 km/s. The explanation of individual curves is the same as in Fig. 2

and the average velocity $\bar{V}$ are shown in Fig. 3. We can see that the anisotropy increased in the lower part of the model. The coefficient of anisotropy δ is nearly constant throughout the model, independent of depth, close to 10%. All the travel-time curves shown in Fig. 4 are constructed in the same way as in the preceding example. Again, when we construct the travel-time curve by linearization from the "average" isotropic medium, differences from exact travel-times do not exceed 0.05 s. In absolute terms, the travel-time difference at an epicentral distance of 120 km is less than 0.25%.

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