ELEMENTARY SEISMOGRAMS OF SEISMIC BODY WAVES IN DISSIPATIVE MEDIA

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Summary: *Approximate expressions for elementary seismograms of seismic body waves propagating in media with small causal absorption are derived. Special attention is devoted to modulated signals with a smooth envelope, for which especially simple formulae were obtained. The derived expressions give a good picture of all important effects of causal absorption, viz., the frequency dependent exponential decrease of amplitudes, the velocity dispersion related to absorption, and the decrease of the prevailing frequency.*

Let us first consider the propagation of a scalar plane wave $U(z, t)$ in a dissipative medium in the direction of the z-axis. Assume that $U(0, t) = x(t)$, where $x(t)$ is known. For $t > 0$, the wave is described by the formula

$$(1) \qquad U(z, t) = 2 \operatorname{Re} \int_0^\infty X(f) \exp\left[-\alpha(f)\, z + \mathrm{i}\, \Phi(f)\, z + \mathrm{i}\, 2\pi f(t - \tau_0)\right] \mathrm{d}f \,,$$

where $X(f)$ denotes the complex spectrum of $x(t)$, $\alpha(f)$ is the absorption coefficient, τ_0 and $\Phi(f)$ are given by formulae

$$(2) \qquad \tau_0 = z/v(f_0)\,, \quad \Phi(f) = 2\pi f[v^{-1}(f_0) - v^{-1}(f)]\,,$$

$v(f)$ being the frequency dependent velocity and f_0 an arbitrary reference frequency.

To evaluate (1) we must first specify the model of absorption. In case of causal absorption, the functions $\alpha(f)$ and $\Phi(f)$ cannot be chosen arbitrarily, they are related by the causality conditions. In this paper, we shall use the Futterman model of absorption [6], which has been used succesfully in many seismological applications. It would, however, be possible to use any other model, such as the model of Lomnitz [8], Kjartansson's model [7], non-causal models, etc.

For Futterman's model, the dispersion relations yield

$$(3) \qquad \alpha(f) = \pi f/(cQ_0)\,, \quad \Phi(f) = [2f/(cQ_0)] \ln (f/f_0)\,,$$

where c and Q_0 are parameters of the model. They can be determined from the known values of the velocity $v(f)$ and the $Q(f)$ factor at any specified frequency f_s (or from the known velocity $v(f)$ and absorption coefficient $\alpha(f)$ at f_s). Note that the product of the velocity $v(f)$ and of the $Q(f)$ factor is independent of frequency in Futterman's model, and it equals cQ_0 in Eqs (3). For details refer to [6].

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Inserting (3) into (1) yields

$$(4) \qquad U(z, t) = 2 \operatorname{Re} \int_0^\infty X(f) \exp\left[-\pi f t^* + 2\mathrm{i} f t^* \ln(f/f_0) + \mathrm{i}\, 2\pi f(t - \tau_0)\right] \mathrm{d}f ,$$

where t^* is a global absorption factor, broadly used in present seismology (see [3, 9, 10]). It is given by formula

$$(5) \qquad t^* = z/(cQ_0) .$$

The factor t^* is the higher, the longer the travel path z and the higher the absorption effects.

The evaluation of (4) is, in principle, simple, but rather time consuming. It is necessary to evaluate the integrand for a system of frequencies and then evaluate the Fourier integral (4) numerically. This procedure will make the computation of ray synthetic seismograms for dissipative media (in which the integrals (4) must be evaluated many times) rather slow. It is, therefore, useful to rewrite the integral (4) in the convolutory form or to look for some approximations directly in the time domain, avoiding the Fourier transform.

We shall first consider various convolutory forms of (4), both exact and approximate. The first version, fully equivalent to (4), is as follows

$$(6) \qquad U(z, t) = x(t) * A(t - \tau_0, t^*) ,$$

where

$$(7) \qquad A(\zeta, t^*) = 2 \operatorname{Re} \int_0^\infty \exp\left[-\pi f t^* + \mathrm{i}\, 2 f t^* \ln(f/f_0) + \mathrm{i}\, 2\pi f \zeta\right] \mathrm{d}f .$$

As f_0 can be chosen arbitrarily, we do not consider it as a parameter. We must, however, keep in mind that the value of τ_0 corresponds to the velocity $v(f_0)$. We can, e.g., take $f_0 = 1$ without any loss of generality.

The function $A(\zeta, t^*)$ represents a one-parameter family of curves. It is simple to show that this one-parameter family of curves can be reduced just to a single curve. Taking ft^* as a new integration variable in (7) we get

$$(8) \qquad A(t - \tau_0, t^*) = B(\Theta)/t^* ,$$

where

$$(9) \qquad B(\Theta) = 2 \operatorname{Re} \int_0^\infty \exp\left(-\pi f + \mathrm{i}\, 2 f \ln f + \mathrm{i}\, 2\pi f \Theta\right) \mathrm{d}f ,$$

$$\Theta = \left[t - \tau_0 - (t^*/\pi) \ln(f_0 t^*)\right]/t^* .$$

As will be shown later the expression for Θ can be also rewritten as follows $\Theta = [t - x/v(f_s)]/t^*$, where $f_s = 1/t^*$. See also [4].

The expression (6) with (7) or (8) is exact and fully equivalent to (4). Now we shall derive an approximate form of (7) under the assumption that the amplitude spectrum $|X(f)|$ of the time function $x(t) = U(0, t)$ is very narrow, highly concentrated in the vicinity of some frequency f_M (which may be different from f_0). We can then approximately write

$$2ift^* \ln (f/f_0) = 2ift^* \ln (f/f_M) + 2ift^* \ln (f_M/f_0) \sim 2ift^* \ln (f_M/f_0) + 2it^*(f - f_M) .$$

This yields

(10) $$2ift^* \ln (f/f_0) \sim 2ift^+ - 2if_M t^* ,$$

where

(11) $$t^+ = (t^*/\pi) [1 + \ln (f_M/f_0)] .$$

Inserting (10) into (7) yields

$$A(\zeta, t^*) \sim 2 \operatorname{Re} \left\{ \exp (-2if_M t^*) \int_0^\infty \exp [-\pi f t^* + i2\pi f(\zeta + \pi^{-1} t^+)] \, df \right\} .$$

The integral in the above formula can be calculated easily. We obtain the following approximate formula

(12) $$A(t - \tau_0, t^*) = (a_1 + a_2 \Theta)/(t^{*2} + 4\Theta^2) ,$$

where

(13) $$\Theta = t - \tau_0 + \pi^{-1} t^* [1 + \ln (f_M/f_0)] ,$$

$$a_1 = (2/\pi) \, t^* \cos (2f_M t^*) , \quad a_2 = (4/\pi) \sin (2f_M t^*) .$$

Here a_1, a_2 and t^* are constants, independent of time. For $f_M = f_0$, the expression (13) for Θ takes a simpler form, $\Theta = t - \tau_0 + \pi^{-1} t^*$. Remember that τ_0 is given by (2) and corresponds to the velocity $v(f_0)$. Taking into account that

$$\tau_0 = z/v(f_0) = z/v(f_M) + (t^*/\pi) \ln (f_M/f_0) ,$$

we can rewrite the expression for Θ given by (13) into the following form

(14) $$\Theta = t - \tau_M + t^*/\pi , \quad \text{where} \quad \tau_M = z/v(f_M) .$$

Thus, the expression (12) is fully independent of the reference frequency f_0, when we use Θ in the form of (14).

Formula (6) with (12) involves all the effects of causal absorption: decrease of amplitudes, decrease of the prevailing frequency and the velocity dispersion. In many cases, it is possible to appreciate roughly certain of these effects by simple approximate formulae, as will be shown later. For example, it is often possible to find some approximate formula for the change of the prevailing frequency f^* of the signal with t^*. The accuracy of (12) will be increased if we substitute this rough approximation $f^* = f^*(t^*)$ for f_M in (13). The accuracy of (12) will be investigated in greater detail elsewhere.

In case of non-causal absorption, with the velocity independent of frequency, we obtain from (7) the well-known expression

$$A(t - \tau_0, t^*) = (2/\pi)\, t^*[t^{*2} + 4(t - \tau_0)^2]^{-1} .$$

It would be possible to rewrite all the above convolutory formulae in a number of other forms. We shall not, however, present them here, we shall try to simplify them even more for a certain class of signals important in seismic applications.

Let us consider the signal

$$x(t) = y(2\pi f_0 t/\gamma) \sin(2\pi f_0 t) , \tag{15}$$

where a harmonic carrier wave with the frequency f_0 is modulated by an envelope $y(2\pi f_0 t/\gamma)$. The parameter γ controls the width of the envelope. Denote the complex spectrum of $y(t)$ by $Y(f)$. Using the similarity and modulation theorem [2] we obtain

$$X(f) = \frac{\gamma}{4\mathrm{i}\pi f_0} \left\{ Y\left[\frac{\gamma(f - f_0)}{2\pi f_0}\right] - Y\left[\frac{\gamma(f + f_0)}{2\pi f_0}\right] \right\} . \tag{16}$$

Thus, for $U(z, t)$ we obtain from (4)

$$U(z, t) = \mathrm{Re} \left[\!\left[\frac{\gamma}{2\mathrm{i}\pi f_0} \int_0^\infty \left\{ Y\left[\frac{\gamma(f - f_0)}{2\pi f_0}\right] - Y\left[\frac{\gamma(f + f_0)}{2\pi f_0}\right] \right\} \times \right.\right.$$
$$\left.\left. \times \exp\left[2\mathrm{i} f t^* \ln(f/f_0) - \pi t^* f + \mathrm{i}\, 2\pi f(t - \tau_0)\right] \mathrm{d}f \right]\!\right] . \tag{17}$$

If the width of the spectrum Y of the envelope y is considerably smaller than the prevailing frequency f_0, we can write approximately for $f > 0$

$$Y[\gamma(f + f_0)/(2\pi f_0)] \sim 0 .$$

We then obtain from (17)

$$U(z, t) \sim \mathrm{Re} \left\{ \frac{\gamma}{2\mathrm{i}\pi f_0} \int_0^\infty Y\left[\frac{\gamma(f - f_0)}{2\pi f_0}\right] \exp\left[-\pi f t^* + \right.\right.$$
$$\left.\left. + 2\mathrm{i} f t^* \ln(f/f_0) + 2\mathrm{i}\pi f(t - \tau_0)\right] \mathrm{d}f \right\} . \tag{18}$$

Now we shall investigate the function $Y[\gamma(f - f_0)/(2\pi f_0)] \exp(-\pi f t^*)$. Even though the spectrum Y is highly concentrated in the vicinity of the frequency f_0, the exponential factor $\exp(-\pi f t^*)$ may shift the prevailing frequency of the spectrum towards lower frequencies. Assume that the maximum of the function $Y[\gamma(f - f_0)/(2\pi f_0)] \times \times \exp(-\pi f t^*)$ is situated at $f^* < f_0$. The value of f^* can often be simply calculated analytically when the envelope $y(t)$ is known. It will depend on the global absorption parameter t^* and on the parameters of the envelope. If it is not possible to determine f^* analytically, f^* can be calculated numerically. In the following, we assume that f^* has been determined.

Due to the exponential factor $\exp(-\pi f t^*)$, the position of the maximum of the spectrum Y is shifted from f_0 to f^*, but the form of the spectrum does not change considerably when Y is narrow (i.e., when γ is large). Thus, we can write approximately

$$(19) \qquad Y[\gamma(f - f_0)/(2\pi f_0)] \exp(-\pi f t^*) \sim [Y(0)]^{-1} \exp(-\pi f^* t^*) \times \times Y[\gamma(f^* - f_0)/(2\pi f_0)] \, Y[\gamma(f - f^*)/(2\pi f_0)] .$$

The constant factor $Y[\gamma(f^* - f_0)/(2\pi f_0)] \exp(-\pi f^* t^*)/Y(0)$ (independent of frequency) has been included to obtain correct amplitudes at $f = f^*$ in (19). To simplify the formulae, we denote

$$(20) \qquad Y[\gamma(f^* - f_0)/(2\pi f_0)]/Y(0) = \varepsilon \exp(i\psi) ,$$

where ε and ψ are constants, independent of frequency.

Similarly, for $f \sim f^*$ we can simplify the term $2ift^* \ln(f/f_0)$ in the exponent in (18). We can write, see (10),

$$(21) \qquad 2ift^* \ln(f/f_0) \sim 2i\pi f \tilde{t} - 2if^* t^* ,$$

where $\tilde{t}$ is now given by the formula

$$(22) \qquad \tilde{t} = (t^*/\pi)[1 + \ln(f^*/f_0)] \sim (t^*/\pi)(f^*/f_0) .$$

Substituting (19), (20) and (21) into (18) yields

$$(23) \qquad U(z, t) \sim \varepsilon \exp(-\pi f^* t^*) \operatorname{Re} \left\{ \gamma(2i\pi f_0)^{-1} \exp(i\psi - 2if^* t^*) \times \times \int_0^\infty Y[\gamma(f - f^*)/(2\pi f_0)] \exp(i\, 2\pi f T) \, df \right\} ,$$

where

$$(24) \qquad T = t - \tau_0 + \tilde{t} = t - \tau_0 + (t^*/\pi)[1 + \ln(f^*/f_0)] .$$

As the spectrum $Y[\gamma(f - f^*)/(2\pi f_0)]$ is highly concentrated in the vicinity of $f = f^*$, we can write approximately

$$\frac{\gamma}{2\pi f_0} \int_0^\infty Y\left[\frac{\gamma(f - f^*)}{2\pi f_0}\right] \exp(i\, 2\pi f T) \, df \sim$$

$$\sim \frac{\gamma}{2\pi f_0} \int_{-\infty}^\infty Y\left[\frac{\gamma(f - f^*)}{2\pi f_0}\right] \exp(i\, 2\pi f T) \, df = y\left(\frac{2\pi f_0 T}{\gamma}\right) \exp(i\, 2\pi f^* T) .$$

Inserting this into (23) yields

(25) $$U(z, t) \sim \varepsilon y(2\pi f_0 T/\gamma) \exp(-\pi f^* t^*) \sin[2\pi f^*(T - \pi^{-1}t^*) + \psi] .$$

This is the final approximate expression for the elementary seismogram of a plane wave propagating in a medium with small causal absorption.

The same procedure can be used to find the approximate expression $U(z, t)$ for, the signal $x(t) = y(2\pi f_0 t/\gamma) \cos(2\pi f_0 t)$. In this case, formula (25) remains valid only the trigonometric function sine must be replaced by cosine. Generally, for $x(t)$ given by the formula

(26) $$x(t) = y(2\pi f_0 t/\gamma) \cos(2\pi f_0 t + v) ,$$

the approximate expression for $U(z, t)$ is as follows:

(27) $$U(z, t) = \varepsilon y(2\pi f_0 T/\gamma) \exp(-\pi f^* t^*) \cos[2\pi f^*(T - \pi^{-1}t^*) + \psi + v] ,$$

where T is given by (24), t^* by (5), ε and ψ by (20).

Let us now consider two special examples of (27).

1) Gaussian envelope.

Now we shall consider the signal (26) with the envelope $y(\zeta) = \exp(-\zeta^2)$, i.e.

(28) $$x(t) = \exp(-2\pi f_0 t/\gamma)^2 \cos(2\pi f_0 t + v) .$$

This signal plays a very important role in many seismological applications [5]. As $Y(f) = \sqrt{\pi} . \exp(-\pi^2 f^2)$, we obtain simply

(29) $$f^* = f_0(1 - 2\pi f_0 t^*/\gamma^2) .$$

For ε and ψ we arrive at

(30) $$\varepsilon = \exp(-\pi^2 f_0^2 t^{*2}/\gamma^2) , \quad \psi = 0 .$$

Equation (27) then yields

(31) $$U(z, t) = \exp[-(2\pi f_0 T/\gamma)^2 - (\pi f_0 t^*/\gamma)^2 - \pi f^* t^*] \times \times \cos\{2\pi f^*(T - \pi^{-1}t^*) + v\} ,$$

where T is given by (24). This can be rewritten in a number of other forms. One of them is as follows:

(32) $$U(z, t) = \exp[-\pi f_0 t^* + (\pi f_0 t^*/\gamma)^2 - (2\pi f_0/\gamma)^2 (t - \tau_0 + t^*/\pi - 2f_0 t^{*2}/\gamma^2)] \times \times \cos[2\pi f^*(t - \tau_0 - 2f_0 t^{*2}/\gamma^2) + v] .$$

A similar expression was obtained as an approximation to the exact solution by Averbukh [1].

2) Berlage signal.

The Berlage signal will be considered in the following form

$$x(t) = Ct^n \exp(-2\pi f_0 t/\gamma) \sin(2\pi f_0 t) \quad \text{for} \quad t > 0\,, \qquad (33)$$
$$= 0 \quad \text{for} \quad t < 0\,,$$

where C is some real positive constant. The envelope $y(\zeta)$ is given for $\zeta > 0$ by formula $y(\zeta) = \zeta^n \exp(-\zeta)$ (the multiplication constant has been ommitted). If we take into account that $Y(f) = n!\,(1 + \mathrm{i}\,2\pi f)^{-(n+1)}$, we obtain a quadratic algebraic equation for f^*,

$$[\pi t^*/(n+1)]\,(f^* - f_0)^2 + (f^* - f_0) + \pi t^* f_0^2/[\gamma^2(n+1)] = 0\,. \qquad (34)$$

For small t^*, we have approximately

$$f^* \sim f_0 - \pi t^* f_0^2/[\gamma^2(n+1)]\,. \qquad (35)$$

The quantities ε and ψ are, in this first approximation, given by the formulae

$$\varepsilon = \left\{\frac{1 + \pi^2 f_0^2 t^{*2}}{(n+1)^2\,\gamma^2}\right\}^{-(n+1)/2}, \quad \psi = (n+1)\arctan\frac{\pi f_0 t^*}{\gamma(n+1)}\,. \qquad (36)$$

Inserting into (27), we finally arrive at

$$U(z, t) \sim \frac{CT^n \exp(-2\pi f_0 T/\gamma - \pi f^* t^*)}{[1 + \pi^2 t^{*2} f_0^2 (n+1)^{-2}\,\gamma^{-2}]^{(n+1)/2}} \sin\left(2\pi f^*\left(T - \frac{1}{\pi}\,t^*\right) + \psi\right). \qquad (37)$$

Here T is given by (20), ψ by (36).

Let us now shortly discuss several properties of elementary seismograms (27) and the accuracy of formula (27). Formula (27) takes into account three effects of causal absorption on seismic signals. The first effect is the exponential, frequency dependent decrease of amplitudes along the ray. This decrease is mainly due to the factor $\exp(-\pi f^* t^*)$, see (27), but it is more complicated, it differs from signal to signal, see (31) and (37). The second effect is the velocity dispersion, which accompanies absorption. We can see from (27), (31) and (37) that the envelope of the signal propagates with a different velocity than the carrier. Thus, the group velocity is different from phase velocity. It would be simple to find expressions for the group and phase velocities for signals under consideration using the formulae presented above, but we shall not do this here. Finally, the third effect consists in the decreasing of the prevailing frequency, see the argument of the cosine in (27).

The accuracy of the above formulae depends substantially on the width of the amplitude spectrum of the signal envelope. For a given f_0, the accuracy decreases when the width of the amplitude spectrum of the envelope increases. It is well known that the width of the amplitude spectrum depends considerably on the smoothness of the function, it decreases with the increasing smoothness of the signal. Thus, the best accuracy of (27) is obtained for signals with an envelope "infinitely smooth", i.e. with all derivatives continuous, such as the Gaussian envelope, see (28). The accuracy of the formulae for the Berlage signal will, of course, be lower, mainly for small n (e.g., $n = 1$). This is due to the fact that the Berlage signal (33) has a discontinuous derivative of the $(n + 1)$st order at $t = 0$.

The properties of seismic signals propagating in media with small causal absorption and the accuracy of the approximate formulae for elementary seismograms, presented above, will be discussed in greater detail elsewhere.

The formulae presented here are derived for plane waves. Similar formulae have, however, also been often approximately applied to seismic body waves propagating in inhomogeneous media. In this case, under the global absorption factor t^* (see (5)) we must consider [10]

$$t^* = \int_L (cQ_0)^{-1} \, ds , \tag{38}$$

where the integral is taken along the ray L; ds denotes the arclength along the ray. Of course, all the presented formulae must be modified according to the well-known principles of the ray method [5] to include geometrical spreading, the reflection and transmission coefficients, etc. These modifications are straightforward.

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