A NEW APPROXIMATION OF THE VELOCITY-DEPTH DISTRIBUTION AND ITS APPLICATION TO THE COMPUTATION OF SEISMIC WAVE FIELDS

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Summary: *A new approximation of the velocity-depth distribution in a vertically inhomogeneous medium is suggested. This approximation guarantees the continuity of velocity and of its first and second derivatives and does not generate false low-velocity zones. It is very suitable for the computations of seismic wave fields in vertically inhomogeneous media by ray methods and its modifications, as it removes many false anomalies from the travel-time and amplitude-distance curves of seismic body waves. The ray integrals can be evaluated in a closed form; the resulting formulae for rays, travel times and geometrical spreading are very simple. They do not contain any transcendental functions (such as* $\ln(x)$ *or* $\sin^{-1}(x)$*) like other approximations; only the evaluation of one square root and of certain simple arithmetic expressions for each layer is required. From a computational point of view, the evaluation of ray integrals and of geometrical spreading is only slightly slower than for a system of homogeneous parallel layers and even faster than for a piece-wise linear approximation.*

1. INTRODUCTION

It is well known that standard methods of interpolating the velocity-depth distribution $v = v(z)$ in vertically inhomogeneous media (such as the piece-wise linear interpolation) do not guarantee the continuity of the first and second derivatives of velocity and thus generate false interfaces of a higher order. These false interfaces cause anomalies in the amplitude-distance curves. The application of the cubic spline interpolation to the velocity-depth distribution $v = v(z)$ removes the false interfaces of second and third order, the ray integrals, however, must be evaluated numerically in this case [5, 6]. It was suggested in [2, 3] to apply the smoothed spline approximation to the function $z = z(v)$, instead of $v = v(z)$. In this case, the ray integrals and ray amplitudes can be evaluated in a closed form. The procedure is as follows. The whole interval of depths is divided into several subintervals with a monotonous increase or decrease of velocities. Also the interfaces of the first order (and optionally the interfaces of a higher order) are taken as the boundaries of sub-intervals. Let the velocity-depth distribution within the sub-interval be specified by N velocity-depth points (z_i, v_i), $i = 1, 2, \ldots, N$, $z_{i+1} > z_i$. The depth z_i corresponds to the top and the depth z_N to the bottom of the sub-interval. The velocity-depth distribution between two points (z_i, v_i) and (z_{i+1}, v_{i+1}), $i = 1, 2, \ldots, N-1$, is now approximated by the formula

$$z = a_i + b_i v + c_i v^2 + d_i v^3\,, \quad i = 1, 2, \ldots, N-1\,. \tag{1}$$

The coefficients a_i, b_i, c_i, d_i $(i = 1, 2, \ldots, N-1)$ can be calculated using the smoothed cubic spline algorithm which guarantees the continuity of the velocity-depth distribution, with its first and second derivatives. Moreover, the application of smoothed splines considerably increases the stability of resulting travel-time curves and amplitude-distance curves in comparison with

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standard interpolating splines. The boundary points z_1 and z_N, of course, must be fixed, without smoothing. As mentioned above, the ray integrals and ray amplitudes can be now evaluated in a closed form. The corresponding formulae can be found in [3], examples of computations in [1, 2].

For the evaluation of ray integrals and ray amplitudes in case of the approximation (1) we must calculate, for each i, the following two elementary transcendental functions: $\sin^{-1}(x)$ and $\ln(x)$. This makes the computation a little slower than for simpler approximations of velocity-depth distributions. For example, for a piece-wise linear approximation we need to calculate for each i only one elementary transcendental function: $\ln(x)$. For a piece-wise constant approximation (homogeneous layers) the situation is even simpler, we do not need to compute any transcendental function (only the computation of one square root for each layer is necessary). It would be very useful to find such an approximation which would have similar properties as the approximation (1), but which would be faster.

Such an approximation is suggested in this paper. It reads

$$z = a_i + b_i v^{-2} + c_i v^{-4} + d_i v^{-6} . \tag{2}$$

The coefficients a_i, b_i, c_i, d_i ($i = 1, 2, \ldots, N - 1$) can be again calculated applying the smoothed spline approximation to points (z_i, v_i^{-2}), $i = 1, 2, \ldots, N$. This guarantees the continuity of the velocity-depth distribution with its first and second derivatives and stability of resultant travel-time curves and amplitude-distance curves. For the evaluations of ray integrals and ray amplitudes, however, we need to calculate only one square root for each i, no transcendental function is needed. Thus, the computations seem to be even faster than for a piece-wise linear approximations, where one needs to compute one elementary function ($\ln(x)$) and one square root for each i. It would be only slightly slower than the computation for a system of homogeneous parallel layers, where also only one square root should be computed for each layer.

2. RAY INTEGRALS

We shall consider a vertically inhomogeneous medium and denote the depth by z. We introduce a Cartesian coordinate system (x, z) so that z-axis is perpendicular to the Earth's surface and points downwards, and the x-axis is situated along the surface. The medium may contain horizontal interfaces of any order.

Let us now investigate the region between the lines $z = z_i$ and $z = z_{i+1}$. We shall call it formally the i-th layer and the lines $z = z_i$ and $z = z_{i+1}$ will be called the top and the bottom interface of the i-th layer. We shall assume that the velocity within the whole layer is a monotonous function of depth and that it is specified by Eq. (2).

Now, we shall consider one element of the ray specified by the ray parameter p, situated between the depths Z_1 and Z_2 within the i-th layer, i.e. $z_i \leqq Z_1 \leqq z_{i+1}$, $z_i \leqq Z_2 \leqq z_{i+1}$. The relation between Z_1 and Z_2 may be arbitrary ($Z_1 < Z_2$ or $Z_1 > Z_2$). It is well known that the ray parameter p is closely connected to $\delta(z)$, the acute angle between the ray and the z-axis at depth z, by the formula

$$p = \sin \delta(z)/v(z) . \tag{3}$$

We denote the travel time along this element of the ray by $T_i(Z_1, Z_2, p)$ and the corresponding range of horizontal distances by $X_i(Z_1, Z_2, p)$. The index i specifies that the element of the ray is fully situated within the i-th layer. We assume that the angle $\delta(z)$ is a monotonous single-valued function of z along the element of the ray. In other words, we do not consider the elements of the ray which have a turning point (defined by $\delta(z) = \frac{1}{2}\pi$) between the initial and the end point of the element. Directly at the initial or end point of the element, however, the possibility $\delta(z) = \frac{1}{2}\pi$ is permitted. Thus the ray with a minimum (maximum) can be constructed as a combination of two elements of the ray, which have $\delta(z) = \frac{1}{2}\pi$ at the initial or end point. The above assumption can be expressed in terms of the ray parameter p as follows

$$p \leqq 1/\mathrm{Max}\,(v(Z_1), v(Z_2))\,. \tag{4}$$

We shall now evaluate the expressions for T_i and X_i. It is well known that T_i and X_i are given by the integrals

$$T_i(Z_1, Z_2, p) = \pm \int_{Z_1}^{Z_2} v^{-1}(1 - p^2v^2)^{-1/2}\,\mathrm{d}z\,, \tag{5}$$

$$X_i(Z_1, Z_2, p) = \pm \int_{Z_1}^{Z_2} pv(1 - p^2v^2)^{-1/2}\,\mathrm{d}z\,,$$

see details in [1, 4]. The upper sign in (5) applies to the case $Z_2 > Z_1$, the lower sign to $Z_2 < Z_1$. In the following, we shall call the element "descending" when $Z_2 > Z_1$, and "ascending", when $Z_2 < Z_1$. It is simple to show that T_i and X_i given by (5) are always positive and that the following relations are valid:

$$T_i(Z_1, Z_2, p) = T_i(Z_2, Z_1, p)\,, \quad X_i(Z_1, Z_2, p) = X_i(Z_2, Z_1, p)\,. \tag{5'}$$

Now we introduce a new integration variable $w(z)$ by the relation

$$w(z) = v^{-2}(z)\,. \tag{6}$$

From (5) we obtain

$$T_i(Z_1, Z_2, p) = \pm \int_{w_1}^{w_2} w(w - p^2)^{-1/2}\,(\mathrm{d}z/\mathrm{d}w)\,\mathrm{d}w\,, \tag{7}$$

$$X_i(Z_1, Z_2, p) = \pm \int_{w_1}^{w_2} p(w - p^2)^{-1/2}\,(\mathrm{d}z/\mathrm{d}w)\,\mathrm{d}w\,.$$

Here we have used the notation

$$w_1 = v^{-2}(Z_1)\,, \quad w_2 = v^{-2}(Z_2)\,. \tag{8}$$

The upper sign again applies to the descending element of the ray ($Z_2 > Z_1$), the lower sign to the ascending element ($Z_2 < Z_1$).

3. CALCULATION OF RAY INTEGRALS

If z is expressed in terms of v^{-2} by Eq. (2), we obtain

$$dz/dw = b_i + 2c_i w + 3d_i w^2 . \tag{9}$$

Inserting this into that ray integrals (7) we obtain

$$T_i(Z_1, Z_2, p) = \pm \int_{w_1}^{w_2} w(w - p^2)^{-1/2} (b_i + 2c_i w + 3d_i w^2)\, dw , \tag{10}$$
$$X_i(Z_1, Z_2, p) = \pm \int_{w_1}^{w_2} p(w - p^2)^{-1/2} (b_i + 2c_i w + 3d_i w^2)\, dw .$$

Integrals (10) can be simply evaluated in a closed form. We shall use a new variable

$$y = (w - p^2)^{1/2} , \tag{11}$$

and denote

$$Y_1 = (v^{-2}(Z_1) - p^2)^{1/2} , \quad Y_2 = (v^{-2}(Z_2) - p^2)^{1/2} . \tag{12}$$

Equations (10) then yield

$$T_i(Z_1, Z_2, p) = \pm \int_{Y_1}^{Y_2} [(2b_i p^2 + 4c_i p^4 + 6d_i p^6) + (2b_i + 8c_i p^2 + 18d_i p^4)\, y^2 + (4c_i + 18d_i p^2)\, y^4 + 6d_i y^6]\, dy , \tag{13}$$
$$X_i(Z_1, Z_2, p) = \pm \int_{Y_1}^{Y_2} [(2b_i p + 4c_i p^3 + 6d_i p^5) + (4c_i p + 12d_i p^3)\, y^2 + 6d_i p y^2]\, dy ,$$

where the upper sign again corresponds to the descending part of the ray ($Z_2 > Z_1$), the lower sign to the ascending part of the ray ($Z_2 < Z_1$).

After a simple integration we obtain

$$T_i(Z_1, Z_2, p) = \pm(\bar{T}_i(Z_2, p) - \bar{T}_i(Z_1, p)) , \tag{14}$$
$$X_i(Z_1, Z_2, p) = \pm(\bar{X}_i(Z_2, p) - \bar{X}_i(Z_1, p)) .$$

The expressions $\bar{T}_i(Z_j, p)$ and $\bar{X}_i(Z_j, p)$ ($j = 1, 2;\ i = 1, 2, \ldots, N$) are given by the following formulae

$$\bar{T}_i(Z_j, p) = Y_j[(2b_i p^2 + 4c_i p^4 + 6d_i p^6) + (\tfrac{2}{3}b_i + \tfrac{8}{3}c_i p^2 + 6d_i p^4)\, Y_j^2 + (\tfrac{4}{5}c_i + \tfrac{18}{5}d_i p^2)\, Y_j^4 + \tfrac{6}{7}d_i\, Y_j^6] , \tag{15}$$
$$\bar{X}_i(Z_j, p) = Y_j[(2b_i p + 4c_i p^3 + 6d_i p^5) + (\tfrac{4}{3}c_i p + 4d_i p^3)\, Y_j^2 + \tfrac{6}{5}d_i p\, Y_j^4] .$$

These are the final formulae for $\bar{T}_i$ and $\bar{X}_i$. To compute T_i and X_i, we do not need to evaluate any elementary transcendental function such as $\ln(x)$ or $\sin^{-1}(x)$, we need only evaluate the square roots Y_1 and Y_2, see (12), and perform certain arithmetic operations.

Let us now assume that the element of the ray under consideration has a turning point at the depth $z = Z_1$. In this case we have $p = v^{-1}(Z_1)$, $Y_1 = 0$ and $\bar{T}_i(Z_1, p) = \bar{X}_i(Z_1, p) = 0$, see (12) and (15). Equations (14) then yield

$$(16) \qquad T_i(Z_1, Z_2, p) = \pm \bar{T}_i(Z_2, p), \quad X_i(Z_1, Z_2, p) = \pm \bar{X}_i(Z_2, p).$$

Similarly, when the turning point is situated at depth $z = Z_2$, we have $p = v^{-1}(Z_2)$, $Y_2 = 0$, $\bar{T}_i(Z_2, p) = \bar{X}_i(Z_2, p) = 0$ and we obtain

$$(17) \qquad T_i(Z_1, Z_2, p) = \mp \bar{T}_i(Z_1, p), \quad X_i(Z_1, Z_2, p) = \mp \bar{X}_i(Z_1, p).$$

It is easy to recognize the physical meaning of $\bar{T}_i(Z_j, p)$ and $\bar{X}_i(Z_j, p)$. The value of $\bar{X}_i(Z_j, p)$ gives the range of horizontal distance from the turning point of the ray to the point where the ray specified by the ray parameter p intersects the depth $z = Z_j$ ($z_i \leqq Z_j \leqq z_{i+1}$). The value of $\bar{X}_i(Z_j, p)$ is negative when the turning points corresponds to the minimum of the ray (i.e., when the turning point lies below the depth $z = Z_j$), and is positive when the turning point corresponds to the maximum. Of course, the turning point may be only fictitious, the real turning point need not exist within the i-th layer, and would exist only if the i-th layer with the velocity-depth distribution (2) were extended beyond the top and/or bottom interface. The value of $\bar{T}_i(Z_j, p)$ gives the travel time along this segment of the ray, again from the turning point to the depth $z = Z_j$. The choice of the sign is the same as in the case of $\bar{X}_i(Z_j, p)$.

4. COMPUTATION OF GEOMETRICAL SPREADING

It is well known that the geometrical spreading can be expressed in terms of the derivatives $\mathrm{d}X_i(Z_1, Z_2, p)/\mathrm{d}p$. All relevant formulae can be found in [1, 4]. Therefore, we shall only give the expressions for $\mathrm{d}X_i(Z_1, Z_2, p)/\mathrm{d}p$ here.

From (14) we obtain

$$(18) \qquad \mathrm{d}X_i(Z_1, Z_2, p)/\mathrm{d}p = \pm(\mathrm{d}\bar{X}_i(Z_2, p)/\mathrm{d}p - \mathrm{d}\bar{X}_i(Z_1, p)/\mathrm{d}p),$$

where $\mathrm{d}\bar{X}_i(Z_j, p)/\mathrm{d}p$ is given by the formula

$$(19) \qquad \mathrm{d}\bar{X}_i(Z_j, p)/\mathrm{d}p = -p^2 Y_j^{-1}(2b_i + 4c_i p^2 + 6d_i p^4) + Y_j(2b_i + 8c_i p^2 + 18d_i p^4) + Y_j^3(\tfrac{4}{3}c_i + 6d_i p^2) + \tfrac{6}{5}d_i Y_j^5.$$

When the element of the ray under consideration has a turning point at the point $z = Z_1$, we obtain from (15)

$$\text{(20)} \qquad \mathrm{d}X_i(Z_1, Z_2, p)/\mathrm{d}p = \pm \mathrm{d}\overline{X}_i(Z_2, p)/\mathrm{d}p\,,$$

where Z_1 again denotes the depth of the turning point. Similarly when the turning point is situated at $z = Z_2$, (17) yields

$$\text{(21)} \qquad \mathrm{d}X_i(Z_1, Z_2, p)/\mathrm{d}p = \mp \mathrm{d}\overline{X}_i(Z_1, p)/\mathrm{d}p\,.$$

The expressions for $\mathrm{d}\overline{X}_i(Z_j, p)/\mathrm{d}p$ in (20) and (21) are again given by (19). The equations (20) and (21) can also be formally obtained from (18) by putting $\mathrm{d}\overline{X}_i(Z_j, p)/\mathrm{d}p = 0$ at the turning point $z = Z_j$ (i.e., for $z = Z_1$ or $z = Z_2$).

5. SIMPLIFIED FORMULAE FOR A SYSTEM OF LAYERS

Let us now consider one sub-interval of depths, specified by N velocity-depth points (z_i, v_i), $i = 1, 2, \ldots, N$; $z_{i+i} > z_i$. We assume that the velocity within the whole sub-interval of depths $z_1 \leqq z \leqq z_N$, is a monotonous smooth function of depth. The individual layers within this sub-interval are separated by plane interfaces of a higher order (the velocity is a continuous function of depth). We shall consider a segment of the ray, composed of ray elements described in the preceeding section. We consider the segment of the ray without any turning point between the initial and end points of the segment. A turning point is permitted only at the initial or end point of the segment of the ray.

Let us assume that the initial point $z = Z_1$ is situated in the K-th layer and the end point $z = Z_2$ in the M-th layer, $1 \leqq K < M \leqq N - 1$. Thus, we consider the descending part of the ray ($K < M$). The whole segment of the ray is composed of the elements from Z_1 to z_{K+1}, from z_{K+1} to $z_{K+2}, \ldots$, from z_{M-1} to z_M, and from z_M to Z_2. The total number of elements is $M - K + 1$. To obtain expressions for the total range of horizontal distance $X(Z_1, Z_2, p)$ along the segment of the ray and corresponding travel-time $T(Z_1, Z_2, p)$, we shall use (14). As we consider the descending segment of the ray, we must take the upper sign in (14). Then we obtain

$$T(Z_1, Z_2, p) = \overline{T}_K(z_{K+1}, p) - \overline{T}_K(Z_1, p) + \sum_{i=K+1}^{M-1} (\overline{T}_i(z_{i+1}, p) - \overline{T}_i(z_i, p)) + \overline{T}_M(Z_2, p) - \overline{T}_M(z_M, p)\,,$$

$$X(Z_1, Z_2, p) = \overline{X}_K(z_{K+1}, p) - \overline{X}_K(Z_1, p) + \sum_{i=K+1}^{M-1} (\overline{X}_i(z_{i+1}, p) - \overline{X}_i(z_i, p)) + \overline{X}_M(Z_2, p) - \overline{X}_M(z_M, p)\,.$$

This can be easily rewritten in the following form

$$(22)\qquad T(Z_1, Z_2, p) = \bar{T}_M(Z_2, p) - \bar{T}_K(Z_1, p) + \sum_{i=K+1}^{M} \tilde{T}_i(p)\,,$$

$$X(Z_1, Z_2, p) = \bar{X}_M(Z_2, p) - \bar{X}_K(Z_1, p) + \sum_{i=K+1}^{M} \tilde{X}_i(p)\,,$$

where

$$(23)\qquad \tilde{X}_i(p) = \bar{X}_{i-1}(z_i, p) - \bar{X}_i(z_i, p)\,,$$

$$\tilde{T}_i(p) = \bar{T}_{i-1}(z_i, p) - \bar{T}_i(z_i, p)\,.$$

Using (15), we obtain

$$(24)\qquad \tilde{T}_i(p) = 2Y_i\{(b_{i-1} - b_i)(p^2 + \tfrac{1}{3}Y_i^2) + (c_{i-1} - c_i)(2p^4 + \tfrac{4}{3}p^2 Y_i^2 + \tfrac{2}{5}Y_i^4) + (d_{i-1} - d_i)(3p^6 + 3p^4 Y_i^2 + \tfrac{9}{5}p^2 Y_i^4 + \tfrac{3}{7}Y_i^6)\}\,,$$

$$\tilde{X}_i(p) = 2p\, Y_i\{(b_{i-1} - b_i) + (c_{i-1} - c_i)(2p^2 + \tfrac{2}{3}Y_i^2) + (d_{i-1} - d_i)(3p^4 + 2p^2 Y_i^2 + \tfrac{3}{5}Y_i^4)\}\,.$$

The formulae (22) with (23) and (24) are quite general. As we can again see, we do not need to evaluate any transcendental function such as $\ln(x)$ or $\sin^{-1}(x)$, we need to evaluate only one square root for each layer, and perform certain arithmetic operations.

The application of Eqs (22)–(24) for a sub-interval of depths simulated by a system of layers will be computationally faster than the successive application of Eq. (14). In case of (14), we need to compute two expressions (15) for each fictitious layer – for its top and bottom boundary. The square roots Y_i must also be evaluated twice for each layer. On the other hand, the application of Eq. (22) requires only one computation of (24) for each layer. Similarly, also the square root Y_i is calculated only once for each layer. Thus, the application of (22) should be approximately twice as fast as the application of (14). These computations must, of course, be supplemented by some computations for the initial and the end point of the segment of the ray, see (22).

Let us now consider special cases of Eq. (22):

1) *First case.* Let the point Z_1 lie on the upper boundary of the sub-interval, and the point Z_2 on the bottom of the subinterval. Then we put $Z_1 = z_1$, $Z_2 = z_N$, $K = 1$, $M = N - 1$ and obtain

$$(25)\qquad X(z_1, z_N, p) = \bar{X}_{N-1}(z_N, p) - \bar{X}_1(z_1, p) + \sum_{i=1}^{N-1} \tilde{X}_i(p)\,,$$

$$T(z_1, z_N, p) = \bar{T}_{N-1}(z_N, p) - \bar{T}_1(z_1, p) + \sum_{i=1}^{N-1} \tilde{T}_i(p)\,.$$

The expressions (25) remain the same both for descending and ascending segments of the ray.

2) *Second case.* Let us assume that the segment of the ray under consideration has a turning point at $z = Z_2$. Similarly as in the case of Eq. (17) we have $\bar{X}_M(Z_2, p) = \bar{T}_M(Z_2, p) = 0$. Equation (22) then yields

$$(26)\qquad X(Z_1, Z_2, p) = -\bar{X}_K(Z_1, p) + \sum_{i=K+1}^{M} \tilde{X}_i(p)\,,$$

$$T(Z_1, Z_2, p) = -\bar{T}_K(Z_1, p) + \sum_{i=K+1}^{M} \tilde{T}_i(p)\,.$$

3) *Third case.* Similarly, when the segment under consideration has a turning point at $z = Z_1$, we put $\bar{X}_K(Z_1, p) = \bar{T}_K(Z_1, p) = 0$. We then obtain

$$(27)\qquad X(Z_1, Z_2, p) = \bar{X}_M(Z_2, p) + \sum_{i=K+1}^{M} \tilde{X}_i(p)\,,$$

$$T(Z_1, Z_2, p) = \bar{T}_M(Z_2, p) + \sum_{i=K+1}^{M} \tilde{T}_i(p)\,.$$

It should be noted that the expressions $X(Z_1, Z_2, p)$ and $T(Z_1, Z_2, p)$ in (22), (26) and (27) are always positive. The negative signs only compensate the negative values of certain auxiliary expressions.

The above formulae can also be used for calculating geometrical spreading along the whole segment of the ray. For this we again need to compute $\mathrm{d}X_i(Z_1, Z_2, p)/\mathrm{d}p$, see details in Sec. 4. For this purpose, we shall use the general expressions (22). We obtain

$$(28)\qquad \frac{\mathrm{d}X(Z_1, Z_2, p)}{\mathrm{d}p} = \frac{\mathrm{d}\bar{X}_M(Z_2, p)}{\mathrm{d}p} - \frac{\mathrm{d}\bar{X}_K(Z_1, p)}{\mathrm{d}p} + \sum_{i=K+1}^{M} \frac{\mathrm{d}\tilde{X}_i(p)}{\mathrm{d}p}\,.$$

As the derivatives of $\bar{X}_M$ and $\bar{X}_K$ are given by (19), we need only to determine $\mathrm{d}\tilde{X}_i(p)/\mathrm{d}p$. We obtain it easily from (24),

$$(29)\qquad \mathrm{d}\tilde{X}_i(p)/\mathrm{d}p = 2Y_i^{-1}\{(b_{i-1} - b_i)(Y_i^2 - p^2) + (c_{i-1} - c_i)(\tfrac{2}{3}Y_i^4 + 4p^2 Y_i^2 - 2p^4) + (d_{i-1} - d_i)(\tfrac{3}{5}Y_i^6 + 3p^2 Y_i^4 + 9p^4 Y_i^2 - 3p^6)\}\,.$$

When the segment of the ray has a turning point at the initial point $z = Z_1$ (or at the end point $z = Z_2$), Eq. (28) remains valid, but we must put $\mathrm{d}\bar{X}_K(Z_1, p)/\mathrm{d}p = 0$ (or $\mathrm{d}\bar{X}_M(Z_2, p)/\mathrm{d}p = 0$). This follows from (26) and (27).

In this section, we have considered only the descending segment of the ray. Due to (5'), however, we obtain the same expressions even for ascending points of the ray, only we must change Z_1 and Z_2. The formulae are not presented here, they are straightforward.

6. CERTAIN PROPERTIES OF THE SUGGESTED VELOCITY-DEPTH APPROXIMATION

In this section, we would like to describe shortly certain properties of the suggested velocity-depth approximation (2).

Before the suggested approximation is applied, the whole interval of depths must be divided into several subintervals of depths with monotonous smooth increase or decrease of velocites, or with a constant velocity.

The boundaries of these subintervals of depths are specified as follows:

1) The interfaces of the first order.
2) Optionally the interfaces of a higher order.
3) The points of local maxima or minima of the velocity-depth distribution $v = v(z)$.

Within these sub-intervals, the velocity-depth distribution is monotonous due to the specification 3). The boundaries of these sub-intervals are specified by their depths, which remain fixed in the following procedure.

Within an sub-interval, the velocity-depth function is specified by N velocity-depth points (z_i, v_i), $i = 1, 2, \ldots, N$. The minimum N is 2.

For $N = 2$ and $v_1 = v_2$ (homogeneous layer), the approximation (2) cannot be used. This is, however, no disadvantage, as we can use the standard formulae for homogeneous layer, see e.g., [4].

For $N = 2$ and $v_1 \neq v_2$ we use (2) and obtain

$$(30) \qquad a_1 = \frac{z_1 v_1^2 - z_2 v_2^2}{v_1^2 - v_2^2}\,, \quad b_1 = \frac{(z_2 - z_1)\, v_1^2 v_2^2}{v_1^2 - v_2^2}\,, \quad c_1 = d_1 = 0\,.$$

In this case, the velocity-depth function is always monotonous, as $\mathrm{d}v/\mathrm{d}z = -v^3/2b_1$. If the velocity gradient is positive, $b_1 < 0$, if negative, $b_1 > 0$.

It should be noted that the approximation (2) with constants given by (30) can also be used throughout the model, in any layer. Of course, this approximation does not yield continuous first and second derivatives of the velocity-depth function, it has similar properties as the well-known piecewise linear approximation. The application of (2) to (30), however, will be computationally faster than the piecewise linear approximation, we do not need to compute the function $\ln(x)$ for each layer as in the case of the piecewise linear approximation.

A similar procedure can be used for $N = 3$ and $N = 4$, but we must solve a quadratic or cubic equation to find the constants a_i, b_i, c_i, d_i. If N is higher than 2 we can use the cubic spline approximations, preferably with smoothing. If we use smoothed splines, the values of z_1, and z_N must remain fixed, as they are the boundaries of the sub-intervals. The procedures for cubic spline approximation with smoothing with fixed end points are now commonly available, see e.g. [7]. We can also use the local splines or fundamental splines. The values of z_i, $i = 2, 3, \ldots, N - 1$ can change during the procedure due to smoothing, but this does not influence the computations suggested in this paper. The values of z_i ($i = 1, 2, \ldots, N$) used in the preceding sections of this paper of course apply to those obtained in the approximation procedure, not to those in the original (non-smoothed) model.

The approximation (2) has the follwoing properties:

1) The velocity-depth function $v = v(z)$ obtained from (2) is always monotonous within the whole sub-interval of depths. The obtained velocity-depth function $v = v(z)$ cannot oscillate within the sub-interval of depths and cannot form false low-velocity zones. This is a very important property, very useful in practical applications.

2) If we use spline approximations to find the coefficients of (2), the velocity-depth function is continuous together with its first and second derivatives. Moreover, if we use smoothed splines the resulting travel-time curves and amplitude-distance curves are very stable and do not depend very much on the number of grid points (see numerical experiments with a similar approximation in [2]). Thus, the number of grid points can be reduced substantially, and this again leads to a considerable saving of computer time. The extent of smoothing can be controlled by a set of parameters.

3) The main disadvantage of the approximation (2) is that z may oscillate as a function of v^{-2} within the sub-interval of depths under consideration. This oscillation would cause the function $v = v(z)$ not to be single-valued. The function $v = v(z)$, however, must be single-valued, the opposite case does not have any physical meaning. Any such oscillation must be removed. This can be done, for example, by introducing an artificial interface of the first or second order. This oscillation, however, can appear only exceptionally, in the case of a very low and very high velocity gradients in the neighbouring layers.

4) The approximations (2) cannot yield $\mathrm{d}v/\mathrm{d}z = 0$ at any depth. Thus, the points of local velocity maxima (or minima) must be introduced as interfaces of the second order, with discontinuous first derivative $\mathrm{d}v/\mathrm{d}z$ (positive from one side and negative from the other side). It should be noted, however, that most of approximations used in practical applications, also have the same property.

The experience with the suggested velocity-depth distribution will be described in detail elsewhere.

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