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Ray amplitudes of seismic body waves in laterally inhomogeneous media

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Summary. The most complicated part in the computation of ray amplitudes of seismic body waves in laterally inhomogeneous media with curved interfaces lies in the evaluation of the geometrical spreading. Geometrical spreading can be simply expressed in terms of the Jacobian J of the transformation from the Cartesian into ray coordinates. Several systems of ordinary differential equations to compute the function J are suggested. For general three-dimensional media, in which the velocity changes with all the three spatial coordinates, a system of three non-linear ordinary differential equations of the first order is derived. If the velocity does not depend on one coordinate, the system of equations reduces to only one non-linear differential equation. The initial conditions for these differential equations at point (or line) source and at points of intersection of the ray with curved interfaces are presented.

1 Introduction

The computation of rays does not now cause any difficulties; it can be performed in several ways. For example, rays can be determined by solving the ray-tracing system

$$d\mathbf{r}/ds = v\mathbf{p}, \qquad d\mathbf{p}/ds = -v^{-2}\,\mathrm{grad}\,v, \tag{1.1}$$

with appropriate initial conditions

$$\mathbf{r}(s_0) = \mathbf{r}_0, \qquad \mathbf{p}(s_0) = \mathbf{p}_0. \tag{1.1'}$$

An alternative form of the system (1.1), more convenient for computational purposes is as follows,

$$d\mathbf{r}/d\tau = v^2\mathbf{p}, \qquad d\mathbf{p}/d\tau = -v^{-1}\,\mathrm{grad}\,v, \tag{1.2}$$

with initial conditions

$$\mathbf{r}(\tau_0) = \mathbf{r}_0, \qquad \mathbf{p}(\tau_0) = \mathbf{p}_0. \tag{1.2'}$$

Here $\mathbf{r} = (x, y, z)$ denotes the radius vector of a point of the ray, $\mathbf{p}$ is the slowness vector at this point, and $v = v(\mathbf{r})$ denotes the velocity and s is the arclength along the ray. In (1.2), the travel time τ along the ray is used instead of s. The symbols $\mathbf{r}_0$ and $\mathbf{p}_0$ denote the radius and the slowness vectors at a reference point s_0 (or τ_0) of the ray. For details and various other forms of the ray-tracing systems see Červený, Molotkov & Pšenčík (1977).

The most complicated part in the computation of ray amplitudes is the evaluation of the geometrical spreading, which can be simply expressed in terms of the Jacobian J of the transformation from Cartesian coordinates x, y, z into ray coordinates s, γ_1, γ_2:

$$J = \frac{D(x, y, z)}{D(s, \gamma_1, \gamma_2)} = \begin{vmatrix} \partial x/\partial s & \partial y/\partial s & \partial z/\partial s \\ \partial x/\partial \gamma_1 & \partial y/\partial \gamma_1 & \partial z/\partial \gamma_1 \\ \partial x/\partial \gamma_2 & \partial y/\partial \gamma_2 & \partial z/\partial \gamma_2 \end{vmatrix} . \tag{1.3}$$

Here the ray coordinates γ_1, γ_2 specify the ray under investigation and the ray coordinate s (as above) specifies the position of a point on a selected ray. If we use τ instead of s in the ray coordinates, then

$$J = v^{-1} D(x, y, z)/D(\tau, \gamma_1, \gamma_2). \tag{1.3'}$$

Relevant formulae for the ray amplitudes and the formulae expressing the geometrical spreading in terms of the Jacobian J introduced by (1.3) can be found in Červený & Ravindra (1971), Červený *et al.* (1977).

In the following, we shall be interested mainly in the evaluation of the function J, which can be computed in several ways, see, e.g. Červený *et al.* (1977). Here we are interested in the methods based on the additional systems of ordinary differential equations. In these methods, the function J is expressed in the terms of certain auxiliary quantities which can be computed from a system of ordinary differential equations, an additional system to the standard ray-tracing system. Such a procedure for the computation of J was first suggested by Belonosova, Tadzhimukhamedova & Alekseyev (1967).

The additional systems can be written in various forms. The simplest derivation of such a system consists in the differentiation of (1.1) with respect to the ray parameters γ_1, γ_2. This gives 12 ordinary differential equations of the first order for the components of the vectors $\partial \mathbf{r}/\partial \gamma_1$, $\partial \mathbf{r}/\partial \gamma_2$, $\partial \mathbf{p}/\partial \gamma_1$ and $\partial \mathbf{p}/\partial \gamma_2$. The quantities $\partial \mathbf{r}/\partial \gamma_1$ and $\partial \mathbf{r}/\partial \gamma_2$ can then be used to compute J, see (1.3). The system can be simply reduced to 10 equations since two components of the above vectors can be expressed in terms of the remaining components.

It is obvious that the numerical solution of such a large system would be very time consuming. As a matter of fact, this is the reason why this method of evaluation of J has not yet been practically used and why considerable effort has been devoted to the reduction of the number of equations in the system. For example, Chen & Ludwig (1973) reduced the system to seven equations only. The right-sides of this system, however, are rather complicated. In Popov & Pšenčík (1976, 1978), the additional system consisting of eight simple linear differential equations of the first order was obtained. This system was reduced to five linear differential equations by Červený (1976). For a medium, in which the distribution of velocity is independent of one spatial coordinate (in the following we shall call such a medium two-dimensional or 2-D), the system can be substantially reduced. A system consisting of three linear ordinary differential equations is presented in Červený, Langer & Pšenčík (1974), a system consisting of only two linear ordinary differential equations of the first order was suggested in Kay (1961), Popov & Pšenčík (1976, 1978).

In this paper, we present several reduced systems of ordinary differential equations for the determination of the function J in three- as well as two-dimensional laterally inhomogeneous media with curved interfaces. To derive these systems we choose a consistent and objective approach, starting from equations (1.1) and (1.3). Using this approach, we first obtain a system consisting of eight linear differential equations, which was already obtained by a different method in Popov & Pšenčík (1976). The system can be reduced to five linear equations, in the same way as in Červený (1976). This seems to be the minimum number of linear equations of the first order in the additional system for 3-D media. In this paper, this system is further reduced to three simple non-linear ordinary differential equations of the first order. For 2-D media, the additional system consisting of two linear equations is first obtained. The system is reduced to only one non-linear differential equation of the first order.

Thus, in a 3-D medium we need to solve only three non-linear ordinary differential equations of the first order to compute the geometrical spreading. Similarly, in 2-D media we have to solve only one non-linear differential equation of the first order to compute the geometrical spreading. At the present time, this is the minimum number of equations for 3-D as well as 2-D media.

For the most important additional systems of differential equations, we also present here the initial conditions at a point (or line) source and the initial conditions at the points at which the ray strikes a curved interface.

Some results presented in Sections 3 and 4.1 were derived in a different form elsewhere (Popov & Pšenčík 1976, 1978; Červený 1976; Červený *et al.* 1977). In this paper we shall try to use the same notation as Červený *et al.* (1977); the notation is slightly different from that used in Popov & Pšenčík (1976, 1978) and in Červený (1976). The additional system consisting of three differential equations for 3-D media can be derived by various approaches and may assume various forms. For example, it can also be derived from the well-known Gel'chinskiy formula (see Gel'chinskiy 1966). A short presentation of this formula is given in Červený & Ravindra (1971, p. 55). This approach was chosen by Goldin (1978, private communication). In this paper, we shall derive all the additional systems using a consistent and simple approach following our earlier investigations in this field.

2 Ray-tracing system for rays in the vicinity of a central ray

Let us call the ray specified by two ray parameters γ_1 and γ_2 the central ray. Furthermore, let us introduce a curvilinear coordinate system s, q_1, q_2 connected with this ray, where s is the arclength measured, e.g. from the source. The symbols q_1, q_2 are used to denote the coordinates in the plane perpendicular to the central ray. The origin of the coordinate system q_1, q_2 in this plane is placed at the central ray, the unit coordinate vectors are denoted by $\mathbf{e}_1$, $\mathbf{e}_2$. The mutually perpendicular vectors $\mathbf{e}_1$, $\mathbf{e}_2$ can be specified, e.g. by the following relations

$$d\mathbf{e}_1/ds = -K \cos \vartheta\, \mathbf{t}, \quad d\mathbf{e}_2/ds = -K \sin \vartheta\, \mathbf{t}. \tag{2.1}$$

Here, $\mathbf{t}$ is a unit vector tangent to the central ray, K denotes the curvature of the central ray. The angle ϑ is given by the relation

$$\vartheta(s) = \int_{s_0}^{s} T(s)ds + \vartheta(s_0), \tag{2.2}$$

where $T(s)$ is the torsion of the ray.

Let us now consider a ray situated in close vicinity of the central ray specified by the ray parameters γ_1', γ_2' and denote the arclength along this ray by σ. Then the radius vector of an arbitrary point of this ray, $\mathbf{r}(\sigma)$, can be expressed as follows:

$$\mathbf{r}(\sigma) = \mathbf{r}(s) + q_1(s)\mathbf{e}_1(s) + q_2(s)\mathbf{e}_2(s), \tag{2.3}$$

where $\mathbf{r}(s)$ is the radius vector of the point of intersection of the central ray with the plane perpendicular to it and containing the point of the ray (γ_1', γ_2') under consideration.

The quantity $d\sigma/ds$ plays an important role in further discussion. Its value can be found from (2.3). By differentiating (2.3) with respect to s and taking into account the relations (2.1) we obtain

$$\frac{d\mathbf{r}}{d\sigma}\frac{d\sigma}{ds} = \frac{d\mathbf{r}}{ds} + q_1'\mathbf{e}_1 + q_2'\mathbf{e}_2 - q_1 K \cos\vartheta\ \mathbf{t} - q_2 K \sin\vartheta\ \mathbf{t}, \tag{2.4}$$

where $q_i' = dq_i/ds$. The determination of the magnitude of the vectorial expression (2.4) yields

$$d\sigma/ds = (h^2 + q_1'^2 + q_2'^2)^{1/2}, \tag{2.5}$$

where

$$h = 1 - q_1 K \cos\vartheta - q_2 K \sin\vartheta. \tag{2.5'}$$

Let us now rewrite the ray-tracing equations (1.1) for the ray (γ_1', γ_2') in the coordinates s, q_1, q_2. For the first set of ray-tracing equations we have from (2.4)

$$d\mathbf{r}/d\sigma = (d\mathbf{r}/ds + q_1'\mathbf{e}_1 + q_2'\mathbf{e}_2 - q_1 K \cos\vartheta\ \mathbf{t} - q_2 K \sin\vartheta\ \mathbf{t})(h^2 + q_1'^2 + q_2'^2)^{-1/2} = v\mathbf{p}. \tag{2.6}$$

Taking scalar products of (2.6) with vectors $\mathbf{e}_i (i = 1, 2)$, we arrive at two equations for q_i':

$$q_i' = v p_i (h^2 + q_1'^2 + q_2'^2)^{1/2}, \tag{2.7}$$

where the symbol p_i is used to denote the components of the slowness vector $\mathbf{p}$ with respect to coordinates q_i in the coordinate system s, q_1, q_2, i.e. $p_i = (\mathbf{e}_i \cdot \mathbf{p})$. It follows immediately from (2.7)

$$q_1'^2 + q_2'^2 + h^2 = h^2 [1 - v^2(p_1^2 + p_2^2)]^{-1}. \tag{2.8}$$

Inserting this relation into the right-side of (2.7) we obtain two ordinary differential equations of the first order for $q_i (i = 1, 2)$

$$dq_i/ds = v p_i h [1 - v^2(p_1^2 + p_2^2)]^{-1/2}. \tag{2.9}$$

This is the first set of differential equations for $q_i(s)$. Now we must derive the second set of equations for dp_i/ds. Taking a scalar product of (2.6) with $\mathbf{t}$ we obtain

$$p_s = v^{-1} [1 - v^2(p_1^2 + p_2^2)]^{1/2}, \tag{2.10}$$

where p_s is used to denote the component of the slowness vector $\mathbf{p}$ with respect to coordinate s, $p_s = (\mathbf{p} \cdot \mathbf{t})$.

Taking into account the Frenet formula $d\mathbf{t}/ds = K\mathbf{n}$, we obtain from (1.1)

$$\begin{aligned} d\mathbf{p}/d\sigma &= (ds/d\sigma)[d(p_1\mathbf{e}_1 + p_2\mathbf{e}_2 + p_s\mathbf{t})/ds] \\ &= (p_1'\mathbf{e}_1 + p_2'\mathbf{e}_2 + p_s'\mathbf{t} - p_1 K \cos\vartheta\ \mathbf{t} - p_2 K \sin\vartheta\ \mathbf{t} + p_s K\mathbf{n})(h^2 + q_1'^2 + q_2'^2)^{-1/2} \\ &= -v^{-2} \operatorname{grad} v, \end{aligned} \tag{2.11}$$

where

$$p_i' = dp_i/ds, \qquad p_s' = dp_s/ds.$$

As above, taking scalar products of (2.11) with vectors $\mathbf{e}_i$ ($i = 1, 2$) we obtain two ordinary differential equations for p_i

$$dp_i/ds = -p_s K(\mathbf{n}\cdot\mathbf{e}_i) - v^{-2}(\text{grad } v \cdot \mathbf{e}_i)(h^2 + q_1'^2 + q_2'^2)^{1/2}. \qquad (2.12)$$

Since p_i (and similarly q_i) are identically zero along the central ray ($p_i = (\mathbf{p}\cdot\mathbf{e}_i)$, and $\mathbf{p} \parallel \mathbf{t}$) and $p_s = v^{-1}$ on the central ray, the relations (2.12) yield the two following identities

$$K(\mathbf{n}\cdot\mathbf{e}_i) + v^{-1}v_i = 0 \qquad (2.13)$$

for $i = 1, 2$. In (2.13), all the quantities are considered on the central ray and v_i denotes $\partial v/\partial q_i$. Using these relations together with (2.8), equations (2.12) can be rewritten in the final form

$$dp_i/ds = [v^{-1}v_0^{-1}v_{i0} - v^{-2}v_i h(1 - v^2(p_1^2 + p_2^2))^{-1}](1 - v^2(p_1^2 + p_2^2))^{1/2}, \qquad (2.14)$$

where the index '0' denotes that the corresponding quantities are taken on the central ray.

Thus, the four ordinary differential equations of the first order (2.9) and (2.14) form ray-tracing equations of rays in the vicinity of the central ray. As was mentioned above, on the central ray $q_i = p_i = 0$ identically for $i = 1, 2$.

The initial values of q_i and p_i for various types of sources can be obtained immediately from the definitions of these quantities. The same holds for the conditions at interfaces. Let us only add that the form of these conditions depends on the choice of vectors $\mathbf{e}_i$.

3 Computation of the function J

Let us now determine the function J in the coordinates s, q_1, q_2. For the ray specified by the parameters γ_1', γ_2', we obtain, (see (1.3)),

$$J(s) = \frac{D(x,y,z)}{D(\sigma,\gamma_1',\gamma_2')} = \frac{D(x,y,z)}{D(s,q_1,q_2)}\cdot\frac{D(s,q_1,q_2)}{D(\sigma,\gamma_1',\gamma_2')} = h\frac{D(s,q_1,q_2)}{D(\sigma,\gamma_1',\gamma_2')}, \qquad (3.1)$$

where h is given by (2.5′). By direct inspection, we can find that the determinant $D(s, q_1, q_2)/D(\sigma, \gamma_1', \gamma_2')$ can be rewritten as follows

$$D(s,q_1,q_2)/D(\sigma,\gamma_1',\gamma_2') = (ds/d\sigma)\cdot D(q_1,q_2)/D(\gamma_1',\gamma_2').$$

Thus, for the central ray ($h = 1, ds/d\sigma = 1$), we have

$$J(s) = J(\tau) = D(q_1,q_2)/D(\gamma_1,\gamma_2) = Q_{11}Q_{22} - Q_{12}Q_{21}. \qquad (3.2)$$

In (3.2), the symbols Q_{ij} are used to denote partial derivatives of coordinates q_j with respect to ray parameters γ_i, $Q_{ij} = \partial q_j/\partial\gamma_i$ ($i, j = 1, 2$), taken on the central ray. Thus, to determine the geometrical spreading along the central ray, it is necessary to determine four quantities Q_{ij}, $i, j = 1, 2$.

3.1 SYSTEM OF DIFFERENTIAL EQUATIONS TO COMPUTE THE GEOMETRICAL SPREADING

The quantities Q_{ij} ($i, j = 1, 2$) can be determined from a system of eight linear ordinary differential equations of the first order. This system is obtained by means of partial differentiation of equations (2.9) and (2.14) with respect to γ_i and specifying the results for the central ray. The system has the following form:

$$dQ_{ij}/ds = vP_{ij}, \qquad dP_{ij}/ds = -v^{-2}(v_{j1}Q_{i1} + v_{j2}Q_{i2}), \qquad (3.3)$$

$(i, j = 1, 2)$. Another form, useful for computation, is as follows (*cf.* (1.2)):

$$
\begin{aligned}
dQ_{11}/d\tau &= v^2 P_{11}, \quad & dP_{11}/d\tau &= -v^{-1}(v_{11}Q_{11} + v_{12}Q_{12}),\\
dQ_{21}/d\tau &= v^2 P_{21}, \quad & dP_{21}/d\tau &= -v^{-1}(v_{11}Q_{21} + v_{12}Q_{22}),\\
dQ_{12}/d\tau &= v^2 P_{12}, \quad & dP_{12}/d\tau &= -v^{-1}(v_{21}Q_{11} + v_{22}Q_{12}),\\
dQ_{22}/d\tau &= v^2 P_{22}, \quad & dP_{22}/d\tau &= -v^{-1}(v_{21}Q_{21} + v_{22}Q_{22}).
\end{aligned} \tag{3.3'}
$$

In (3.3), P_{ij} denotes the partial derivatives of p_j with respect to ray parameters γ_i, $P_{ij} = \partial p_j/\partial \gamma_i$, $v_{ij} = \partial^2 v/\partial q_i \partial q_j$ $(i, j = 1, 2)$.

To solve system (3.3) or (3.3′), it is necessary to know the initial conditions at a source and the initial conditions at interfaces, which are discussed in Sections 3.2 and 3.3. We note that the details concerning these initial conditions and the evaluation of partial derivatives v_{ij} (see (3.3) and (3.3′)) can be found in Popov & Pšenčík (1976, 1978).

3.2 INITIAL CONDITIONS AT A SOURCE

The derivation of initial conditions at a source is not complicated. Therefore only the final results will be presented here.

Let us start with a point source. As ray parameters, the polar angles, azimuth ϕ_0 and declination $\delta_0 (0 \leq \phi_0 < 2\pi,\ 0 \leq \delta_0 \leq \pi)$ at the point source $\mathbf{r} = \mathbf{r}_0 (\tau = \tau_0)$ are taken. We choose them in such a way that the tangent to the ray at the point source $\mathbf{t}(\tau_0)$ can be expressed as follows

$$\mathbf{t}(\tau_0) \equiv (\sin\delta_0 \cos\phi_0, \sin\delta_0 \sin\phi_0, \cos\delta_0). \tag{3.4}$$

At the source, in the plane perpendicular to the ray specified by ϕ_0 and δ_0, we choose the vectors $\mathbf{e}_1$, $\mathbf{e}_2$ in such a way that the vector $\mathbf{e}_1$ is horizontal and the vector $\mathbf{e}_2$ is lying in the vertical plane containing the vector $\mathbf{t}$. Thus, the vectors $\mathbf{e}_1$, $\mathbf{e}_2$ at the source can be expressed as follows

$$
\begin{aligned}
\mathbf{e}_1(\tau_0) &= (-\sin\phi_0, \cos\phi_0, 0),\\
\mathbf{e}_2(\tau_0) &= (-\cos\delta_0\cos\phi_0, -\cos\delta_0\sin\phi_0, \sin\delta_0).
\end{aligned} \tag{3.5}
$$

Then the initial conditions at the point source are given by the following relations:

$$
\begin{aligned}
&Q_{ij}(\tau_0) = 0 \quad \text{for } i, j = 1, 2,\\
&P_{11}(\tau_0) = v_0^{-1}\sin\delta_0, \quad P_{12}(\tau_0) = 0, \quad P_{21}(\tau_0) = 0, \quad P_{22}(\tau_0) = -v_0^{-1}.
\end{aligned} \tag{3.6}
$$

It should be mentioned that the identity $Q_{ij}(\tau_0) = 0$ $(i, j = 1, 2)$ will hold for an arbitrary selection of vectors $\mathbf{e}_1$, $\mathbf{e}_2$ in the plane perpendicular to the investigated ray at the source. The initial conditions for the quantities $P_{ij} (i, j = 1, 2)$, however, will change with the change of the vectors $\mathbf{e}_1$, $\mathbf{e}_2$ at a source.

In the following also the initial conditions for the system (3.3), for the case of a line source are given. We assume that the medium is locally homogeneous in the close vicinity of the line source. The initial conditions can be derived in the same way as in (3.6).

Without any loss of generality, we can choose the Cartesian coordinate system in such a way that, say, the y axis is parallel to the line source. As ray parameters, we take the arclength L_0 along the line source and the declination δ_0 defined as above, i.e. the tangent to the ray leaving the line source can be expressed in the following way

$$\mathbf{t}(\tau_0) = (\sin\delta_0, 0, \cos\delta_0). \tag{3.7}$$

At the source, in the plane perpendicular to the ray specified by L_0 and δ_0, the vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ are chosen in such a way that the vector $\mathbf{e}_1$ coincides with the line source and the vector $\mathbf{e}_2$ is perpendicular to the source. Thus, the vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ can be expressed as follows

$$\mathbf{e}_1(\tau_0) = (0, 1, 0), \quad \mathbf{e}_2(\tau_0) = (-\cos\delta_0, 0, \sin\delta_0). \tag{3.8}$$

Then the initial conditions at the line source are given by the following relations

$$\begin{aligned} &Q_{11}(\tau_0) = 1, \quad Q_{12}(\tau_0) = 0, \quad Q_{21}(\tau_0) = 0, \quad Q_{22}(\tau_0) = 0, \\ &P_{11}(\tau_0) = 0, \quad P_{12}(\tau_0) = 0, \quad P_{21}(\tau_0) = 0, \quad P_{22}(\tau_0) = -v_0^{-1}. \end{aligned} \tag{3.9}$$

3.3 CONDITIONS AT INTERFACES

In the case of transmission or reflection of the ray at an interface of the first or even second order, some of the quantities Q_{ij}, P_{ij} $(i, j = 1, 2)$ change discontinuously at the point of incidence. Thus, new values of Q_{ij}, P_{ij} – we denote them by $\tilde{Q}_{ij}$, $\tilde{P}_{ij}$ – must be determined at the point of incidence. The values of $\tilde{Q}_{ij}$, $\tilde{P}_{ij}$ will serve as the initial values for the solution of the system (3.3) along a ray of a transmitted or reflected wave. The relations between Q_{ij}, P_{ij} and $\tilde{Q}_{ij}$, $\tilde{P}_{ij}$ depend on the orientation of vectors $\mathbf{e}_1$, $\mathbf{e}_2$ at the point of incidence. Here, we choose these vectors in the same way as in Červený *et al.* (1977).

Let us define a local Cartesian coordinate system x, y, z at the point of incidence in the following way. The z axis lies along the normal to the interface at the point of incidence and it is directed towards the medium from which the incident wave impinges on the interface. The y axis is perpendicular to the plane of incidence, i.e. a plane determined by the normal to the interface and tangent to the ray of the incident wave at the point of incidence. The positive direction of the y axis is taken so that the local y axis makes an obtuse angle with the general y axis. The x axis lies along the intersection of the plane of incidence with the plane tangent to the interface at the point of incidence. Its positive direction is determined so that the local system x, y, z is right-handed.

We shall present here the relations between Q_{ij}, P_{ij} and $\tilde{Q}_{ij}$, $\tilde{P}_{ij}$ for the vectors $\mathbf{e}_1$, $\mathbf{e}_2$ which are oriented at the point of incidence so that the vector $\mathbf{e}_1$ corresponds to the unit vector along the local y axis, perpendicular to the plane of incidence and the vector $\mathbf{e}_2$ lies in the plane of incidence, the direction of $\mathbf{e}_2$ being determined so that the system of vectors $\mathbf{t}$, $\mathbf{e}_1$, $\mathbf{e}_2$ is right-handed.

The orientation of vectors $\mathbf{e}_1$, $\mathbf{e}_2$ determined along the ray from (2.1), however, generally differ from the above specified orientation. Thus, before the formulae relating $\tilde{Q}_{ij}$, $\tilde{P}_{ij}$ with Q_{ij}, P_{ij} are applied, it is necessary to perform the corresponding rotation of vectors $\mathbf{e}_1$, $\mathbf{e}_2$ in the plane perpendicular to the ray and to transform the quantities Q_{ij}, P_{ij}, $\partial v/\partial q_i$ at the point of incidence in accordance with the rotation. The final formulae have the following form:

$$\begin{aligned} &\tilde{Q}_{j1} = Q_{j1}, \quad \tilde{Q}_{j2} = Q_{j2}\sin\beta/\sin\alpha, \\ &\tilde{P}_{j1} = P_{j1} - 2Q_{j1}D_{22}R - Q_{j2}S_2/\sin\alpha, \\ &\tilde{P}_{j2} = P_{j2}\sin\alpha/\sin\beta - Q_{j1}S_2/\sin\beta - Q_{j2}S_2/(\sin\alpha\sin\beta), \end{aligned} \tag{3.10}$$

$(j = 1, 2)$, where

$$\begin{aligned} &S_1 = 2V^{-1}\cos\alpha(\kappa_2\sin\alpha - \tilde{\kappa}_2\sin\beta) + 2D_{11}R + V^{-2}(V_s - \tilde{V}_s)\cos^2\alpha, \\ &S_2 = V^{-1}(\kappa_1 - \tilde{\kappa}_1)\cos\alpha + 2D_{12}R, \quad R = V^{-1}\sin\alpha - \tilde{V}^{-1}\sin\beta. \end{aligned} \tag{3.10'}$$

Here, α and β are the angles between the positive direction of the local x axis and the tangent to the ray of the incident and generated wave, respectively, at the point of incidence. The angles are measured positively clockwise from the local x axis. If we denote by ψ and $\tilde{\psi}$ the angles of incidence and reflection (transmission), i.e. the acute angles between the local z axis and tangent to the corresponding ray, then $\cos\psi = |\sin\alpha|$, $\cos\tilde{\psi} = |\sin\beta|$. The symbols $\kappa_j (j = 1, 2)$ have the following meaning: $\kappa_j = V^{-1}V_j$. The symbols V, V_s and V_j denote the velocity and its derivatives with respect to s and q_j at the point of incidence, measured from the side of the incident wave. The symbols $\tilde{\kappa}_j$, $\tilde{V}$, $\tilde{V}_s$ and $\tilde{V}_j$ have the same meaning on that side of the interface where the transmitted or reflected wave propagates. The symbols D_{11}, D_{12}, D_{22} in (3.10) denote the coefficients of the approximation of the equation of the interface in the vicinity of the point of incidence in the local coordinates,

$$z \sim D_{11}x^2 + 2D_{12}xy + D_{22}y^2. \tag{3.11}$$

For unconverted reflected waves and for waves transmitted at an interface of the second order, the formulae (3.10) simplify considerably.

4 Reduction of the number of differential equations in the additional system: 3-D medium

The system of eight linear ordinary differential equations of the first order (3.3′) for Q_{ij} and $P_{ij} (i, j = 1, 2)$ is simple and convenient for computation. The number of differential equations in the system however, can be decreased. The new systems are as a rule more complicated than (3.3′). In this chapter, we shall present several systems with the reduced number of equations. It would be necessary to perform some numerical tests to determine which of the systems is most convenient for computations.

4.1 SYSTEM OF FIVE LINEAR EQUATIONS

Let us introduce the quantities A, B, C, D, E by the following formulae

$$\begin{aligned} A &= Q_{11}Q_{22} - Q_{12}Q_{21}, \qquad B = Q_{11}P_{22} - Q_{21}P_{12}, \\ C &= Q_{22}P_{11} - Q_{12}P_{21}, \qquad D = P_{11}P_{22} - P_{12}P_{21}, \\ E &= Q_{22}P_{12} - Q_{12}P_{22} + Q_{11}P_{21} - Q_{21}P_{11}. \end{aligned} \tag{4.1}$$

Using (3.3), we obtain a system of five linear ordinary differential equations of the first order for these quantities:

$$\begin{aligned} &dA/d\tau = v^2(C + B), \quad dB/d\tau = v^2 D - v^{-1}v_{22}A, \quad dC/d\tau = v^2 D - v^{-1}v_{11}A, \\ &dD/d\tau = -v^{-1}(v_{11}B + v_{22}C - v_{12}E), \quad dE/d\tau = -2v^{-1}v_{12}A. \end{aligned} \tag{4.2}$$

The function J is then given by the formula

$$J = A. \tag{4.3}$$

The initial conditions for the system (4.2) for a point source situated at $\mathbf{r} = \mathbf{r}_0 (\tau = \tau_0)$ are as follows, see (3.6):

$$A(\tau_0) = B(\tau_0) = C(\tau_0) = E(\tau_0) = 0, \quad D(\tau_0) = -v_0^{-2}\sin\delta_0. \tag{4.4}$$

For a line source at $\mathbf{r} = \mathbf{r}_0 (\tau = \tau_0)$, defined in Section 3.2, we obtain

$$A(\tau_0) = C(\tau_0) = D(\tau_0) = E(\tau_0) = 0, \quad B(\tau_0) = -v_0^{-1}. \tag{4.4'}$$

The initial conditions for the system (4.2) at the point of reflection (transmission) at a curved interface can be easily obtained from (3.10).

It should be noted that the system (4.2) consists of five linear ordinary differential equations of the first order, which seems to be the minimum known number of such equations for the computation of J in a 3-D medium. The number of equations can be reduced further, but the equations will become non-linear. Only in special cases can we find systems with a smaller number of linear equations, but not in general.

4.2 SYSTEM FOR FOUR NON-LINEAR EQUATIONS

Consider a quantity

$$G = AD - BC + \tfrac{1}{4}E^2, \tag{4.5}$$

where A, B, C, D, E are the quantities introduced in Section 4.1. Using (4.2), it is not difficult to show that

$$dG/d\tau = 0. \tag{4.6}$$

Hence it follows that $G(\tau)$ = constant along the whole ray. Taking into account the initial conditions (4.4), respectively (4.4′) at the source and the initial conditions at individual interfaces, we obtain

$$G = 0 \tag{4.7}$$

along the whole ray, even if it crosses any number of interfaces. Thus, we have

$$AD - BC + \tfrac{1}{4}E^2 = 0, \tag{4.7'}$$

identically along any ray. Formula (4.7′) can be used to eliminate one equation from (4.2). There are several possible ways in which one of the variables can be eliminated. We shall present here one version of such a system, other possibilities are straightforward.

From (4.7′) we can express A in terms of the quantities B, C, D, E (assuming $D \neq 0$),

$$A = (BC - \tfrac{1}{4}E^2)/D. \tag{4.8}$$

The auxiliary quantities B, C, D, E can be determined from the system of four ordinary differential equations of the first order (three non-linear and one linear),

$$\begin{aligned} dB/d\tau &= v^2 D - v^{-1} v_{22}(BC - \tfrac{1}{4}E^2)/D, \\ dC/d\tau &= v^2 D - v^{-1} v_{11}(BC - \tfrac{1}{4}E^2)/D, \\ dE/d\tau &= -2v^{-1} v_{12}(BC - \tfrac{1}{4}E^2)/D, \\ dD/d\tau &= -v^{-1}(v_{11}B + v_{22}C - v_{12}E). \end{aligned} \tag{4.9}$$

The function J is then obtained from the formula

$$J = A = (BC - \tfrac{1}{4}E^2)/D, \tag{4.10}$$

see (4.3) and (4.8).

For a point source we have the following initial conditions

$$B(\tau_0) = C(\tau_0) = E(\tau_0) = 0, \qquad D(\tau_0) = -v_0^{-2}\sin\delta_0. \tag{4.11}$$

The system (4.9) is not convenient for a line source, as then $D(\tau_0) = 0$. It would be, however, easy to write an equivalent system suitable for a line source, using $B = (AD + \tfrac{1}{4}E^2)/C$ instead of (4.8).

The initial conditions for the system (4.9) at the points of reflection (transmission) at curved interfaces are just the same as the initial conditions for the system (4.2).

4.3 SYSTEM OF THREE NON-LINEAR EQUATIONS

After some simple manipulations we obtain from (4.9) a fully independent system of three non-linear differential equations of the first order. The remaining fourth equation in (4.9) can be solved by quadratures along the ray, as soon as the solution of the three equations is found. Let us introduce

$$b = B/D, \qquad c = C/D, \qquad e = E/D. \tag{4.12}$$

From (4.9) we obtain without difficulty a system of three ordinary differential equations of the first order for b, c, e and one linear equation for D. The system for b, c, e reads:

$$\begin{aligned} db/d\tau &= v^2 - v^{-1}(v_{12}eb - \tfrac{1}{4}v_{22}e^2 - v_{11}b^2), \\ dc/d\tau &= v^2 - v^{-1}(v_{12}ec - \tfrac{1}{4}v_{11}e^2 - v_{22}c^2), \\ de/d\tau &= v^{-1}[v_{11}be + v_{22}ce - 2v_{12}(bc + \tfrac{1}{4}e^2)]. \end{aligned} \tag{4.13}$$

The equation for D now reads

$$dD/d\tau = -v^{-1}D(v_{11}b + v_{22}c - v_{12}e). \tag{4.14}$$

The initial conditions for a point source situated at $\mathbf{r} = \mathbf{r}_0(\tau = \tau_0)$ are as follows

$$b(\tau_0) = c(\tau_0) = e(\tau_0) = 0, \qquad D(\tau_0) = -v_0^{-2}\sin\delta_0. \tag{4.15}$$

The initial conditions for the system (4.13) and for the equation (4.14) at the points of reflection (transmission) at a curved interface can be obtained from (3.10). Since the system (4.13) might be important in applications, we shall present here these conditions in a complete form,

$$\begin{aligned} \tilde{b} &= [b\sin^2\alpha - (bc - \tfrac{1}{4}e^2)S_1]/\Delta, \\ \tilde{c} &= [c - 2(bc - \tfrac{1}{4}e^2)D_{22}R]\sin^2\beta/\Delta, \\ \tilde{e} &= [e\sin\alpha - 2(bc - \tfrac{1}{4}e^2)S_2]\sin\beta/\Delta, \\ \tilde{D} &= D\Delta/(\sin\alpha\sin\beta), \end{aligned} \tag{4.16}$$

where

$$\Delta = (bc - \tfrac{1}{4}e^2)(2D_{22}RS_1 - S_2^2) - (2bD_{22}R - 1)\sin^2\alpha - cS_1 + eS_2\sin\alpha. \tag{4.16'}$$

All the symbols used in the right-sides of (4.16) and (4.16′) are defined in (3.10) and in the following discussion.

Assume now that the quantities $b(\tau)$, $c(\tau)$, $e(\tau)$ were determined from (4.13) along the whole ray. Equation (4.14) can be then solved by quadratures to give

$$D(\tau) = D(\tau_0)W\exp\left[-\int_{\tau_0}^{\tau} v^{-1}(v_{11}b + v_{22}c - v_{12}e)d\zeta\right], \tag{4.17}$$

where

$$W = \prod_{i=1}^{N} \Delta_i/(\sin\alpha_i\sin\beta_i). \tag{4.17'}$$

In (4.17′), Δ_i, α_i, β_i, denote the quantities Δ (see (4.16′)), α and β, respectively, corresponding to the ith point of incidence of the investigated ray on an interface, N denotes the number of interfaces struck by the investigated ray.

The function $J(\tau)$ is finally obtained from the formula

$$J(\tau) = D(\tau_0)W[b(\tau)c(\tau) - \tfrac{1}{4}e^2(\tau)] \exp\left\{-\int_{\tau_0}^{\tau} v^{-1}(v_{11}b + v_{22}c - v_{12}e)d\zeta\right\}, \tag{4.18}$$

see (4.10) and (4.12).

For a general case of a three-dimensional medium with the coordinates q_1, q_2 along the vectors $\mathbf{e}_1$, $\mathbf{e}_2$ specified by (2.1), the above presented system (4.13) cannot be reduced further. Thus the minimum number of ordinary differential equations of the first order in the additional system is three.

It would, however, be possible to use a special coordinate system q_1, q_2, which would keep $v_{12} = 0$ along the whole ray. The system of ordinary differential equations of the first order could then be reduced to two equations only.

5 Reduction of the number of differential equations in the additional system: 2-D medium

In some special situations the additional systems can be considerably simplified. The most important example is a two-dimensional medium. Under a two-dimensional medium we shall understand the medium in which the velocity does not depend on one coordinate. Let us consider a coordinate plane Σ perpendicular to, say, the coordinate axis y, along which the velocity does not change, and let us denote the orthogonal coordinates in the plane by x and z (x denotes the horizontal distance, z the depth). When the initial directions of the rays lie in the plane Σ, the rays are plane curves and lie fully in Σ. We choose the coordinate system s, q_1, q_2 such that the vector $\mathbf{e}_1$ is perpendicular to Σ. Then we have $v_1 = \partial v/\partial q_1 = 0$ and $v_{12} = \partial^2 v/\partial q_1 \partial q_2 = 0$ along the whole ray. It would be possible to obtain the additional system of equations for a 2-D medium from the systems presented in Section 4. It is, however, more natural and simpler to start directly from the system of eight equations (3.3). Taking into account the initial conditions at the source and the boundary conditions at interfaces, we readily obtain

$$Q_{12}(\tau) = Q_{21}(\tau) = P_{12}(\tau) = P_{21}(\tau) = 0, \tag{5.1}$$

identically along the ray. The system (3.3′) is now reduced to two independent systems, each of them consisting of two linear ordinary differential equations of the first order. The first system reads

$$dQ_{11}/d\tau = v^2 P_{11}, \quad dP_{11}/d\tau = -v^{-1}v_{11}Q_{11}. \tag{5.2}$$

The second system is as follows

$$dQ_{22}/d\tau = v^2 P_{22}, \quad dP_{22}/d\tau = -v^{-1}v_{22}Q_{22}. \tag{5.3}$$

The initial conditions for P_{11}, P_{22}, Q_{11}, Q_{22} at the source and at the point of reflection (transmission) at a curved interface for these systems can be deduced from those given in (3.6), (3.9) and (3.10).

Let us denote

$$J_{\parallel}(\tau) = Q_{22}, \quad J_{\perp}(\tau) = Q_{11}. \tag{5.4}$$

The function $J(\tau)$ is then given by the formula

$$J(\tau) = J_{\parallel}(\tau) J_{\perp}(\tau). \tag{5.5}$$

Geometrically, the quantity $J_{\parallel}(\tau)$ describes the spreading in the plane Σ, $J_{\perp}(\tau)$ in the plane perpendicular to the plane Σ and to the ray.

5.1 COMPUTATION OF $J_{\parallel}(\tau)$

The system (5.3) can be used to compute $J_{\parallel}(\tau)$. The system, however, can be rewritten in several equivalent forms. We shall present three of these versions. It would be necessary to perform numerical tests to find which version is the most convenient for computations.

5.1.1 Two linear equations of first order

Inserting $J_{\parallel}(\tau) = Q_{22}$ into (5.3) and denoting $P_{\parallel} = P_{22}$, we obtain the final form of two linear differential equations of the first order to compute $J_{\parallel}$,

$$dJ_{\parallel}/d\tau = v^2 P_{\parallel}, \qquad dP_{\parallel}/d\tau = -v^{-1} v_{22} J_{\parallel}. \tag{5.6}$$

The initial conditions for the above system at a point source situated at $\mathbf{r} = \mathbf{r}_0 (\tau = \tau_0)$ (or a line source defined in Section 3.2, and situated at the same point) are as follows, see (3.6) and (3.9),

$$J_{\parallel}(\tau_0) = 0, \qquad P_{\parallel}(\tau_0) = -v_0^{-1}. \tag{5.7}$$

The initial conditions at the points of reflection (transmission) at curved interfaces can be obtained from (3.10). We again denote by a tilde quantities corresponding to reflected (transmitted) waves, the quantities corresponding to the incident wave being without a tilde. Then, $\tilde{J}_{\parallel}$ and $\tilde{P}_{\parallel}$ are given by the following formulae:

$$\tilde{J}_{\parallel} = J_{\parallel} \sin\beta/\sin\alpha, \qquad \tilde{P}_{\parallel} = (P_{\parallel} \sin\alpha - J_{\parallel} S_1/\sin\alpha)/\sin\beta, \tag{5.8}$$

where the meaning of the symbols α, β and S_1 is the same as in (3.10) and (3.10′). For the equation of the interface $f(x, z) = 0$ the quantity D_{11} in the expression for S_1 (see (3.10′)) is given by the formula:

$$D_{11} = -\tfrac{1}{2} A (f_{xx} f_z^2 - 2 f_{xz} f_x f_z + f_{zz} f_x^2)/(f_x^2 + f_z^2)^{3/2}. \tag{5.9}$$

In (5.9), $f_x = \partial f/\partial x$, $f_z = \partial f/\partial z$, $f_{xx} = \partial^2 f/\partial x^2$, $f_{xz} = \partial^2 f/\partial x \partial z$, $f_{zz} = \partial^2 f/\partial z^2$, the constant A assumes the value of $+1$ or -1 depending on the desired orientation of the normal. If we choose, for example, the orientation of the normal so that it is directed toward the medium in which the incident wave propagates, then $A = -\operatorname{sign}(f_x \sin\delta + f_z \cos\delta)$. Here δ is the angle between the tangent to the ray and the positive direction of the general z axis at the point of incidence, $-\pi \leqslant \delta \leqslant \pi$.

Finally, the partial derivatives of velocity with respect to s and q_2 appearing in (5.6) and (5.8) can be expressed as follows

$$\begin{aligned} &\partial v/\partial s = v_x \sin\delta + v_z \cos\delta, \qquad v_2 = v_x \cos\delta - v_z \sin\delta, \\ &v_{22} = v_{xx} \cos^2\delta - 2 v_{xz} \cos\delta \sin\delta + v_{zz} \sin^2\delta. \end{aligned} \tag{5.10}$$

In (5.10), $v_x = \partial v/\partial x$, $v_z = \partial v/\partial z$, $v_{xx} = \partial^2 v/\partial x^2$, $v_{xz} = \partial^2 v/\partial x \partial z$, $v_{zz} = \partial^2 v/\partial z^2$. The angle δ has the same meaning as above.

5.1.2 One linear equation of the second order

It is not complicated to rewrite the system of two ordinary differential equations of the first order into one linear differential equation of the second order for $J_{\parallel}$:

$$d(v^{-2}dJ_{\parallel}/d\tau)/d\tau + v^{-1}v_{22}J_{\parallel} = 0. \tag{5.11}$$

The initial conditions for $J_{\parallel}$ at the source and at the points of reflection (transmission) are given in (5.7) and (5.8). The initial conditions for $dJ_{\parallel}/d\tau$ are easily obtained from the same equations, when we consider $dJ_{\parallel}/d\tau = v^2 P_{\parallel}$.

5.1.3 One non-linear equation of the first order

We introduce

$$q = J_{\parallel}/P_{\parallel}. \tag{5.12}$$

Then we obtain for q one fully independent non-linear ordinary differential equation of the first order

$$dq/d\tau = v^2 + v_{22}v^{-1}q^2. \tag{5.13}$$

The equation for $P_{\parallel}$ now reads

$$dP_{\parallel}/d\tau = -v^{-1}v_{22}P_{\parallel}q. \tag{5.14}$$

The initial conditions for a point as well as line source situated at $\mathbf{r} = \mathbf{r}_0(\tau = \tau_0)$ are as follows:

$$q(\tau_0) = 0, \quad P_{\parallel}(\tau_0) = v_0^{-1}. \tag{5.15}$$

The initial conditions at the point of reflection (transmission) at a curved interface simply follow from (5.8) and (5.12)

$$\tilde{q} = q \sin^2\beta/\bar{\Delta}, \quad \tilde{P}_{\parallel} = P_{\parallel}\bar{\Delta}/(\sin\alpha\sin\beta), \tag{5.16}$$

where

$$\bar{\Delta} = \sin^2\alpha - qS_1. \tag{5.16'}$$

All the symbols used in (5.16) and (5.16′) are defined in (3.10) and in the following discussion.

Assume now that the quantity $q(\tau)$ was determined from (5.13) along the whole ray. Equation (5.14) can be then solved by quadratures to give

$$P_{\parallel}(\tau) = P_{\parallel}(\tau_0)\bar{W}\exp\left\{-\int_{\tau_0}^{\tau} v^{-1}v_{22}q\,d\zeta\right\}, \tag{5.17}$$

where

$$\bar{W} = \prod_{i=1}^{N} \bar{\Delta}_i/(\sin\alpha_i\sin\beta_i). \tag{5.17'}$$

The symbols $\bar{\Delta}_i$, α_i, β_i denote the quantities $\bar{\Delta}$, α, β, respectively, corresponding to the ith point of incidence of the ray on an interface; N denotes the number of interfaces struck by the investigated ray.

Function $J_{\|}$ is finally given by the formula

$$J_{\|}(\tau) = P_{\|}(\tau_0)\bar{W}q(\tau)\exp\left\{-\int_{\tau_0}^{\tau} v^{-1}v_{22}q\,d\zeta\right\}. \tag{5.18}$$

Thus, the additional system of ordinary differential equations reduces to only one non-linear differential equation of the first order for a 2-D situation (see (5.13)). The procedure must be supplemented by one numerical quadrature along a ray.

5.2 COMPUTATION OF $J_{\perp}(\tau)$

Denoting $J_{\perp} = Q_{11}$, $P_{\perp} = P_{11}$, we have a system of two differential equations to determine $J_{\perp}$ and $P_{\perp}$ (see (5.2))

$$dJ_{\perp}/d\tau = v^2 P_{\perp}, \quad dP_{\perp}/d\tau = -v^{-1}v_{11}J_{\perp}. \tag{5.19}$$

The initial conditions for $J_{\perp}$ and $P_{\perp}$ for a point (or line) source situated at the point $\mathbf{r} = \mathbf{r}_0$ ($\tau = \tau_0$) are as follows:

$$J_{\perp}(\tau_0) = 0, \quad P_{\perp}(\tau_0) = v_0^{-1}\sin\delta_0 \tag{5.20}$$

for a point source,

$$J_{\perp}(\tau_0) = 1, \quad P_{\perp}(\tau_0) = 0 \tag{5.21}$$

for a line source.

The initial conditions at points of reflection (transmission) can be simply obtained from (3.10).

We shall now consider three different situations, which can have important practical applications.

5.2.1 *Cartesian coordinates – point source*

In this case, $v_{11} = 0$ and $P_{\perp}(\tau) = P_{\perp}(\tau_0) = v_0^{-1}\sin\delta_0$. Inserting this expression into the equation for $J_{\perp}$ we obtain $dJ_{\perp}/d\tau = v^2\sin\delta_0 v_0^{-1}$, which gives

$$J_{\perp}(\tau) = v_0^{-1}\sin\delta_0\int_{\tau_0}^{\tau} v^2\,d\zeta. \tag{5.22}$$

5.2.2 *Cartesian coordinates – line source*

In this case, $v_{11} = 0$ and $P_{\perp}(\tau) = P_{\perp}(\tau_0) = 0$ along the whole ray. From the first equation in (5.19), we then obtain $J_{\perp}(\tau) = J_{\perp}(\tau_0)$. Taking into account (5.21), we have finally

$$J_{\perp}(\tau) = 1. \tag{5.23}$$

5.2.3 *Cylindrical coordinates – point source*

Let us consider cylindrical coordinates r, z, ϕ and assume that the velocity does not depend on ϕ. Assume that a point source is situated at $r = 0$, thus r is the epicentral distance. As the coordinate plane Σ, let us select any vertical plane containing the axis of symmetry (z axis) and consider that the initial directions of rays are again in the plane Σ. In this case again $v_1 = 0$, the value of v_{11}, however, does not vanish, $v_{11} = r^{-1}\partial v/\partial r$. Then we can rewrite the system (5.19) into the following form

$$dJ_\perp/d\tau = v^2 P_\perp, \quad dP_\perp/d\tau = -v^{-1}r^{-1}\partial v/\partial r J_\perp. \tag{5.24}$$

By direct inspection, we readily find the solution of these equations

$$J_\perp(\tau) = r, \quad P_\perp(\tau) = v^{-1}\sin\delta. \tag{5.25}$$

(Let us note that equations (5.24) correspond in fact to two equations in the ray-tracing system: $dx/d\tau = v^2 p_x$, $dp_x/d\tau = -v^{-1}\partial v/\partial x$, where p_x is the x component of the slowness vector, $p_x = v^{-1}\sin\delta$.)

6 Dependence of velocity on one spatial coordinate only

When the velocity depends only on one coordinate, say z, the differential equation for $J_\parallel$ can be solved in closed-form integrals. We shall consider equation (5.11). We choose a new variable z instead of τ, $dz = v\cos\delta\, d\tau$. Taking into account that $v_{22} = v_{zz}\sin^2\delta$ (see (5.10)), we obtain

$$\cos\delta\, d(v^{-1}\cos\delta\, dJ_\parallel/dz)/dz + v^{-2}v_{zz}\sin^2\delta\, J_\parallel = 0. \tag{6.1}$$

We introduce a new quantity $\bar{J}$ by the relation

$$J_\parallel = \bar{J}\cos\delta. \tag{6.2}$$

For $\bar{J}$, we obtain from (6.1) a new differential equation

$$d(-p^2 v_z\bar{J} + v^{-1}\cos^2\delta\, d\bar{J}/dz)/dz + p^2 v_{zz}\bar{J} = 0, \tag{6.3}$$

where $p = v^{-1}\sin\delta$. After some manipulation this equation can be rewritten in the following form

$$d(\ln d\bar{J}/dz)/dz = v_z v^{-1}\cos^2\delta(2\sin^2\delta + 1) = d[\ln(v\cos^{-3}\delta)]/dz.$$

From this, an ordinary differential equation of the first order for $\bar{J}$ follows immediately,

$$d\bar{J}/dz = Cv\cos^{-3}\delta, \qquad C = v_0^{-1}\cos\delta_0.$$

The equation can be integrated to give

$$\bar{J}(z) = C\int_{z_0}^{z} v\cos^{-3}\delta\, d\zeta. \tag{6.4}$$

Finally, we obtain for $J_\parallel(z)$, see (6.2) and (6.4),

$$J_\parallel(z) = C\cos\delta\int_{z_0}^{z} v(\zeta)\cos^{-3}\delta\, d\zeta. \tag{6.5}$$

This is a well-known integral expression for $J_\parallel(z)$ in a vertically inhomogeneous medium.

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