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Seismic Rays and Ray Intensities in Inhomogeneous Anisotropic Media

V. Červený

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Summary

A ray series expansion for seismic body waves propagating in inhomogeneous anisotropic media is studied. Methods for calculation of rays and amplitude coefficients of the ray series are suggested. A seismic ray is described by a system of ordinary differential equations of first order which can be solved by standard numerical techniques. Another system of ordinary differential equations can be used to compute amplitude coefficients. The method may be applied to general anisotropic media in which the elastic parameters are arbitrary continuous functions of all three co-ordinates.

1. Introduction

In connection with a more detailed study of the structure of the Earth's crust and upper mantle greater attention has recently been paid to the anisotropy of the velocities of seismic waves and to its influence on travel-time curves of seismic waves. The theory of propagation of elastic waves in an anisotropic medium is well developed in the case of plane waves in a homogeneous medium (see e.g. Musgrave 1954; Fedorov 1965; Hearmon 1961). However, the problem of propagation of elastic waves in inhomogeneous anisotropic media has not been given much consideration until now although it is evident that in seismology and seismic prospecting this problem is of considerable importance.

Because the methods for separation of the equation of motion for a general inhomogeneous anisotropic medium are not known, it is necessary, when solving this problem, to make use of approximate methods, e.g. the so-called ray method. The theory of ray series has been well developed for an isotropic medium (Alekseyev, Babich & Gel'chinskiy 1961; Karal & Keller 1959; Yanovskaya 1968; Červený & Ravindra 1971) and it has brought about a number of very valuable results. The theory of ray series was first applied to anisotropic media by Babich (1961) who derived differential equations for the wave fronts and for the amplitude coefficients of a ray series. The principles of Babich's approach will essentially be followed in the present paper. However, the possibilities of numerical calculation of individual quantities, important from the viewpoint of application, are not studied in detail in Babich's work. The purpose of the present paper is to derive systems of equations for the most important quantities which will enable a numerical solution by means of standard numerical procedures.

From the viewpoint of applications in seismology the kinematic description of elastic waves (i.e. rays, wave fronts, and travel-time curves) and the calculation of the zeroth amplitude coefficient of a ray series are of the greatest importance. In this

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paper numerical procedures for the calculation of these quantities are proposed. A number of results valuable from the viewpoint of kinematic description of wave processes in some special cases of an inhomogeneous anisotropic medium have already been presented in papers by Gassmann (1964), Helbig (1966), Vlaar (1968, 1969).

In the ray theory for an inhomogeneous anisotropic medium, a very important role is played by the eigenvalues of the matrix $\Gamma_{jk} = p_i p_l c_{ijkl}/\rho$, where c_{ijkl} are the elastic parameters, ρ the density and p_i the components of the slowness vector. The eigenvalues of the matrix Γ_{jk} can be written in a simple form only in the case of an isotropic medium and certain few simple anisotropic media (e.g. a transversely isotropic medium). For most anisotropic media, however, the analytical expressions for the eigenvalues are very complicated. This fact has been a limiting factor in previous papers. Therefore, the main effort in the present paper is to derive such equations for seismic rays for which a knowledge of the eigenvalues of the matrix Γ_{jk} is not necessary. By means of the theorem on implicit functions these equations have been derived. A seismic ray is described by a system of ordinary differential equations of first order which can be solved by standard numerical procedures, e.g. Runge–Kutta's method or Hamming's predictor-corrector method. The system may be used for calculation of rays of quasi-compressional as well as quasi-shear waves in an arbitrary anisotropic inhomogeneous medium (with the exception of singular situations). In simpler situations when analytical expressions for the eigenvalues of the matrix Γ_{jk} are known, the system reduces to the cases studied previously. A similar system of ordinary differential equations for a seismic ray in an inhomogeneous *isotropic* medium has been known for some time and has helped in the study of a large variety of problems connected with the kinematics of seismic waves in an isotropic inhomogeneous medium (Gassmann 1964; Yeliseyevnin 1964; Belonosova, Tadzhimukhamedova & Alekseyev 1967; Chernoff 1967; Eby 1967, 1970; Marcinkovskaya & Krasavin 1968; Burmakov & Oblogina 1968; Miri–Zade 1968, 1970; Belonosova 1969; Omel'chenko 1969; Wesson 1970, 1971; Červený & Ravindra 1971). The system of differential equations for a seismic ray in an inhomogeneous anisotropic medium proposed here, can serve as the basis for solving analogous problems in an anisotropic medium.

Further, in this paper a system of ordinary differential equations of first order is derived which enables the study of the amplitudes of seismic body waves (in the zeroth approximation of the ray theory) along the ray by the same numerical methods. The resulting system of differential equations can be applied to an arbitrary anisotropic medium (with the exception of some singular situations). Some difficulties could arise in the case of quasi-shear waves. However, the system is universal for quasi-compressional waves. An analogous system for an isotropic inhomogeneous medium was derived by Wesson (1970), see also Uginčius (1969, 1970). The proposed system does not overcome the fundamental limitations introduced by the ray theory.

For other approaches to the investigation of elastic wave propagation in anisotropic media see, e.g. Backus (1965, 1970), Gilbert & Backus (1966), Crosson & Christensen (1969), Crampin (1970), Sato & Lapwood (1968a, b), Sato, Lapwood & Singh (1971), where other references can be found.

2. Ray theory for inhomogeneous anisotropic media

The propagation of elastic waves in an inhomogeneous anisotropic medium is investigated. The equation of motion in rectangular Cartesian coordinates x_i can be written in the form

$$\frac{\partial}{\partial x_i}\left(c_{ijkl}\frac{\partial U_k}{\partial x_l}\right) = \rho\frac{\partial^2 U}{\partial t^2} \quad , \qquad (1)$$

where t is the time, ρ is the density, U_j are the components of the displacement vector $\mathbf{U}$, c_{ijkl} are the elastic parameters. The elastic parameters c_{ijkl}, the density ρ and their derivatives are assumed to be continuous functions of coordinates.

The solutions of the equation of motion (1) which are non-analytic along certain moving surfaces are sought. These surfaces are called wave fronts. Let us suppose that a wave front is described by the equation

$$t = \tau(x_i). \tag{2}$$

It is known that in this case it is convenient to seek the solution in the form of a ray series

$$U_k(x_i, t) = \sum_{n=0}^{\infty} U_k^{(n)}(x_i) f_n(t-\tau(x_i)), \tag{3}$$

where the functions $f_n(\vartheta)$ satisfy the relation

$$df_{n+1}(\vartheta)/d\vartheta = f_n(\vartheta). \tag{4}$$

The ray series includes the solutions which are discontinuous on wave fronts. It follows from (4) that the order of discontinuity of function f_{n+1} on the wave front (at $t = \tau$) is one less than that of f_n.

If (3) is substituted into (1) and the coefficients of the functions $f_n(t-\tau)$ on the left-hand and right-hand sides of the equation compared, a system of equations is obtained

$$\mathbf{N}(\mathbf{U}^{(n)}) - \mathbf{M}(\mathbf{U}^{(n-1)}) + \mathbf{L}(\mathbf{U}^{(n-2)}) = 0, \tag{5}$$

for $n = 0, 1, 2, \ldots$, with $\mathbf{U}^{(-1)} = \mathbf{U}^{(-2)} = 0$. Vector operators $\mathbf{N}$, $\mathbf{M}$ and $\mathbf{L}$ are given by the formulae

$$\left.\begin{aligned} N_j(\mathbf{U}^{(n)}) &= \Gamma_{jk} U_k^{(n)} - U_j^{(n)}, \\ M_j(\mathbf{U}^{(n)}) &= p_i a_{ijkl} \frac{\partial U_k^{(n)}}{\partial x_l} + \rho^{-1} \frac{\partial}{\partial x_i} (\rho a_{ijkl} p_l U_k^{(n)}), \\ L_j(\mathbf{U}^{(n)}) &= \rho^{-1} \frac{\partial}{\partial x_i} \left(\rho a_{ijkl} \frac{\partial U_k^{(n)}}{\partial x_l}\right), \end{aligned}\right\} \tag{6}$$

where

$$\Gamma_{jk} = p_i p_l a_{ijkl}, \; a_{ijkl} = c_{ijkl}/\rho, \; p_i = \frac{\partial \tau}{\partial x_i}. \tag{7}$$

The system of equations (5) is the basic system of equations of ray theory for an inhomogeneous anisotropic medium. A similar system was first derived by Babich (1961). This system can be used, when certain initial conditions are known, to determine $\tau(x_i)$ and $\mathbf{U}^{(n)}(x_i)$ $(n = 0, 1, 2, \ldots)$. The system is recurrent.

Before solving (5) let us note some properties of the matrix Γ_{jk}. Its eigenvalues can be determined as roots of the characteristic equation

$$\operatorname{Det}(\Gamma_{jk} - G\delta_{jk}) = 0. \tag{8a}$$

This equation may be rewritten in the following form

$$G^3 - PG^2 + QG - R = 0 \tag{8b}$$

where the values P, Q and R are the invariants of a symmetric matrix Γ_{jk}. It is evident from (8b) that the matrix Γ_{jk} has three eigenvalues. We shall denote them G_1, G_2 and G_3. It is known that the matrix Γ_{jk} is definite positive. Therefore, the eigenvalues of Γ_{jk} are always real and positive. The eigenvalues $G_m(p_i, x_i)$ are homogeneous functions of second order in p_i. Using Euler's theorem on homogeneous functions, the equation

$$p_i \frac{\partial G_m}{\partial p_i} = 2G_m \qquad (m = 1, 2, 3), \tag{9}$$

is obtained. One eigenvector $\mathbf{g}^{(m)}$ corresponds to each eigenvalue G_m of the matrix Γ_{jk}. Consequently, the matrix Γ_{jk} has three eigenvectors, $\mathbf{g}^{(1)}$, $\mathbf{g}^{(2)}$ and $\mathbf{g}^{(3)}$, corresponding to three eigenvalues G_1, G_2 and G_3, respectively. The eigenvectors $\mathbf{g}^{(m)}$ will be taken as unit vectors. If $G_1 \neq G_2 \neq G_3$, they can be determined (except for the sign) from equations

$$(\Gamma_{jk} - G_m \delta_{jk})\, g_k^{(m)} = 0, \tag{10}$$

(no summation over m). In the degenerate case of two identical eigenvalues the direction of the two corresponding eigenvectors cannot be determined from (10); only the plane in which they lie is determined (perpendicular to the other eigenvector).

Note that $G_1 = \alpha^2 p_i p_i$ and $G_2 = G_3 = \beta^2 p_i p_i$ for isotropic media. α and β are the velocities of compressional and shear waves, respectively.

3. Calculation of rays and wave fronts

Let us return to the basic equations (5). Equation (5) yields $\mathbf{N}(\mathbf{U}^{(0)}) = 0$ for $n = 0$, i.e.

$$(\Gamma_{jk} - \delta_{jk})\, U_k^{(0)} = 0. \tag{11}$$

(11) represents a system of three algebraic equations for $U_1^{(0)}$, $U_2^{(0)}$ and $U_3^{(0)}$. By comparison with (8a) it is easily seen that system (11) has a non-trivial solution only in the case when any of the eigenvalues of the matrix Γ_{jk} (G_1, G_2 or G_3) is equal to one. If $G_1 \neq G_2 \neq G_3$, (11) has a non-trivial solution in three cases: (a) $G_1(p_i, x_i) = 1 (G_2 \neq 1,\ G_3 \neq 1)$, (b) $G_2(p_i, x_i) = 1 (G_1 \neq 1, G_3 \neq 1)$, and (c) $G_3(p_i, x_i) = 1 (G_1 \neq 1, G_2 \neq 1)$.

Each of the above mentioned equations

$$G_m(p_1, p_2, p_3, x_1, x_2, x_3) = 1, \tag{12}$$

is a non-linear partial differential equation for $\tau(x_i)$ which describes the propagation of a wave front. Thus, in an inhomogeneous anisotropic medium (if a_{ijkl} and their derivatives are continuous) three independent wave-fronts can propagate. One of them corresponds to the so-called quasi-compressional waves, the other two wave fronts to two different quasi-shear waves (which are generally independent of each other). In the degenerate case of two identical eigenvalues (e.g. $G_2 = G_3$) there will be only two independent wave fronts.

The non-linear partial differential equation $G_m(p_i, x_i) = 1$ can be solved by means of characteristics (see, e.g. Courant & Hilbert 1962). If (9) is also used, the equations of characteristics corresponding to the partial differential equation $G_m(p_i, x_i) = 1$ can be written in the form

$$\frac{dx_i}{d\tau} = \frac{1}{2}\frac{\partial G_m}{\partial p_i}, \quad \frac{dp_i}{d\tau} = -\frac{1}{2}\frac{\partial G_m}{\partial x_i}, \tag{13}$$

($i = 1, 2, 3$).

The basic difficulty in using the system (13) in anisotropic media lies in the fact that the analytical expressions for G_m are very complicated (the solution of cubic equation (8b)). Therefore, it is possible to use the system (13) without additional rearrangements only in the simpler situations when (8) can be rewritten in a factorized form (e.g. as a product of one linear and one quadratic factor).

Examples are the isotropic and transversely isotropic media. For example, for compressional waves propagating in inhomogeneous isotropic media ($G_1 = \alpha^2 p_i p_i = 1$), the well-known system of ordinary differential equations is obtained from (13):

$$\frac{dx_i}{d\tau} = \alpha^2 p_i, \; \frac{dp_i}{d\tau} = -\frac{1}{\alpha}\frac{\partial\alpha}{\partial x_i}.$$

However, in equations (13) the analytical expression for the eigenvalues need not be known explicitly; only their partial derivatives must be known. Analytical expressions for these partial derivatives can be written without difficulty, using equations (8a) or (8b) and the theorem on implicit functions. We obtain

$$\left.\begin{aligned} \frac{\partial G_m}{\partial p_i} &= h_i(p_j, x_j) = \frac{\partial \Gamma_{jk}}{\partial p_i} D_{jk}/D, \\ \frac{\partial G_m}{\partial x_i} &= H_i(p_j, x_j) = \frac{\partial \Gamma_{jk}}{\partial x_i} D_{jk}/D, \end{aligned}\right\} \tag{14a}$$

where

$$\left.\begin{aligned} D_{11} &= (\Gamma_{22}-1)(\Gamma_{33}-1)-\Gamma_{23}{}^2, \\ D_{22} &= (\Gamma_{11}-1)(\Gamma_{33}-1)-\Gamma_{13}{}^2, \\ D_{33} &= (\Gamma_{11}-1)(\Gamma_{22}-1)-\Gamma_{12}{}^2, \\ D_{12} &= D_{21} = \Gamma_{13}\Gamma_{23}-\Gamma_{12}(\Gamma_{33}-1), \\ D_{13} &= D_{31} = \Gamma_{12}\Gamma_{23}-\Gamma_{13}(\Gamma_{22}-1), \\ D_{23} &= D_{32} = \Gamma_{12}\Gamma_{13}-\Gamma_{23}(\Gamma_{11}-1), \\ D &= tr D_{jk} = D_{11}+D_{22}+D_{33}. \end{aligned}\right\} \tag{14b}$$

When deriving (14) it was taken into consideration that $G_m(p_i, x_i) = 1$. Partial derivatives of Γ_{jk} which are in (14) are determined easily from the relation (7) for Γ_{jk},

$$\left.\begin{aligned} \frac{\partial \Gamma_{jk}}{\partial x_i} &= \frac{\partial a_{ljks}}{\partial x_i} p_l p_s, \\ \frac{\partial \Gamma_{jk}}{\partial p_i} &= (a_{ijkl}+a_{ikjl}) p_l. \end{aligned}\right\} \tag{14c}$$

Using equations (14a), (14b) and (14c), the system (13) may be rewritten as follows

$$\left.\begin{aligned} \frac{dx_i}{d\tau} &= a_{ijkl} p_l D_{jk}/D, \\ \frac{dp_i}{d\tau} &= -\frac{1}{2}\frac{\partial a_{ljks}}{\partial x_i} p_l p_s D_{jk}/D, \end{aligned}\right\} \tag{15}$$

($i = 1, 2, 3$).

Equations (15) are the final system of ordinary differential equations for characteristics of the non-linear partial differential equation (12). These equations are also the equations for seismic rays in an inhomogeneous anisotropic medium. When using system (15) it is not necessary to know the analytic expressions for the eigenvalues G_m.

In order to solve system (15), we must know six initial conditions which determine the co-ordinates of the point on the ray at the initial time $\tau = t_0$ and also the components of slowness vector at that point:
for $\tau = t_0$

$$x_i = (x_i)_0,\ p_i = (p_i)_0. \tag{16a}$$

The initial values $(x_i)_0$ and $(p_i)_0$ must satisfy the relation

$$G_m((p_i)_0, (x_i)_0) = 1. \tag{16b}$$

If the initial conditions (16a), satisfying equation (16b) are known, the system of differential equations (15) for a seismic ray can be solved by standard numerical techniques. For example, Runge–Kutta's method or Hamming's predictor-corrector method can be used. Thus the construction of a ray does not present any difficulties. As a solution of (15) we obtain a ray, specified by the initial conditions (16a) and (16b), in the parametric form $x_1 = x_1(\tau)$, $x_2 = x_2(\tau)$, $x_3 = x_3(\tau)$. The parameter along the ray is τ ($\tau = t$, where t is the time). The resulting values x_i and p_i along the ray will satisfy the equation $G_m(p_i, x_i) = 1$ for any time t.

System (15) is quite identical for all three types of waves which can propagate in an anisotropic inhomogeneous medium (the quasi-compressional and the two quasi-shear waves). The type of wave whose ray is to be calculated must be specified by means of the initial conditions (equation (16b)).

The system of equations (15) for the seismic ray cannot be used if the eigenvalue corresponding to the wave whose ray is to be determined is equal to one of the two other eigenvalues (in this case $D = 0$). This restriction applies both when two eigenvalues are identically equal (as in the case of shear waves in an isotropic medium) and in the case when two eigenvalues are equal to each other for certain directions (for some p_i). These possibilities can occur only in the case of quasi-shear waves. System (15) is universal for quasi-compressional waves.

The initial conditions (16a) and (16b), especially (16b), play an important role in the calculation of rays. The condition (16b) determines the type of the wave whose ray is to be calculated. In order to satisfy the condition (16b) it is helpful to take $(x_i)_0$ and $(N_i)_0$ as the initial conditions, where $(N_i)_0$ are direction cosines of the normal to the wave front at the point $(x_i)_0$. From the known $(x_i)_0$ and $(N_i)_0$ three normal velocities V_1, V_2, V_3 (corresponding to the quasi-compressional and two quasi-shear waves) can be determined by well-known methods. Knowing these values, the initial conditions (16a) satisfying (16b) are easily determined, using the relation $p_i = N_i/V$ (where V is the normal velocity).

Let us note that the direction cosines $(N_i)_0$ $(i = 1, 2, 3)$ may be fully described by two parameters, e.g. by two take-off angles. Let these parameters be denoted by q_1 and q_2, respectively, and be called the parameters of the ray. If $(x_i)_0$ is given, then each ray is fully determined by two parameters q_1 and q_2.

The number of equations in system (15) will naturally be reduced in the case that a_{ijkl} depend on one or two coordinates only. When a_{ijkl} depend on one coordinate only, e.g. on x_3, and the equations $G_m(p_i, x_i) = 1$ can be rewritten in the form $p_3 = \tilde{G}_m(p_1, p_2, x_i)$, system (15) can be solved in closed-form integrals. This fact is well known from the ray theory for isotropic media; similar integrals can be written easily for the anisotropic medium.

By comparing (11) with (10) it can be seen that the eigenvector $\mathbf{g}^{(m)}$ determines the direction of the vector $\mathbf{U}^{(0)}$. Thus, the determination of the direction of the vector of displacement $\mathbf{U}^{(0)}$ at each point of the ray does not cause difficulties. Further, the values $p_i = p_i(\tau)$, $v_i = dx_i/d\tau$ and $dp_i/d\tau$ will be obtained at each point of the ray. These quantities enable us to follow a number of physical parameters along the ray: the v_i's give the components of ray velocity, $v = (v_i v_i)^{\frac{1}{2}}$ the ray velocity and $n_i = v_i/v$

the direction cosines of the tangent to the ray. The normal velocity V is obtained from the relation $V = (p_i p_i)^{-\frac{1}{2}}$ and the direction cosines of the normal to the wave front N_i by means of the formula $N_i = V p_i$. If the angle between the direction of the ray and of the normal to the wave front is denoted by v, we get the relation $\cos v = V/v$ from the relation $p_i v_i = 1$. For the construction of wave fronts it is necessary to determine a series of rays characterized by different ray parameters. Then the wave fronts will be obtained by connecting the points lying on various rays corresponding to the same time t. The possibility of constructing theoretical travel-time curves along various profiles is evident.

Numerical examples of the computation of rays and travel-time curves in inhomogeneous anisotropic media using the system of ordinary differential equations, see above, are given in Červený & Pšenčík (in press).

It should again be stressed that the system (15) enables the calculation of rays and time-distance curves for an arbitrary anisotropic medium (described by 21 elastic parameters), inhomogeneous in all three coordinates. As the laws of reflection and refraction in anisotropic media are known (Fedorov 1965; Gassmann 1964) the existence of interfaces in this medium does not cause any major difficulty in the calculation.

4. Calculation of ray amplitudes

This section deals with the calculation of the coefficient of the ray series $\mathbf{U}^{(n)}$. As has been shown in the preceding part of the paper, three types of wave front propagate in the medium. The propagation of every one of them is described by the non-linear partial differential equation $G_m(p_i, x_i) = 1$ $(m = 1, 2, 3)$. We shall choose one of them, e.g. one for which the following differential equation is valid

$$G_1(p_1, p_2, p_3, x_1, x_2, x_3) = 1. \quad (17)$$

Let it be assumed that $G_2 \neq 1$ and $G_3 \neq 1$. The results will naturally be applicable (by a mere interchange of indices) to the other two types of waves.

It was found that the vector $\mathbf{U}^{(0)}$ for the wave described by the differential equation (17) has the direction of the eigenvector $\mathbf{g}^{(1)}$ which is determined by the relations (10). The vector $\mathbf{U}^{(n)}$ $(n \geqslant 1)$, however, may already have an arbitrary direction. Therefore, we can write

$$\mathbf{U}^{(n)} = \phi_n(x_i)\,\mathbf{g}^{(1)} + \mathbf{W}^{(n)}, \quad (18a)$$

where $\mathbf{W}^{(n)}$ lies in the plane perpendicular to $\mathbf{g}^{(1)}$,

$$\mathbf{W}^{(n)} = W_2^{(n)}\,\mathbf{g}^{(2)} + W_3^{(n)}\,\mathbf{g}^{(3)}. \quad (18b)$$

The component $\phi_n\,\mathbf{g}^{(1)}$, which has the same direction as $\mathbf{U}^{(0)}$, will be called the principal component of $\mathbf{U}^{(n)}$, the component $\mathbf{W}^{(n)}$ which is perpendicular to $\mathbf{U}^{(0)}$, will be called the additional component of $\mathbf{U}^{(n)}$. First, the additional component will be found, then the principal one. It is assumed that all the vectors $\mathbf{U}^{(0)}, \mathbf{U}^{(1)}, \ldots, \mathbf{U}^{(n-1)}$ have already been found (system (5) is recurrent).

(a) *Determination of the additional component of* $\mathbf{U}^{(n)}$

Let us denote

$$\mathbf{B}_n = \mathbf{M}(\mathbf{U}^{(n-1)}) - \mathbf{L}(\mathbf{U}^{(n-2)}). \quad (19)$$

It is supposed that the function $\mathbf{B}_n$ is known because it contains only the known vectors $\mathbf{U}^{(n-1)}$ and $\mathbf{U}^{(n-2)}$. (5) can be then written as follows

$$\mathbf{N}(\phi_n\,\mathbf{g}^{(1)}) + \mathbf{N}(\mathbf{W}^{(n)}) = \mathbf{B}_n. \quad (20)$$

It is easy to see that the first term on the left-hand side vanishes. Taking into account (10) and (18b), equation (20) yields

$$W_2^{(n)}(G_2-1)\,\mathbf{g}^{(2)}+W_3^{(n)}(G_3-1)\,\mathbf{g}^{(3)} = \mathbf{B}_n. \tag{21}$$

Because the unit vectors $\mathbf{g}^{(2)}$ and $\mathbf{g}^{(3)}$ are orthogonal, it is concluded that

$$W_2^{(n)} = (G_2-1)^{-1}\,(\mathbf{B}_n.\mathbf{g}^{(2)}),\; W_3^{(n)} = (G_3-1)^{-1}\,(\mathbf{B}_n.\mathbf{g}^{(3)}). \tag{22}$$

Note $W_2^{(0)} = W_3^{(0)} = 0$ for $n = 0$ because $\mathbf{B}_0 = 0$.

(b) *Determination of the principal component of* $\mathbf{U}^{(n)}$

Taking a scalar product of (21) with $\mathbf{g}^{(1)}$ the relation $\mathbf{B}_n.\mathbf{g}^{(1)} = 0$ is obtained. This relation can be rewritten as follows

$$\mathbf{M}(\phi_{n-1}\,\mathbf{g}^{(1)}).\mathbf{g}^{(1)} = \big(\mathbf{L}(\mathbf{U}^{(n-2)})-\mathbf{M}(\mathbf{W}^{(n-1)})\big).\mathbf{g}^{(1)}. \tag{23}$$

The index $n-1$ is changed into n and the following notation is introduced

$$\xi_n = \big(\mathbf{L}(\mathbf{U}^{(n-1)})-\mathbf{M}(\mathbf{W}^{(n)})\big).\mathbf{g}^{(1)}. \tag{24}$$

The function ξ_n is assumed to be known because it contains only the known vector $\mathbf{U}^{(n-1)}$ and the vector $\mathbf{W}^{(n)}$ determined earlier. Note that $\xi_0 = 0$. Substituting (24) into (23) yields

$$\mathbf{M}(\phi_n\,\mathbf{g}^{(1)}).\mathbf{g}^{(1)} = \xi_n. \tag{25}$$

Taking into account the definition of the operator $\mathbf{M}$ given by (6), the left-hand side of (25) may be rewritten as follows

$$\mathbf{M}(\phi_n\,\mathbf{g}^{(1)}).\mathbf{g}^{(1)} = 2\,\frac{\partial\phi_n}{\partial x_i}\,w_i+\frac{\phi_n}{\rho}\,\frac{\partial}{\partial x_i}\,(\rho w_i),$$

where $w_i = a_{ijkl}\,p_l\,g_k^{(1)}\,g_j^{(1)}$. It can be shown from (10) and (13) that w_i equal the components of the ray velocity $v_i = dx_i/d\tau$. Because $v_i(\partial\phi_n/\partial x_i) = d\phi_n/d\tau$, (25) may be rewritten in the following form

$$\frac{d\phi_n}{d\tau}+\frac{\phi_n}{2\rho}\,\frac{\partial}{\partial x_i}\,(\rho v_i) = \tfrac{1}{2}\xi_n. \tag{26}$$

The ordinary differential equation (26) for ϕ_n is the transport equation of higher order for an inhomogeneous anisotropic medium.

Equation (26) will be simplified if the so-called ray coordinates τ, q_1, q_2 are introduced, where q_1 and q_2 are the parameters of the ray. q_1 and q_2 characterize the given ray, τ characterizes the position of the point on the ray. q_1 and q_2 can be considered the curvilinear coordinates on the wave front. In the ray coordinates, the expression $\partial v_i/\partial x_i$ can be rewritten as follows

$$\frac{\partial v_i}{\partial x_i} = \frac{1}{VJ}\,\frac{d}{d\tau}\,(VJ), \tag{27}$$

where V is the normal velocity $\big(V = (p_i\,p_i)^{-\frac{1}{2}}\big)$ and $J = |\mathbf{x}_{q_1}\times\mathbf{x}_{q_2}|$, where $\mathbf{x}_{q_1}$ denotes $(\partial x_1/\partial q_1, \partial x_2/\partial q_1, \partial x_3/\partial q_1)$ and $\mathbf{x}_{q_2}$ is similar. Later we shall discuss the physical meaning and computational possibilities of J.

Substituting of (27) into (26) yields

$$\frac{d\phi_n}{d\tau}+\frac{\phi_n}{2\rho VJ}\,\frac{d}{d\tau}\,(V\rho J) = \tfrac{1}{2}\xi_n. \tag{28}$$

The above differential equation can be simply integrated. The solution of (28) is

$$\phi_n(\tau) = \phi_n(t_0)\left[\frac{(V\rho J)_{t_0}}{(V\rho J)_\tau}\right]^{\frac{1}{2}} + \frac{1}{2(V\rho J)_\tau^{\frac{1}{2}}}\int_{t_0}^{\tau}(V\rho J)_{\tilde{\tau}}^{\frac{1}{2}}\,\xi_n(\tilde{\tau})\,d\tilde{\tau}. \tag{29a}$$

The integral is taken along a ray.

From this formula, ϕ_n at an arbitrary point of the ray can be determined if its values at an earlier point $\tau = t_0$ of the ray is known.

In many cases it is sufficient to consider only the zeroth (leading) term in the ray series (3). This term is the most important in seismological applications. For $n = 0$ (29a) yields

$$\phi_0(\tau) = \phi_0(t_0)\left[\frac{(V\rho J)_{t_0}}{(V\rho J)_\tau}\right]^{\frac{1}{2}}. \tag{29b}$$

As the values of V and ρ can be determined simply along a ray, the main difficulty in calculations is caused by the function J. The function J has direct physical meaning because it characterizes some features of the so-called elementary ray tube. By an elementary ray tube, the family of rays with parameters between the limits q_1, q_1+dq_1; q_2, q_2+dq_2 is understood. The surface element of the wave front within the corresponding ray tube can be expressed by the relation $d\sigma = J dq_1 dq_2$. Therefore, the function J measures the expansion or contraction of the ray tube along a ray. However, it should be stressed that in an anisotropic medium the wave front is not perpendicular to the rays; consequently the surface element $d\sigma$ is not a perpendicular cross-section of the ray tube like in the isotropic medium. Note that the formula for $d\sigma$ shown above can be used for approximate computation of J. When the ray paths are known for parameters (q_1,q_2), $(q_1,q_2+\Delta q_2)$, $(q_1+\Delta q_1,q_2+\Delta q_2)$, $(q_1,q_2+\Delta q_2)$, where Δq_1 and Δq_2 are small, then $\Delta\sigma$ can be calculated and $J \sim \Delta\sigma/\Delta q_1\,\Delta q_2$.

The relation for J may be rewritten in the form

$$J = (EF - G^2)^{\frac{1}{2}}, \tag{30}$$

where

$$E = \frac{\partial x_i}{\partial q_1}\frac{\partial x_i}{\partial q_1},\quad F = \frac{\partial x_i}{\partial q_2}\frac{\partial x_i}{\partial q_2},\quad G = \frac{\partial x_i}{\partial q_1}\frac{\partial x_i}{\partial q_2}. \tag{31}$$

Provided it is possible to determine the quantities $\partial x_i/\partial q_1$ and $\partial x_i/\partial q_2$ along the ray, the problem of numerically calculating the function J along the ray can be solved from this. Later, a system of ordinary differential equations of first order is presented with which it is possible to determine the quantities $\partial x_i/\partial q_1$ and $\partial x_i/\partial q_2$ along the ray. (Of course, the system must be solved simultaneously with the system of differential equations for the rays (15).) (13) yields

$$\left.\begin{aligned}
\frac{d}{d\tau}\left(\frac{\partial x_i}{\partial q_r}\right) &= \frac{1}{2}\left(\frac{\partial^2 G_m}{\partial p_i \partial x_j}\frac{\partial x_j}{\partial q_r} + \frac{\partial^2 G_m}{\partial p_i \partial p_j}\frac{\partial p_j}{\partial q_r}\right),\\
\frac{d}{d\tau}\left(\frac{\partial p_i}{\partial q_r}\right) &= -\frac{1}{2}\left(\frac{\partial^2 G_m}{\partial x_i \partial x_j}\frac{\partial x_j}{\partial q_r} + \frac{\partial^2 G_m}{\partial x_i \partial p_j}\frac{\partial p_j}{\partial q_r}\right),
\end{aligned}\right\} \tag{32}$$

$(i = 1, 2, 3;\ r = 1, 2)$.

When the analytical expressions for the eigenvalues G_m are known, the expressions for the second derivatives of G_m can be determined and the system is ready for computation. Taking into consideration that the wave front is described by the partial differential equation $G_m(p_i, x_i) = 1$, the following is valid

$$\frac{\partial G_m}{\partial x_i}\frac{\partial x_i}{\partial q_r} + \frac{\partial G_m}{\partial p_i}\frac{\partial p_i}{\partial q_r} = 0. \tag{33}$$

For isotropic media the system (32) immediately produces the formulae obtained by Wesson (1970) (when (33) is also used to obtain the final form).

In inhomogeneous anisotropic media, the analytic expressions for G_m are very complicated. However, the second derivatives of G_m can also be determined without difficulty. Using (33), the system (32) can then be rewritten in the following form

$$\left.\begin{aligned}\frac{d}{d\tau}\left(\frac{\partial x_i}{\partial q_r}\right) &= \left(F_{ij}^{(1)}\frac{\partial x_j}{\partial q_r} + F_{ij}^{(2)}\frac{\partial p_j}{\partial q_r}\right)\Big/2D,\\ \frac{d}{d\tau}\left(\frac{\partial p_i}{\partial q_r}\right) &= -\left(F_{ij}^{(3)}\frac{\partial x_j}{\partial q_r} + F_{ij}^{(4)}\frac{\partial p_j}{\partial q_r}\right)\Big/2D,\end{aligned}\right\} \tag{34}$$

($i = 1, 2, 3$; $r = 1, 2$). The functions F_{ij} are given by the formulae

$$\left.\begin{aligned}F_{ij}^{(1)} &= D_{kl}\frac{\partial^2 \Gamma_{kl}}{\partial p_i \partial x_j} + \frac{\partial \Gamma_{kl}}{\partial p_i}\frac{\partial D_{kl}}{\partial x_j} + h_i E_{kl}\frac{\partial \Gamma_{kl}}{\partial x_j},\\ F_{ij}^{(2)} &= D_{kl}\frac{\partial^2 \Gamma_{kl}}{\partial p_i \partial p_j} + \frac{\partial \Gamma_{kl}}{\partial p_i}\frac{\partial D_{kl}}{\partial p_j} + h_i E_{kl}\frac{\partial \Gamma_{kl}}{\partial p_j},\\ F_{ij}^{(3)} &= D_{kl}\frac{\partial^2 \Gamma_{kl}}{\partial x_i \partial x_j} + \frac{\partial \Gamma_{kl}}{\partial x_i}\frac{\partial D_{kl}}{\partial x_j} + H_i E_{kl}\frac{\partial \Gamma_{kl}}{\partial x_j},\\ F_{ij}^{(4)} &= D_{kl}\frac{\partial^2 \Gamma_{kl}}{\partial x_i \partial p_j} + \frac{\partial \Gamma_{kl}}{\partial x_i}\frac{\partial D_{kl}}{\partial p_j} + H_i E_{kl}\frac{\partial \Gamma_{kl}}{\partial p_j},\end{aligned}\right\} \tag{35}$$

where $E_{kl} = (2-\Gamma_{nn})\delta_{kl}+\Gamma_{kl}$, h_i and H_i are given by (14a). The quantities h_i and H_i have already been determined in the calculation of seismic rays; therefore, it is not necessary to redetermine them when calculating the quantity J. Relations for first derivatives of Γ_{kl} are given by (14c). From (14c) the relations for second derivatives of Γ_{kl} are easily determined as well. The determination of the derivatives of the functions D_{kl} does not cause any difficulties either, when using equations (14b) for D_{kl}.

Equations (34) represent a system of 12 ordinary differential equations on the basis of which $\partial x_i/\partial q_r$ and $\partial p_i/\partial q_r$ may be calculated by standard numerical methods. For the determination of J only $\partial x_i/\partial q_r$ need be known; however, these quantities cannot be found without simultaneously calculating $\partial p_i/\partial q_r$.

Of course, for the solution of system (34) the initial conditions must be known: for $\tau = t_0$

$$\frac{\partial x_i}{\partial q_r} = \left(\frac{\partial x_i}{\partial q_r}\right)_0, \quad \frac{\partial p_i}{\partial q_r} = \left(\frac{\partial p_i}{\partial q_r}\right)_0. \tag{36}$$

If the quantities $\partial x_i/\partial q_r$ and $\partial p_i/\partial q_r$ are known at one point of the ray (for $\tau = t_0$), they can be determined along the whole ray using system (34). In this manner J is found using relation (30). It should be noted that in the case of a point source no difficulties arise in determining the initial conditions (36) at the point of the source.

The quantity J can then be investigated along the whole ray, beginning at the source. However, at the point of the source $(\partial x_i/\partial q_r)_0 = 0$; hence it follows that $J = 0$. Therefore, in this case, equation (29b) for ϕ_0 cannot be used without further arrangement. It can be rewritten in the following form

$$\phi_0 = \frac{\psi(q_1, q_2)}{(V\rho J)_\tau^{\frac{1}{2}}}.$$

The function $\psi(q_1, q_2)$ must be determined by other means: by a study of the problem of a point source in an anisotropic homogeneous medium. Nevertheless, the above relation may serve for solving a number of problems of dynamics of body seismic waves in an inhomogeneous anisotropic medium, even if the function $\psi(q_1, q_2)$ is not known exactly.

The above relations may be used only in the case when a_{ijkl} are continuous functions of coordinates. When the ray under study strikes an interface, the quantity J changes discontinuously. This situation needs special investigation.

5. Conclusions

Systems of ordinary differential equations of first order for rays and ray intensities (amplitudes) in inhomogeneous anisotropic media are derived. The systems can be used to compute rays and ray intensities in general anisotropic media in which the elastic parameters may be arbitrary continuous functions of all three coordinates.

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Institute of Geophysics,
Charles University,
Ke Karlovu 3, Prague 2,
Czechoslovakia

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